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WAITING TIMES WITHOUT MEMORY

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Abstract

A waiting time without memory, or age-independent residual life-time, is a positive-valued random variable T with the property that for any $x, y > 0$, given that $T > x$, the conditional probability of $T > x + y$ is the same as the unconditional probability of $T > y$; in other words, the physical process operates as if it has no memory concerning the successive occurrences of a certain event. The paper investigates the consequences of defining the property of lack of memory on more general time-domains than the positive reals. As a side issue, there is discussion of a stochastic variation of Cauchy's functional equation.

WAITING TIMES; RESIDUAL LIFE-TIMES; LACK OF MEMORY; CAUCHY FUNCTIONAL EQUATIONS

1. Introduction

If T is a positive random variable which represents the waiting time till the recurrence of a certain event in some stochastic process, it is said to be 'memory-less' iff

$$\Pr\{T > s + t \mid T > s\} = \Pr\{T > t\}. \tag{1.1}$$

This condition holds for all $s, t > 0$ iff there exists a $\lambda > 0$ such that $\Pr\{T > t\} = e^{-\lambda t}$ for all $t > 0$; the condition holds only for all positive integers s, t if T has a geometric distribution on the positive integers. These are the only known examples of memory-less waiting time distributions, each being unique on its particular domain. Are there other domains for 'waiting times without memory' (or 'age-dependent residual life-times') and corresponding distributions?

This, of course, is a vague question without a general definition of the property of lack of memory. We propose to fill this gap with the following definition.

Definition 1. A positive-valued random variable T will be said to be a *waiting time without memory* iff there exists a measurable subset S of the positive reals such that

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(1.2) $\Pr\{T \in S\} = 1$, and $s \in S$, $t > 0 \Rightarrow \Pr\{T > s + t \mid T > s\} = \Pr\{T > t\}$.

In a considerable portion of literature in applied probability, the lack-of-memory property of the exponential distribution has played a major role in simplifying work. For example, in queuing theory, if inter-arrival times and/or service times are not exponential, the problems can be almost intractable. However, in some situations, it is physically impossible to schedule inter-occurrence times to be exponentially distributed. As an example, we may consider the problem of scheduling a night-watchman's rounds through a warehouse complex; to minimize the chance of burglary, it might be desirable to achieve maximum unpredictability of the watchman's presence at any location. On the other hand, physical limitations permit only certain sets of time-points for presence at a particular location. It might be of use to know whether, for any given set of possible occurrence times, there is a probability distribution without memory for inter-occurrence times. For reasons such as these, a question of some practical interest is: What are all the possible sets S and distributions on them that satisfy (1.2)? However, before proceeding to answer this question, if we stop to take a second look at (1.2), we notice that the formal equation is symmetric in the two variables, but the domain is not. Hence, it is natural to inquire into the consequences of relaxing (1.2) and requiring it to hold only for $s, t \in S$. This suggests another definition.

Definition 2. A positive-valued random variable T will be said to be a *partially memory-less waiting time of Type* A iff there exists a measurable subset A of the positive reals such that

(1.3) $\Pr\{T \in A\} = 1$ and $s, t \in A \Rightarrow \Pr\{T > s + t \mid T > s\} = \Pr\{T > t\}$.

Referring to the night-watchman problem mentioned above, if A is the set of possible times of appearance of a watchman at a certain location, and these appearances are scheduled in accordance with a probability law satisfying (1.3), then a potential burglar who is watching the situation will get little help from his observations in determining how much safe time he has for his operation. Also, the functional equation resulting from (1.3) is of some interest in itself, since it is a variation on the classical Cauchy equation. Whereas in Cauchy's equation, the domain is predetermined to be the reals and different conditions are imposed on the function, in the present problem, we are interested in functions of predetermined type (tail-probability functions) and the domain is restricted only by a stochastic condition.

Finally, there is another aspect of lack of memory of waiting times which merits some attention. The following example will serve as motivation: Suppose that two consecutive scheduled arrival times for buses at a bus-stop are 5:10 and 5:25; a person who is waiting for the 5:10 bus finds that the bus does not arrive on schedule; there are then two possibilities, namely either the bus is late and will arrive at some time later or it has been cancelled and there will be no arrival till the next scheduled arrival time. This suggests a third definition.

Definition 3. A positive real-valued random variable T will be said to be a *partially memory-less waiting time of Type* B iff there exists a measurable subset B of the positive reals such that

$$\text{(1.4)} \quad \Pr\{T \in B\} = 1 \text{ and } s, t \in B,\ s < t \Rightarrow \Pr\{T > t \mid T > s\} = \Pr\{T > t - s\}.$$

It is obvious that if T is a waiting time without memory (Definition 1), then it is partially memory-less of both Types A and B. The question of the converse implication does not seem to be trivial.

In Section 2, we investigate Definition 2 in detail and consider the problem of all possible domains and distributions having the required property.

In Section 3, we do the same for Definition 3 (an easier task). It is then easily seen that if T has the properties required in both Definitions 2 and 3, then its distribution is either exponential on the positive reals or geometric on a lattice of the positive reals. It is thus shown that Definitions 2 and 3 together are exactly equivalent to Definition 1.

2. Partial lack of memory (Type A)

Concentrating on the purely mathematical aspects of Definition 2, we encounter a functional equation which is a variation on the classical Cauchy functional equation. With the identification $f(t) = \Pr\{T > t\}$, we state the following problem.

Problem. Let f be a right-continuous, monotone non-increasing function on the reals, with $f(x) = 1$ for $x \leqq 0$ and $f(\infty) = 0$, and let P be the measure on one-dimensional Borel sets which is uniquely defined by $P((a, b]) = f(a) - f(b)$. Let A be a Borel set such that

$$\text{(2.1)} \qquad P(A) = 1 \text{ and } s, t \in A \Rightarrow f(s + t) = f(s)f(t).$$

Then what are all possible pairs (A, f) satisfying these conditions?

In Cauchy's functional equation, there is no probabilistic condition, A is the real line and f has been shown to be of exponential form under quite weak conditions on it. In our case, $f(x) = e^{-\lambda x}$, $x \geqq 0$, with some $\lambda > 0$, is a possibility, as is also $A = \{nh, n = 1, 2, \cdots\}$ and $f(nh) = e^{-\lambda n}$; and $A = \{a\}$, $f(a-) = 1$ and $f(a) = 0$ is another solution, the trivial one which will always be excluded.

While looking for solutions of the problem, repeated use will be made of the fact that $P(A) = 1$, which implies, in particular, that every point of discontinuity of f belongs to A; more generally, for all x and y such that $x < y$ and

$f(x) > f(y)$, we have $(x, y] \cap A \neq \varnothing$. Keeping this property in mind, we can expect to effect some simplification by eliminating inessential points (such as interior points of intervals of zero P-measure). So, in (2.1) we replace A by $A' = a - \cup\{(a, b): 0 < a < b, f(a) = f(b)\}$. This aspect can be illustrated by looking at the geometric distribution: $f(x) = \alpha^n$ for $x \in [n, n+1)$, $n = 0, 1, 2, \cdots$, with some $\alpha \in (0, 1)$; here A can be any set such that $\{1, 2, 3, \cdots\} \subseteq A \subseteq \{1, 2, 3, \cdots\} \bigcup_{k=1}^{\infty} [k, k+\frac{1}{2})$, but $A' = \{1, 2, 3, \cdots\}$.

Further A' can be augmented by throwing in limit-points, so that we are dealing with the closed minimal support of P. That (2.1) is satisfied at right-limit points of A' follows immediately from the right-continuity of f. As regards left-limit points, let $\{a_n, n = 1, 2, \cdots\}$ A' and $a_n \uparrow a$ as $n \to \infty$. Then $f(a_n) \to f(a-)$ and $f(2a_n) = [f(a_n)]^2$, so that $f(2a-) = [f(a-)]^2$. If $f(a-) > f(a)$, then $a \in A'$, and so $f(a + a_n) = f(a)f(a_n)$; taking limits, we have $f(2a-) = f(a)f(a-) < [f(a-)]^2 = f(2a-)$. Hence, it is impossible that $f(a-) > f(a)$; and so, if a is the limit of a bounded increasing sequence of elements of A', then f is continuous at a. Consequently, if $\{a_n, n = 1, 2, \cdots\}$ A' and $a_n \to a$ as $n \to \infty$, then $f(a_n) \to f(a)$. Thus (2.1) holds with A replaced by the closure A^* of A'; note that A^* is the minimal closed support of P. We have thus established two lemmas.

Lemma 2.1. Without loss of generality, we may assume that A is closed and that $f(a) > f(b)$ if $(a, b) \cap A \neq \varnothing$.

Lemma 2.2. The following two properties are equivalent, and either implies the existence of a $\lambda > 0$ such that $f(x) = e^{-\lambda x}$ for all $x > 0$:

(I) f is continuous on $[0, \infty)$,

(II) there does not exist an $x > 0$ such that $f(x) = 1$.

Proof. (i) I $\Rightarrow$ II. For if I $\not\Rightarrow$ II, then f is continuous and there exists an $x > 0$ such that $f(x) = 1$, then there exists an $a > 0$ such that $f(a) = f(a-) = 1$ and $f(x) < 1$ for every $x > a$; hence, every right-neighbourhood of a has positive probability and contains a point of A, so that a is a limit point of A and, by Lemma 2.1, is in A; consequently, $f(2a) = [f(a)]^2 = 1$, which contradicts the definition of a.

Note that the argument used here shows that if f satisfies (2.1) and there exists an $x > 0$ such that $f(x) = 1$, then there exists an $a > 0$ such that $f(a-) = 1 > f(a)$; this fact will be used later.

(ii) II implies that f is strictly decreasing on $[0, \infty)$. To prove this, we start with the observation that II implies that every right-neighbourhood of 0 is of positive probability, and hence there exists $\{a_n, n = 1, 2, \cdots\} \subset A$ such that $a_n \downarrow 0$. Now suppose that II holds but not the implication (ii); then there exist b, c such that $0 < b < c$, $1 > f(c) = f(b) < f(x)$ for $x < b$. Hence, every left-neighbourhood of b

has positive probability, so that b is the left-limit of points of A and is therefore in A; consequently, $f(b+a_n)=f(b)f(a_n)<f(b)$, which contradicts the assumption that $f(b)=f(c)$, since $b+a_n<c$ for sufficiently large n.

(iii) Finally, if f is strictly decreasing on $[0,\infty)$, then $A=[0,\infty)$ by Lemma 2.1; it is well-known that if f is a monotone solution of Cauchy's functional equation, then $f(X)=e^{-\lambda x}$ for some $\lambda>0$ (which implies I).

Lemma 2.3. If there is an additive semi-group $S\subset A$, then there exists a $\lambda>0$ such that $f(s)=e^{-\lambda s}$ for all $s\in S$. (This is undoubtedly a well-known result, and we give a proof only because we do not have a reference.)

Proof. Let $a,b\in S$ with $0<a<b$, and let n be a positive integer. Choose N such that $Na\leqq nb<(N+1)a$. Since f is monotone, $f(Na)\geqq f(nb)>f[(N+1)a]$. Now, for each $s\in S$ and each positive integer k, we have $f(ks)=[f(s)]^k$, so that $[f(a)]^{N/n}\geqq f(b)\geqq[f(a)]^{(N+1)/n}$. Letting $n\to\infty$, we get $f(b)=[f(a)]^{b/a}$, which leads to the conclusion.

Lemma 2.4. If f satisfies (2.1), then the range of $f=e^{-\Sigma}$, where Σ is an additive semi-group with $\inf\Sigma=0$; conversely, if Σ is any closed additive semi-group with $\inf\Sigma=0$, then there exists a pair (A,f) satisfying (2.1) such that the range of $f=e^{-\Sigma}$.

Proof. First of all, the converse is the immediate consequence of the construction: $A=\Sigma$, $f(\sigma)=e^{-\sigma}$ for all $\sigma\in A$.

So now suppose f satisfies (2.1), and for $x>0$ define $m(x)=\sup\{a:a\in A, a\leqq x\}$. Then $m(x)\in A$ by Lemma 2.1; also, $(m(x),x)\cap A=\emptyset$, so that $P(m(x),x)=0$, or $f[m(x)]=f(x)$. Hence, to each point τ in the range of f, there corresponds a point t of A such that $f(t)=\tau$; i.e. the range of f is $f(A)$.

If $\tau_1,\tau_2\in f(A)$ and $t_1,t_2\in A$ such that $f(t_i)=\tau_i$, $i=1,2$, then $f(t_1+t_2)=f(t_1)f(t_2)=\tau_1\tau_2$; but $f(t_1+t_2)\in f(A)$. Thus the range of f is closed under multiplication.

Theorem 2.1. Let Σ be any closed additive semi-group other than the singleton $\{0\}$; let $\inf\Sigma=0$ and $\delta(\Sigma)=\inf\{|\sigma-\tau|:\sigma,\tau\in\Sigma,\sigma\neq\tau\}$.

(I) If $\delta(\Sigma)=0$, and if (A,f) be any pair which satisfies (2.1) with range of $f=e^{-\Sigma}$, A closed, $f(a)>f(b)$ if $(a,b)\cap A\neq\emptyset$, then there exists a $\lambda>0$ such that $\lambda A=\Sigma$ and $a\in A\Rightarrow f(a)=e^{-\lambda a}$.

(II) If $\delta(\Sigma)=\delta>0$, then there exists an additive semi-group $J=\{j_n, n=1,2,\cdots\}$ of positive integers such that $\Sigma=\{0\}\cup(\delta J)$; further, there exists a continuum of countable discrete sets $A=\{a_n, n=1,2,\cdots\}$ such that if f is a pure jump-function whose discontinuity points are $\{a_n\}$ and if $f(a_n)=e^{-\delta j_n}$, $n=1,2,\cdots$, then (A,f) satisfies (2.1); in particular, there always exists an A for which there exists no λ such that if (A,f) satisfies (2.1), then $f(a_n)=e^{-\lambda a_n}$ for two or more n. The possible choices for A are subject only to the constraints described below.

Choose $b_1 > 0$ arbitrarily, and the successive b's arbitrarily subject to the following sequential constraints:

Let m_n and M_n be respectively the minimum and the maximum of $S_n = \{b_1 + b_{n-1}, b_2 + b_{n-2}, \cdots, b_{n-1} + b_1\}$, $n = 2, 3, \cdots$. Then

$$M_{n-1} < b_n \leqq m_n, \ n = 2, 3, \cdots, \ (M_1 = b_1), \tag{2.2}$$

and $a_n = b_{j_n}$, $n = 1, 2, \cdots$.

Proof. Since Lemma 2.2 has already disposed of the case in which $f(x) < 1$ for every $x > 0$, we may now assume that there exists an $x > 0$ such that $f(x) = 1$, and let a be the supremum of all such x's; then, by the note at the end of the proof of Lemma 2.2, $f(a-) > f(a)$, $a \in A$ and $a = \inf\{A\}$. Let $\alpha = -\ln f(a)$.

(I) $\delta(\Sigma) = 0$. For $\sigma \in \Sigma$, $\sigma > \alpha$ and any positive integer n, let N be such that $n\sigma \leqq N\alpha < (n+1)\sigma$ and let $s(n) = \inf\{x : f(x) = e^{-n\sigma}\}$, $n = 1, 2, \cdots$, and $a(N) = \inf\{x : f(x) = e^{-N\alpha}\}$; then $a(N), s(n) \in A$. Now, $f[s(1) + s(n)] = e^{-(n+1)\sigma} = f[s(n+1)]$; and by the monotonicity of f, we then have $s(n) \leqq a(N) < s(n+1) \leqq s(1) + s(n)$, so that, in particular, $s(n) \leqq ns(1) < s(n+1)$. Consequently, we see that $(n-1)s(1) < s(n) \leqq a(N) \leqq Na(1)$, and as $n \to \infty$, we get $\alpha / a(1) \leqq \sigma / s(1)$. Now, by definition, $a(1) = a$, and writing s for $s(1)$, we get, for all $\sigma \in \Sigma$, the inequality $\sigma / s \geqq \alpha / a = \lambda$, say.

Next we note that Σ being closed under addition and $\delta(\Sigma) = 0$ imply that Σ is dense in the reals at $+\infty$. This is seem by observing that, given any $h > 0$, there exist $\sigma, \tau \in \Sigma$ such that $0 < \tau - \sigma = h' \leqq h$; and if N is such that $Nh' \geqq \sigma$, then for every $n \geqq N$, $n\sigma + jh' = (n-j)\sigma + j\tau \in \Sigma$ for $j = 0, 1, \cdots, n$; that is to say, for each $h > 0$, there exists a $\sigma_h \in \Sigma$ such that, to the right of σ_h, the elements of Σ are at most distance h apart.

An immediate consequence of these results is that $\exists c_n, d_n \in A$, $n = 1, 2, \cdots$, such that $c_n, d_n \to \infty$ and $|c_n - d_n| \to 0$ as $n \to \infty$, which is seen as follows: to every positive integer k, there corresponds a $\sigma(k) \in \Sigma$ such that, to the right of $\sigma(k)$, the points of Σ are at most distance $1/k$ apart. Hence, between $\sigma(k)$ and $2\sigma(k)$, there are more than $k\sigma(k) - 1$ points of Σ; but $f[s(k)] = e^{-\sigma(k)}$ and $f[2s(k)] = e^{-2\sigma(k)}$ imply that between $s(k)$ and $2s(k)$ there are more than $k\sigma(k) - 1$ points of A, and thus more than $\lambda ks(k) - 1$; consequently, the minimum distance between points of $A \cap [s(k), 2s(k)]$ is at most $s(k)/\{\lambda ks(k) - 1\}$, which goes to zero as $k \to \infty$.

Actually, we can now conclude that A is dense in the reals at $+\infty$: If possible, assume the contrary, so that there exist an $h > 0$ and sequences $\{a_n\}$, $\{b_n\}$ in A such that $a_n + h \leqq b_n$, $a_n \to \infty$ and $(a_n, b_n) \cap A = \varnothing$. Then $f(a_n) = e^{-\alpha_n}$, $f(b_n) =$

$e^{-\beta_n}$, $(\alpha_n, \beta_n) \cap \Sigma = \varnothing$, and hence $\alpha_n - \beta_n \to 0$ as $n \to \infty$; i.e., $f(a_n)/f(b_n) \to 1$ as $n \to \infty$. By the previous paragraph, we can choose $c, d \in A$ such that $c < d < c + h$. Now we have $a_n + d < b_n + c$, so that $f(a_n)f(d) = f(a_n + d) > f(b_n + c) = f(b_n)f(c)$; i.e. $f(a_n)/f(b_n) > f(c)/f(d) > 1$, for all n, which contradicts the earlier conclusion that the limit of the left-hand ratio is 1.

Now, for the final step, let $b > a$ be an arbitrary element of A, and for each positive integer n, let N be such that $Na \leqq nb < (N+1)a$. Having chosen an n, let us choose a positive $h < a - nb/(N+1)$, and observe that there exists an $M \in A$ such that $x > M$ implies $(x - h, x] \cap A \neq \varnothing$. In the argument that follows, we shall obtain a relation satisfied by f for every n and then let $n \to \infty$.

Let $f(b) = \beta$, $f(M) = \mu$, and for any $c \in A$, let $f(c) = \gamma$. If k is any integer in $(0, N+1]$, we observe the following:

$$f(M + c) = f(M)f(c) = \mu\gamma.$$

If $M + c \notin A$, there exists a nearest left-neighbour $M + c - h_1$ of $M + c$ in A, so that $f(M + c) = f(M + c - h_1)$ where $h_1 \in (0, h)$, so that

$$f(M + 2c - h_1) = f(M + c - h_1 + c) = f(M + c - h_1)f(c) = f(M + c)f(c) = \mu\gamma^2.$$

Proceeding in this fashion, we obtain

$$\text{(2.3)} \qquad \mu\gamma^k = f(M + c - H_k), \text{ where } H_k = h_1 + h_2 + \cdots + h_{k-1} < kh.$$

Now let $r, s, t \in A$ be such that $f(r) = \mu e^{-(N-1)\alpha}$, $f(s) = \mu\beta^n$ and $f(t) = \mu e^{-(N+1)\alpha}$. Substituting in (2.3) successively $(e^{-\alpha}, N-1)$, (β, n) and $(e^{-\alpha}, N+1)$ for (γ, k), and noting that the maximum H_k involved is less than $(N+1)a - nb$, we obtain

$$\text{(2.4)} \quad r \leqq M + (N-1)a \leqq M + nb - a < s \leqq M + nb < t \leqq M + (N+1)a,$$

and hence

$$\text{(2.5)} \qquad \mu e^{-(N-1)\alpha} < \mu\beta^n < \mu e^{-(N+1)\alpha},$$

from which, as $n \to \infty$, we see that $e^{-\alpha b} = \beta^a$.

Thus, for each $b \in A$, $f(b) = \beta = e^{-\alpha b/a} = e^{-\lambda b}$.

(II) $\delta(\Sigma) = \delta > 0$. Since Σ is closed under addition, therefore all integer multiples and sums of integer multiples of elements of Σ are in Σ. From this, we see that there is a pair of elements of Σ which are a distance δ apart; for otherwise, there would have to be a sequence of pairs of points of Σ such that the distance between the nth pair decreases to δ as $n \to \infty$; hence there would have to be $a_1, b_1, a_2, b_2 \in \Sigma$ such that $b_1 = a_1 + \delta_1$, $b_2 = a_2 + \delta_2$ and $\delta < \delta_2 < \delta_1 < 2\delta$, which would mean that Σ contains the points $a_1 + b_2 = a_1 + a_2 + \delta_2$ and $a_2 + b_1 = a_1 + a_2 + \delta_1$ whose distance apart, $\delta_2 - \delta_1$, is positive but less than δ. Thus there exists a pair $a, b \in \Sigma$ such that $b = a + \delta$. Next we conclude that the distance between any pair of elements of Σ is an integral multiple of δ; for if there exists an integer k and points $a_k, b_k \in \Sigma$ such that $k\delta < b_k - a_k < (k+1)\delta$, then $ka + b_k$ and $a_k + kb$ would be distinct points of Σ whose distance apart is less than δ. Finally, if follows that $\alpha = \inf\{\sigma : \sigma \subset \Sigma, \sigma > 0\}$ is an integral multiple of δ, since it is the distance between α and 2α which are in Σ. Hence, we see that $\Sigma = \{0\} \cup \{\delta J\}$, where J is an additive semi-group of positive integers.

Consequently, f is a pure jump-function whose points of discontinuity are $A = \{a_n, n = 1, 2, \cdots\}$, with $0 < a_1 < a_2 < \cdots$, and $f(a_n) = e^{-\delta j_n}$, $n = 1, 2, \cdots$. We shall first resolve the problem when J is the whole set of positive integers; after that, dealing with a proper subset J is merely a matter of redefining f by making it constant through the integers which are no longer there.

So we now consider a sequence $A = \{b_n\}$ such that $0 < b_1 < b_2 < \cdots$, $f(b_n) = e^{-n\delta}$, $n = 1, 2, \cdots$, and (A, f) satisfy (2.1). We shall now proceed to show that the construction given by (2.2) is valid; i.e., $M_{n-1} < m_n$, $n = 2, 3, \cdots$, and any sequence $\{b_n\}$ satisfying it is admissible in (2.1).

Note that $b_1 > 0$ is arbitrary, $b_1 < b_2 \leqq ab_1$, so that $M_1 < m_2$; now make the induction hypothesis $\mathcal{H}_k$ that $M_{n-1} < m_n$, $n = 2, 3, \cdots, k$, with $b_1, \cdots, b_{k-1}$ chosen to satisfy (2.2), and choose a b_k so as to satisfy (2.2). Then we wish to show that $M_k < m_{k+1}$; i.e., we have to show that $b_l + b_{k-l}$, $l = 1, 2, \cdots, k-1$, are all less than $b_m + b_{k-m+1}$, $m = 1, 2, \cdots, k$. The case $l = m$ presents no problem, since the construction ensures that $\{b_n\}$ is an increasing sequence; so we consider the two cases: (i) $l < m$, (ii) $l > m$.

In Case (i), we have $b_l + b_{m-l-1} < b_m$, and $b_{k-l} \leqq b_{k-m+1} + b_{m-l-1}$ by $\mathcal{H}_k$, so that

$$b_l + b_{k-l} + b_{m-l-1} < b_m + b_{k-m+1} + b_{m-l-1}.$$

In Case (ii), $b_l \leqq b_m + b_{l-m}$ and $b_{k-l} + b_{l-m} < b_{k-m+1}$ by $\mathcal{H}_k$, so that $b_l + b_{k-l} + b_{l-m} < b_m + b_{k-m+1} + b_{l-m}$.

Hence $\mathcal{H}_k \Rightarrow \mathcal{H}_{k+1}$, and so we have $M_{n-1} < m_n$, $n = 2,3, \cdots$. (Incidentally, it is easily seen that $\{(m_n - M_{n-1})\}$ is a monotone non-increasing sequence; we have not looked at the matter of its limiting value.)

Having chosen such a sequence $\{b_n\}$, we now define f by

$$f(x) = \begin{cases} 1, & 0 \leqq x < b_1, \\ e^{-n\delta}, & b_n \leqq x < b_{n+1}, \quad n = 1, 2, \cdots. \end{cases} \tag{2.6}$$

It is now easily verified that (2.1) is satisfied, because the conditions (2.2) are precisely the conditions needed to verify that

$$b_{m+n} \leqq b_m + b_n < b_{m+n+1}, \qquad m, n = 1, 2, \cdots.$$

Finally, the existence of integers m, n and a $\lambda > 0$ such that $f(b_m) = e^{-\lambda b_m}$ and $f(b_n) = e^{-\lambda b_n}$ is exactly equivalent to the rationality of b_m/b_n; and our construction makes it clear that each b_n can be chosen so as to be relatively irrational to the previous ones, since we have a continuum of choices.

Having disposed of the case $J = N^+$, it remains to state precisely what happens when J is a proper subset of N^+. It this case $\Sigma = \{0\} \cup \{\delta J\}$ where $J = \{j_n\}$ is an additive semi-group of positive integers, with $0 < j_1 < j_2 < \cdots$; if we take any sequence $\{b_n\}$ constructed as above and define

$$(2.7) \qquad a_n = b_{j_n}, \; n = 1, 2, \cdots, \text{ and } f(x) = \begin{cases} 1, & 0 \leqq x < a_1 \\ e^{-\delta j_n}, & a_n \leqq x < a_{n+1}, \; n = 1, 2, \cdots, \end{cases}$$

it is easily verified that $(\{a_n\}, f)$ satisfy (2.1).

3. Total lack of memory

To return to the original problem of waiting times without memory, we shall first deal with Definition 3 and Property (1.5) of Section 1, and then with random variables which have both Properties (1.4) and (1.5); finally, Definition 1 will be discussed independently by itself.

Concentrating on the purely mathematical aspect of (1.5), we encounter a functional equation which is complementary to (2.1), but turns out to be much easier to deal with. This is primarily because $f(t) = \Pr\{T > t\}$ being a right-continuous function, Equation (3.1) below gives us more of a hold on it than (2.1) does.

Problem. Let f be a right-continuous, monotone non-increasing function on the reals, with $f(x) = 1$ for $x \leqq 0$ and $f(\infty) = 0$, and let P be the measure on one-dimensional Borel sets which is uniquely defined by $P((a, b]) = f(a) - f(b)$. Let B be a Borel set such that

$$(3.1) \qquad P(B) = 1 \quad \text{and} \quad s, t \in B, \qquad s < t \Rightarrow f(t) = f(s) f(t - s).$$

Then what are all possible (B, f) satisfying these conditions?

Theorem 3.1. If (B, f) satisfies (3.1), then either

(I) there exists a $\lambda > 0$ such that $f(x) = e^{-\lambda x}$, $x > 0$, and (3.1) is satisfied by this f and $B = R^+$, or

(II) there exists a $\lambda > 0$ and B is a countable set $\{b_n\}$, with $0 < b_1 < b_2 < \cdots$, and $f(b_n) = e^{-n\lambda}$, $n = 1, 2, \cdots$; the possible choices for B are subject only to the constraints described below:

Choose $b_1 > 0$ arbitrarily, and the successive b's arbitrarily subject to

$$(3.2) \qquad M_n \leqq b_n < m_{n-1}, \qquad n = 2, 3, \cdots,$$

where m_n and M_n are as defined for (2.2).
(It is understood that the trivial case, in which f is a pure jump-function with a single jump of magnitude 1, is excluded.)

Proof. As in Section 2, first consider the two mutually exclusive and exhaustive possibilities: (I) $f(x) < 1$ for all $x > 0$, or (II) $\exists b_1 > 0 \ni f(b_1 -) = 1$ and $f(x) < 1$ for all $x > b_1$. In Case (I), $\exists a_n \to 0$ such that $\{a_n\} \subset B$ and $f(a_n) > f(a_{n+1})$, $n = 1, 2, \cdots$. Hence $t \in B \Rightarrow f(t) = f(a_n)f(t - a_n)$, so that $f(t) = f(t-)$, and also t is the limit from the left of points at which f exceeds $f(t)$. Thus there are no discontinuity points in B (and hence no discontinuity points at all) and every point in B is a point of decrease from the left. Thus f decreases continuously on R^+, and hence we have conclusion (I) of the theorem.

In Case (II), $b_1 \in B$ either because it is a discontinuity point of B or because it is the limit from the right of points in B (same reasoning as in Section 2); $t \in B$, $t > b_1 \Rightarrow f(t) = f(b_1)f(t - b_1)$, so that $f(2b_1 -) = f(b_1)$. Thus, b_1 is a discontinuity point of f and f is constant in $[b_1, 2b_1)$. It is now easily verified that the right-hand end-point, b_2, of this interval of constancy is again a discontinuity-point; and continuing in this way, we see that f is a pure jump-function with $f(b_n) = e^{-n\lambda}$, $n = 1, 2, \cdots$, where $\lambda > 0$ is arbitrary. The constraints on $\{b_n\}$ follow from the fact that (3.1) implies

$$b_m + b_n \leqq b_{m+n} < b_{m+n+1}. \tag{3.3}$$

Theorem 3.2. T is a waiting time which is partially memory-less of both Types A and B (Definitions 2 and 3 of Section 1) iff either

(I) $\exists \lambda > 0$ such that $\Pr\{T > t\} = e^{-\lambda t}$, $t > 0$, or

(II) $\exists \lambda, h > 0$ such that

$$\Pr\{T > t\} = \begin{cases} 1, & 0 \leqq t < h \\ e^{-n\lambda}, & nh \leqq t < (n+1)h, \ n = 1, 2, \cdots. \end{cases}$$

Proof. The result follows immediately from the fact that $f(t) = \Pr\{T > t\}$ must satisfy both (2.1) and (3.1), so that if $C = A \cap B$, then we can replace A by C in (2.1) and B by C in (3.1). Then Part (I) of Theorem 3.1 gives us the first part of the present theorem, and the intersection of Parts (II) of Theorems 2.1 and 3.1 is the present Part (II).

Theorem 3.3. T is a waiting time without memory (Definition 1) iff it is partially memory-less of both Types A and B (Definitions 2 and 3).

Proof. The theorem asserts the equivalence of two sets of assumptions; since the consequences of one of these have been disposed of in Theorem 3.2, it remains only to study the consequences of Property (1.2). If P is the probability measure on the Borel sets of real members which is defined by $P((a,b]) = f(a) - f(b)$, we have $P(S) = 1$ and hence $f(a) > f(b)$ implies that $(a, b] \cap S \neq \varnothing$. Note that since T is a positive random variable, $f(0) = 1$. We have now two possibilities: (I) $f(x) < 1$ for all $x > 0$, or (II) $h > 0$ such that $f(h -) = 1$ and $f(x) < 1$ for all $x > h$.

In Case (I), every right-neighbourhood of 0 has positive probability and hence there exists an infinite sequence $\{s_n\} \subset S$ such that $s_n \downarrow 0$ as $n \to \infty$ and $f(s_n) < f(s_{n+1}) < 1$. Hence for all $t > 0$, we have $f(s_n + t) = f(s_n)f(t) < f(t)$, so that every right-neighbourhood of t has positive probability. Thus to each $t > 0$ there corresponds an infinite sequence $\{t_n\} \subset S$ such that $t_n \downarrow t$; and so for all t, $u > 0$, we have $f(t_n + u) = f(t_n)f(u)$, giving in the limit, $f(t + u) = f(t)f(u)$. It is well known that this implies consequence (I) of Theorem 3.2.

In Case (II), either $f(h) < f(h-)$ in which case $h \in S$ or $f(h) = f(h-)$ in which case, every right-neighbourhood of h has positive probability and hence, as in the previous paragraph, Equation (1.2) is satisfied with s replaced by h. Hence $f(2h-) = f(h)$ and $f(2h) = [f(h)]^2$, which means $f(h) < 1$ and $2h$ is also a discontinuity point of f. Proceeding this way, we come to conclusion (II) of Theorem 3.2.