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Conditioned Limit Theorems for Random Walks with Negative Drift*

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Summary. In this paper we will solve a problem posed by Iglehart. In (1975) he conjectured that if S_n is a random walk with negative mean and finite variance then there is a constant α so that $(S_{[n.]}/\alpha n^{1/2} | N > n)$ converges weakly to a process which he called the Brownian excursion. It will be shown that his conjecture is false or, more precisely, that if $ES_1 = -a < 0$, $ES_1^2 < \infty$, and there is a slowly varying function L so that $P(S_1 > x) \sim x^{-q} L(x)$ as $x \to \infty$ then $(S_{[n.]}/n | S_n > 0)$ and $(S_{[n.]}/n | N > n)$ converge weakly to nondegenerate limits. The limit processes have sample paths which have a single jump (with d.f. $(1-(x/a)^{-q})^+$) and are otherwise linear with slope $-a$. The jump occurs at a uniformly distributed time in the first case and at $t=0$ in the second.

Introduction

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables which have $EX_1 = -a < 0$, $EX_1^2 < \infty$, and a distribution which is regularly varying at ∞ – that is, there is a slowly varying function L so that $P(X_1 > x) \sim x^{-q} L(x)$ as $x \to \infty$. Let $S_n = X_1 + \ldots + X_n$ and let $N = \inf\{m \geqq 1 : S_m \leqq 0\}$. In this paper we will obtain limit theorems for the sequences of stochastic processes defined by

$$Y_n(t) = (S_{[nt]}/n | S_n > 0) \qquad 0 \leqq t \leqq 1$$

and

$$Z_n(t) = (S_{[nt]}/n | N > n) \qquad 0 \leqq t \leqq 1$$

(here $[nt]$ denotes the greatest integer $\leqq nt$). The key to determining the limit behavior of these processes is the following asymptotic formula for $P(S_n > 0)$.

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Theorem 2.1. *As* $n \to \infty$

$$P(S_n > 0)/n P(S_1 > n a) \to 1.$$

From this result we get that $n^2 P(S_1 > n a)^2 \to 0$ and see that if $S_n > 0$ then it is because there was one jump bigger than $n a$ in the first n steps. Since the distribution of $(X_1, \dots X_n | S_n > 0)$ is exchangeable the jump occurred at a uniformly distributed time. Combining the last observation with the fact that $P(S_1 > x) \sim x^{-q} L(x)$ leads easily to

Theorem 3.1. *As* $n \to \infty$

$$(S_{[n.]}/n | S_n > 0) \Rightarrow J_{a,q} 1_{(U \leqq .)} - a.$$

where $J_{a,q}$ *and* U *are independent random variables with*

$$P(J_{a,q} \geqq x) = (x/a)^{-q} \quad \text{for } x \geqq a$$

and

$$P(U \leqq t) = t \quad \text{for } 0 \leqq t \leqq 1.$$

Using this result it is easy to guess what the limit theorem for $(S_{[n.]}/n | N > n)$ should be. The last result shows that $S_n > 0$ only if there is a jump bigger than $n a$ in the first n steps so if we have $S_k > 0$ for all $1 \leqq k \leqq n$ then there must have been a large jump which occurred very early, i.e. $o(n)$ in the sequence. The next result shows this reasoning is correct and in fact that the jump occurs at a time which is $O(1)$.

Theorem 3.2. *If* $U_n = \inf\{j : S_{j-1} > n a\}$ *then*

$$P(U_n = j | N > n) \to P\{S_i > 0 \ 1 \leqq i < j\}/EN$$

and $(S_{[n.] \vee U_n}/n | N > n) \Rightarrow J_{a,q} - a.$

The reader should note that the last result needs to be carefully formulated so that there is weak convergence

$$\lim_{n \to \infty} (S_0/n | N > n) = 0$$

while

$$\lim_{t \downarrow 0} \lim_{n \to \infty} (S_{[nt]}/n | N > n) = J_{a,q} \geqq a > 0.$$

Since $\theta_t f = f(t)$ is continuous for $t = 0$ and for all but a countable number of $t \in (0, 1)$ this means that $(S_{[n.]}/n | N > n)$ cannot converge weakly in D. This defect could be remedied by embedding D in the class of compact subsets of $R \times [0, 1]$ and using the topology which arises from Hausdorf's metric on that space (see Kelley (1955) p. 131) but this would not improve the result.

Finally we would like to comment on the assumptions we have used. The proof of Theorem 2.1 relies on the assumed shape of $P(X_1 > x)$ and uses the fact that $EX_1 < 0$ and $EX_1^2 < \infty$. The moment assumptions can be weakened (by considering convergence to stable laws) but the assumption that $P(X_1 > x) \sim x^{-q} L(x)$ for some $q \in (0, \infty)$ is necessary for $(X_1/n \mid X_1 > na)$ to converge to a nondegenerate limit and hence is necessary for $(S_{[n.]}/n \mid S_n > 0)$ and $(S_{[n.]}/n \mid N > n)$ to converge to the limits indicated above. We believe that if $E(X_1^+)^p = \infty$ for some $p < \infty$ and $P\{X_1 > x\}$ is not regularly varying as $x \to \infty$ then $(S_{[n.]}/n \mid S_n > 0)$ and $(S_{[n.]}/n \mid N > n)$ do not converge in distribution. This result would imply that Iglehart's conjecture is false when $E(X_1^+)^p = \infty$ for some $p < \infty$, and suggest that the results of Kao (1976) and Durrett (1977), who verified the conjecture for distributions satisfying Cramér's condition, are close to the best possible.

Section 2

In this section we will prove

Theorem 2.1. *If S_n is a random walk with $ES_1 < 0$, $ES_1^2 < \infty$, and $P(S_1 > x) \sim x^{-q} L(x)$ then as $n \to \infty$*

$$P(S_n > 0)/n\, P(S_1 > na) \to 1.$$

In the next section, which may be read before this one, we will apply this result to obtain our conditioned limit theorems.

The proof of Theorem 2.1 will be accomplished in two steps. We will first prove that the lim inf$\geqq 1$ and then in Lemma 2.3 that the lim sup$\leqq 1$. For the first step we need only assume that $E|S_1| < \infty$.

Lemma 2.1. *Let S_n be a random walk with $ES_1 = 0$. If a and ε are positive then*

$$\liminf_{n \to \infty} \frac{P(S_n > na)}{n\, P(S_1 > n(a+\varepsilon))} \geqq 1.$$

Proof. If we let $b = a + \varepsilon$ and let N_n^{nb} be the number of $j \leqq n$ with $S_j - S_{j-1} > nb$ then

$$P(S_n > na) \geqq \sum_{k=1}^{n} P(S_n > na \mid N_n^{nb} = k)\, P(N_n^{nb} = k). \tag{1}$$

It follows from the definition of N_n^{nb} that

$$P(N_n^{nb} = k) = \binom{n}{k} (1 - F(nb))^k\, F(nb)^{n-k}$$

where $F(x) = P(X_1 \leqq x)$. If we let

$$G^{nb}(x) = (F(x) - F(nb))/(1 - F(nb))$$

and

$$H^{nb}(x) = (F(x) \wedge F(nb))/F(nb)$$

then

$$P(S_n > na \mid N_n^{nb} = k) = 1 - (G_k^{nb} * H_{n-k}^{nb})(na)$$

where G_k^{nb} and H_{n-k}^{nb} denote the kth and $(n-k)$th convolutions of G^{nb} and H^{nb}. Now $a=b-\varepsilon$ so

$$1-(G_k^{nb}*H_{n-k}^{nb})(na)\geqq(1-G_k^{nb}(nb))(1-H_{n-k}^{nb}(-n\varepsilon)).$$

From the definition of G^{nb} we have $G_k^{nb}(nb)=0$ for all $k\geqq 1$ so the first term on the right hand side is 1. To see that the other one converges to 1 we observe that $EX_1=0$ so if we pick M large enough then $\int x\,dH^M(x)>-\varepsilon$. From the weak law of large numbers we have $1-H_n^M(-n\varepsilon)\to 0$. From this it follows that for each fixed $k\geqq 1$

$$\liminf_{n\to\infty} 1-H_{n-k}^{nb}(-n\varepsilon)\geqq\lim_{n\to\infty} 1-H_{n-k}^M(-n\varepsilon)=1.$$

At this point we have shown that if $k\geqq 1$ then $P(S_n>na\,|\,N_n^{nb}=k)\to 1$ so to complete the proof it suffices to observe that $E|X_1|<\infty$ so we have $nP(X_1>nb)\to 0$ and

$$\frac{P(N_n^{nb}=1)}{nP(X_1>nb)}=F(nb)^{n-1}\to e^0=1.$$

The reader should observe that

$$\frac{P(N_n^{nb}=1)}{nP(X_1>nb)}\to 0 \quad \text{for } k\geqq 2$$

so we cannot get a better lower bound by using more than the first term in (1).

Lemma 2.1 shows that the lim inf of the expression in Theorem 2.1 is $\geqq 1$. To prove that the $\limsup\leqq 1$ we need an upper bound on $P(S_n>0)$. This is furnished by the following result which is an easily proved extension of a theorem of Nagaev (1965).

Lemma 2.2. *Let S_n be a random walk with mean zero and variance one. If $p\geqq 2$ and $c_p\ E(X_1^+)^p<\infty$ then there is a constant $K_p<\infty$ such that whenever $x>0$ and y is sufficiently large (that is, $y>p\log y>\log nK_p$) we have*

$$P(S_n>x)\leqq n(1-F(y))+\exp\left(1+(2+c_p)\,n\left(\frac{p\log y-\log(nK_p)}{y}\right)^2\right)\left(\frac{nK_p}{y^p}\right)^{x/y}.$$

If we let $x=na$ and $y=nb$ in Lemma 2.2 then since

$$\exp\left(1+(2+c_p)\,n\left(\frac{p\log(nb)-\log(nK_p)}{nb}\right)^2\right)\to e$$

we have that if n is sufficiently large

$$1-F_n(na)\leqq n(1-F(nb))+3\left(\frac{nK_p}{n^p b^p}\right)^{a/b}.$$

If we let $K'_p = 3(K_p/b^p)^{a/b}$ then we have

$$\limsup_{n\to\infty} \frac{1-F_n(na)}{n(1-F(nb))} \leqq 1 + K'_p \limsup_{n\to\infty} \frac{n^{(1-p)a/b}}{n(1-F(nb))}. \tag{7}$$

With this inequality we can easily prove

Lemma 2.3. *Let S_n be a random walk with mean zero and finite variance. If there is a slowly varying function L so that $P(X_1 > x) \sim x^{-q} L(x)$ as $x \to \infty$ then*

$$\limsup_{n\to\infty} \frac{P(S_n > na)}{nP(X_1 > n(a-\varepsilon))} \leqq 1$$

whenever $a > \varepsilon > 0$.

Proof. Let $a > \varepsilon > 0$. $a/(a-\varepsilon) > 1$ and $q \geqq 2$ so we can pick $p < q$ so that $(p-1)\,a/(a-\varepsilon) > q-1$. If we do this then

$$\limsup_{n\to\infty} \frac{n^{(1-p)a/(a-\varepsilon)}}{n(1-F(n(a-\varepsilon)))} = \limsup_{n\to\infty} n^{\frac{(1-p)a}{a-\varepsilon}-(1-q)} L(n) = 0$$

since L is slowly varying (see Feller (1971), p. 277). The desired result now follows from (7).

Combining Lemmas 1 and 3 proves Theorem 2.1.

Section 3

In this section we will apply the results of the last section to obtain limit theorems for $(S_{[n.]}/n \mid S_n > 0)$ and $(S_{[n.]}/n \mid N > n)$. The first result is an easy consequence of Theorem 2.1.

Theorem 3.1. *Let S_n be a random walk with mean zero, finite variance and $P(S_1 > x) \sim x^{-q} L(x)$. Then for all $a > 0$*

$$(S_{[n.]}/n \mid S_n > na) \Rightarrow J_{a,q}\, 1_{(U \leqq .)}$$

where

$$P(J_{a,q} > x) = (x/a)^{-q} \quad \text{for } x \geqq a$$

and

$$P(U \leqq t) = t \quad \text{for } 0 \leqq t \leqq 1.$$

Proof. We start by observing that

$$\begin{aligned} nP\{X_1 > na\} &\geqq P\{\max_{1\leqq j\leqq n} S_j - S_{j-1} > na\} \\ &\geqq nP\{X_1 > na\} - \frac{n^2}{2}(P\{X_1 > na\})^2 \end{aligned}$$

and $nP\{X_1 > na\} \leqq a^{-1} E[X_1; X_1 > na] \to 0$ so

$$\frac{P\{\max_{1\leqq j\leqq n} S_j - S_{j-1} > na\}}{nP\{X_1 > na\}} \to 1.$$

From the computations above it also follows that

$$P\{N_n^{na}=1\}/P\{\max_{1\leqq j\leqq n} S_j-S_{j-1}>na\}\to 1$$

so

$$P\{N_n^{na}=1\}/P\{S_n>na\}\to 1.$$

The next thing we want to show is

$$P\{N_n^{na}=1, S_n>na\}/P\{S_n>na\}\to 1.$$

To do this we observe that if $M_n=\max_{1\leqq j\leqq n} S_j-S_{j-1}$ then for $b>a$

$$P(M_n>nb\,|\,N_n^{na}=1)=\frac{1-F(nb)}{1-F(na)}\to\left(\frac{b}{a}\right)^{-q}$$

and

$$P(S_n>na\,|\,N_n^{na}=1, M_n>nb)\geqq 1-H_{n-1}^{nb}(n(a-b))$$

where H_{n-1}^{nb} is the $(n-1)$th convolution of $H^{nb}(x)=(F(x)\wedge F(nb))/F(nb)$. From the proof of Lemma 2.1 we have that $1-H_{n-1}^{nb}(n(a-b))\to 1$ if $a<b$ so using the results above we have

$$\liminf_{n\to\infty} P(S_n>na\,|\,N_n^{na}=1)\geqq(b/a)^{-q}$$

for all $b>a$. Since $P\{S_n>na\}/P\{N_n^{na}=1\}\to 1$ the last result implies

$$P(S_n>na, N_n^{na}=1)/P(S_n>na)\to 1.$$

From this it follows that to prove Theorem 3.1 it suffices to show

$$(S_{[n.]}/n\,|\,N_n^{na}=1)\Rightarrow J_{a,q}1_{(U\leqq .)}.$$

Let $U_n^{na}=\inf\{j: S_j-S_{j-1}>na\}$.

Let $J_n^{na}=S_{U_n^{na}}-S_{U_n^{na}-1}$.

Since S_k, $k\geqq 1$ is a random walk, U_n^{na} and J_n^{na} are independent. It is easy to see that

$$P(U_n^{na}=k\,|\,N_n^{na}=1)=1/n$$

and

$$P(J_n^{na}>nb\,|\,N_n^{na}=1)=P(J_n^{na}>nb)=\frac{1-F(nb)}{1-F(na)}$$

so

$$\left(\frac{U_n^{na}}{n},\frac{J_n^{na}}{n}\middle|\,N_n^{na}=1\right)\Rightarrow(U,J_{a,q})$$

and consequently

$$\left(\frac{J_n^{na}}{n}1_{(U_n^{na}\leqq n.)}\middle|\,N_n^{na}=1\right)\Rightarrow J_{a,q}1_{(U\leqq .)}.$$

To complete the proof we want to show that if we let

$$R_k^n = S_k - J_n^{na} 1_{(U_n^{na} \leqq k)}$$

then

$$(R_{[n.]}^n / n \,|\, N_n^{na} = 1) \Rightarrow 0.$$

To do this we observe that the process $(R_k^n\, 1 \leqq k \leqq n \,|\, N_n^{na} = 1)$ can be constructed by taking a random walk which takes steps with distribution H^{na} and deleting a step chosen at random. From this we see that if S_k^{na} is a random walk which takes steps with distribution H^{na} then on $\{N_n^{na} = 1\}$

$$\sup_{0 \leqq t \leqq 1} \left| \frac{R_{nt}^n}{n} - t \int x H^{na}(dx) \right| \leqq \sup_{1 \leqq j \leqq n-1} \left| \frac{S_j^{na}}{n} - \frac{j}{n} \int x H^{na}(dx) \right| + \frac{2 |\int x H^{na}(dx)|}{n}.$$

Now from Doob's inequality (see Chung (1974), p. 116) we have that

$$P\left(\sup_{1 \leqq j \leqq n-1} \left| \frac{S_j^{na}}{n} - \frac{j}{n} \int x H^{na}(dx) \right| > \varepsilon \right) \leqq \frac{(n-1) \int x^2 H^{na}(dx)}{n^2} \leqq \frac{E S_1^2}{n F(na)} \to 0.$$

Since $\int x H^{na}(dx) \to 0$ this implies

$$(R_{[n.]}^n / n \,|\, N_n^{na} = 1) \Rightarrow 0$$

and completes the proof of Theorem 3.1.

Our next goal is to compute the limiting behavior of $(S_{[n.]}/n \,|\, N > n)$. Our main result is

Theorem 3.2. *Let S_n be a random walk with mean $-a$ and finite variance and let $U_n^{na} = \min\{j: S_j > na\}$. If there is a slowly varying function L so that $P(X_1 > x) \sim x^{-q} L(x)$ as $x \to \infty$ then for all $j \geqq 1$*

$$P(U_n^{na} = j \,|\, N > n) \to P(S_i > 0\; 1 \leqq i < j)/EN,$$

$$P(N > n) \sim P(X_1 > na) \sum_{k=1}^{\infty} P(S_i > 0\; 1 \leqq i < k)$$

and

$$(S_{[n.] \vee U_n^{na}}/n \,|\, N > n) \Rightarrow J_{a,q} - a.$$

Proof. We will begin by showing that if $c < a$ and $c > a/2$

$$P(J_n^{nc} > n \,|\, N > n) \to 0.$$

To do this we observe that

$$\begin{aligned}P(J_n^{nc}>n, N>n) &\leqq P(J_n^{nc}>n, S_{n/2}>0, S_n>0)\\ &= P(J_n^{nc}>n/2, S_{n/2}>0)\\ &\quad\cdot \sum_x P(S_n>0 \mid S_{n/2}=nx, J_n^{nc}>n)\\ &\quad\cdot P(S_{n/2}=nx \mid J_n^{nc}>n/2, S_{n/2}>0).\end{aligned}$$

To obtain an estimate for the right hand side we observe that from the proof of Theorem 2.1 we have that if $a'=a/2$

$$\frac{P(J_n^{nc}>n/2, S_{n/2}>0)}{\frac{n}{2} P(nc>X_1>na')} \to 1 \tag{1}$$

and from the proof of Theorem 3.1 we have that $\varepsilon>0$

$$\frac{P(J_n^{nc}>n/2, n(c-a'+\varepsilon)>S_{n/2}>0)}{P(J_n^{nc}>n/2, S_{n/2}>0)} \to 1. \tag{2}$$

Now $P(S_n>0 \mid S_{n/2}=nx, J_n^{nc}>n)$ is an increasing function of x so if $\varepsilon>0$

$$\begin{aligned}\sum_x P(S_n>0 \mid S_{n/2}=nx, J_n^{nc}>n)\, P(S_{n/2}=nx \mid J_n^{nc}>n/2, S_{n/2}>0)\\ \leqq P(S_n>0 \mid S_{n/2}=n(c-a'+\varepsilon), J_n^{nc}>n)\\ +P(S_{n/2}>n(c-a'+\varepsilon) \mid J_n^{nc}>n/2, S_{n/2}>0).\end{aligned}$$

From (1) and (2) above we have that if $a-c-\varepsilon<c$ and $\varepsilon<c$

$$\frac{P(S_{n/2}>-n(c-a'+\varepsilon) \mid J_{n/2}^{nc}>n)}{\frac{n}{2} P(nc>X_1>n(a-c-\varepsilon))} \to 1$$

$$\lim_{n\to\infty} \frac{P(S_{n/2}>n(c-a'+\varepsilon) \mid J_n^{nc}>n/2, S_{n/2}>0)}{\frac{n}{2} P(nc>X_1>na')} = 0.$$

Letting $\varepsilon=(a-c)/2$ and using the formulas above gives

$$\limsup_{n\to\infty} \frac{P(J_n^{nc}>n, S_{n/2}>0, S_n>0)}{\frac{n}{2} P(nc>X_1>na')\frac{n}{2} P(X_1>n\varepsilon)} \leqq 1.$$

From the proof of Lemma 2.1 we have that if $b>a$

$$\liminf_{n\to\infty} P(N>n)/P(X_1>nb) \geqq 1.$$

Combining this with previous inequalities gives that

$$\begin{aligned}&\limsup_{n\to\infty} P(J_n^{nc}>n \mid N>n)\\ &\qquad \leqq \limsup_{n\to\infty} \frac{\frac{n}{2}P(nc>X_1>na')\,\frac{n}{2}P(X_1>n\varepsilon)}{P(X_1>nb)}\\ &\qquad \leqq \limsup_{n\to\infty} \left(\frac{a'}{b}\right)^{-q}\frac{n^2}{4}P(X_1>n\varepsilon)=0\end{aligned}$$

which proves the desired result.

Having established the preliminary result we now begin the proof of the theorem. To prove the first statement we pick $c<a$ and write

$$\begin{aligned}&P(U_n^{nc}=j,\, N>n)\\ &\qquad =P(S_i^{nc}>0\ 1\leqq i<j,\ S_j-S_{j-1}>nc,\ N_j>n).\end{aligned}$$

Now $P(U_n^{nc}=j,\, N>n)\leqq P(S_i>0\ 1\leqq i<j)\, P(S_j-S_{j-1}>nc)$ and if $b>a$ it is

$$\geqq P(S_i^{nc}>0\ 1\leqq i<j)\, P(S_j-S_{j-1}>nb)\, P(\inf_{j\leqq k\leqq n} S_k-S_j>nb).$$

Since we have shown in the first part of the proof that

$$\sum_{j=1}^{n} P(U_n^{nc}=j,\, N>n)/P(N>n)\to 1$$

it follows from the two estimates above that if $M<\infty$

$$\liminf_{n\to\infty}\frac{P(U_n^{nc}=j,\, N>n)}{P(N>n)}\geqq \frac{P(S_i^{m}>0\ 1\leqq i<j)\,(b/c)^{-q}}{\sum_{k=1}^{\infty} P(S_j>0,\, 1\leqq i<k)} \tag{3}$$

(the sum in the denominator is finite since we have from Theorem 2.1 that there is a $C<\infty$ so that

$$\begin{aligned}\sum_{k=1}^{\infty} P(S_i>0,\ 1\leqq i<k)&\leqq \sum_{k=1}^{\infty} P(S_k>0)\\ &\leqq C\sum_{k=1}^{\infty} kP(X_1>ka)<\infty \quad \text{since } EX_1^2<\infty).\end{aligned}$$

Now if we let $M\uparrow\infty$, $b\downarrow a$, and $c\uparrow a$ then the right hand side of (10) converges to

$$p_j=P(S_i>0\ 1\leqq i<j)\Big/\sum_{k=1}^{\infty} P(S_i>0,\ 1\leqq i<k).$$

Since $\sum_{j=1}^{\infty} p_j=1$ this shows that $P(U_n^{na}=j \mid N>n)\to p_j$ which proves the first result.

To obtain the asymptotic formula for $P(N>n)$ which is given in the theorem we observe that

$$P(N>n)=\frac{P(U_n^{nc}=1,\, N>n)}{P(U_n^{nc}=1\,|\,N>n)}$$

so if $c<a<b$ then it follows from (10) and a similar upper bound that

$$\limsup_{n\to\infty}\frac{P(N>n)}{P(X_1>na)}\leqq\left(\frac{b}{c}\right)^q\sum_{k=1}^{\infty}P(S_j>0,\ 1\leqq i<k)$$

and

$$\liminf_{n\to\infty}\frac{P(N>n)}{P(X_1>na)}\geqq\left(\frac{c}{b}\right)^q\sum_{k=1}^{\infty}P(S_i>0,\ 1\leqq i<k).$$

To prove the last result we begin by observing that if h is a bounded measurable function from D to R

$$\begin{aligned}&E(h(S_{[n.]\vee U_n^{na}}/n)\,|\,N>n,\ U_n^{na}\leqq n)\\&\quad=\sum_{j=1}^{n}E(h(S_{[n.]\vee j}/n)\,|\,N>n,\ U_n^{na}=j)\ P(U_n^{na}=j\,|\,N>n,\ U_n^{na}\leqq n)\end{aligned}$$

and from the previous section of the proof we have that $(U_n^{na}\,|\,N>n)$ converges in distribution so to compute the limit of the left hand side above it suffices to consider the limit of $(S_{[n.]\vee j}/n\,|\,N>n,\ U_n^{na}=j)$ for each fixed j.

The first step in doing this is consider the distribution of $(S_j/n\,|\,N>j,\ U_n^{na}=j)$. From previous results we have that if $b>a$

$$P(S_j-S_{j-1}>nb\,|\,U_n^{na}=j)\to(b/a)^{-q}$$

and it is clear that

$$P(N>j,\ U_n^{na}=j)/P(N>j-1,\ U_n^{na}=j)\to 1$$

so we have

$$P(S_j-S_{j-1}>nb\,|\,U_n^{na}=j,\ N>j)\to(b/a)^{-q}.$$

To estimate the size of S_{j-1}/n we observe that

$$P(S_{j-1}>x\,|\,N>j,\ U_n^{na}=j)\leqq\frac{P(S_{j-1}>x)}{P(N>j)}$$

so

$$(S_{j-1}/n\,|\,N>j,\ U_n^{na}=j)\Rightarrow 0.$$

Combining this with the previous result gives

$$(S_j/n\,|\,N>j,\ U_n^{na}=j)\Rightarrow J_{a,q}.$$

To complete the proof at this point it suffices to show that

$$P(N>j,\ U_n^{na}=j)/P(N>n,\ U_n^{na}=j)\to 1$$

and that if $x_n \to x > a$ then

$$(S_{[n.]\vee j}/n \mid S_j = x_n n,\ N_j > n) \Rightarrow x - a.$$

The second result follows from the obvious fact that if $x_n \to x > a$

$$(S_{[n.]\vee j}/n \mid S_j = x_n n) \Rightarrow x - a$$

and

$$P(N_j > n \mid S_j = x_n n) \to 1.$$

To prove $P(N>j,\ U_n^{na}=j) \sim P(N>n,\ U_n^{na}=j)$ we use the last two facts and observe that for all $b>a$

$$P(S_j - S_{j-1} > nb \mid U_n^{na}=j) \to (b/a)^{-q}.$$

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