*Journal of Theoretical Probability, Vol. 13, No. 2, 2000*

# Boundary Modified Contact Processes[1]

**Rick Durrett[2] and Rinaldo B. Schinazi[3]**

*Received September 4, 1998; revised April 5, 1999*

We introduce a one dimensional contact process for which births to the right of the rightmost particle and to the left of the leftmost particle occur at rate $\lambda_e$ (where $e$ is for external). Other births occur at rate $\lambda_i$ (where $i$ is for internal). Deaths occur at rate 1. The case $\lambda_e = \lambda_i$ is the well known basic contact process for which there is a critical value $\lambda_c > 1$ such that if the birth rate is larger than $\lambda_c$ the process has a positive probability of surviving. Our main motivation here is to understand the relative importance of the external birth rates. We show that if $\lambda_e \leqslant 1$ then the process always dies out while if $\lambda_e > 1$ and if $\lambda_i$ is large enough then the process may survive. We also show that if $\lambda_i < \lambda_c$ the process dies out for all $\lambda_e$. To extend this notion to $d > 1$ we introduce a second process that has an epidemiological interpretation. For this process each site can be in one of three states: infected, a susceptible that has never been infected, or a susceptible that has been infected previously. Furthermore, the rates at which the two types of susceptible become infected are different. We obtain some information about the phase diagram about this case as well.

**KEY WORDS:** Contact processes; Poisson process.

## 1. INTRODUCTION AND RESULTS

There are two motivations for our work. The first is the theoretical question: Can changing the two external birth rates of the one dimensional contact process significantly affect its chances of survival? To formulate this question precisely and to prepare for our second question, we will define the basic contact process in $d$ dimensional space. Among the several notational possibilities (see Liggett,[10] Durrett[4, 5]), we will choose to define the contact process $\zeta_t$ to be a Markov process in which:

---

[1] Partially supported by National Science Foundation Grants DMS 96-26201 and DMS 96-25126.

[2] Department of Mathematics, Cornell University, 224 Rhodes Hall, Ithaca, New York 14853.

[3] Department of Mathematics, University of Colorado, Colorado Springs, Colorado 80933.

(i) The state at any time $t \geqslant 0$ is a function from $\mathbf{Z}^d$ to $\{0, 1\}$ with $\zeta_t(x) = 1$ indicating that the site $x$ is occupied by a particle, while $\zeta_t(x) = 0$ indicates that the site is empty.

(ii) Particles die at rate 1 and for each vacant nearest neighbor give birth to a new particle there at rate $\lambda$.

The first modification of the basic rules we will consider here pertains only to the one dimensional contact process started from a finite set of occupied sites. In what we will call the *boundary contact process*, if the leftmost particle is at $\ell_t$ and the rightmost particle is at $r_t$ then births at $\ell_t - 1$ and at $r_t + 1$ occur at rate $\lambda_e$, where $e$ is for external. To symmetrize the notation we introduce $\lambda_i$, with $i$ for internal, to denote the common value of all the other neighbor birth rates.

Let $\xi_t^0$ denote the boundary contact process started from a single occupied site at the origin, let $|\xi_t^0|$ be the number of occupied sites at time $t$, and let $\Omega_0 = \{|\xi_t^0| > 0, \text{ for all } t \geqslant 0\}$ be the event that the contact process does not end up in the all 0's state. The following monotonicity property is intuitively clear, though it does require some work to prove:

**Proposition 1.** The survival probability $P_{\lambda_i, \lambda_e}(\Omega_0)$ is an increasing function of each of its two variables $\lambda_e$ and $\lambda_i$.

Proposition 1 implies that if we examine the behavior of the system as a function of $(\lambda_i, \lambda_e)$ there is a *critical curve* so that parameters above the curve correspond to *survival*, i.e., $P_{\lambda_i, \lambda_e}(\Omega_0) > 0$, while the points below the curve correspond to *dying out*, i.e., $P_{\lambda_i, \lambda_e}(\Omega_0) = 0$. Let $\lambda_c = \inf\{\lambda\colon P_{\lambda, \lambda}(\Omega_0) > 0\}$ be the critical value for the contact process. It is clear from Proposition 1 that if $\lambda_i > \lambda_c$ and $\lambda_e > \lambda_c$ then the system survives, while if $\lambda_i < \lambda_c$ and $\lambda_e < \lambda_c$ then the system dies out.

This leaves us with two quadrants of parameter space to explore. See Fig. 1 to follow our progress. Theorem 1 explores the lower right quadrant: $\lambda_i > \lambda_c$, $\lambda_e < \lambda_c$. The next result shows that even though the outside rates are used at only two sites, changing them can reduce the survival probability to 0.

**Theorem 1.** (a) If $\lambda_e \leqslant 1$ then the boundary contact process dies out from any $0 \leqslant \lambda_i \leqslant \infty$.

(b) If $\lambda_e > 1$ then for $\lambda_i > \Lambda(\lambda_e)$ the boundary contact process survives. Furthermore,

$$\liminf_{t \to \infty} P(\zeta_t^0(x) = 1) > 0$$

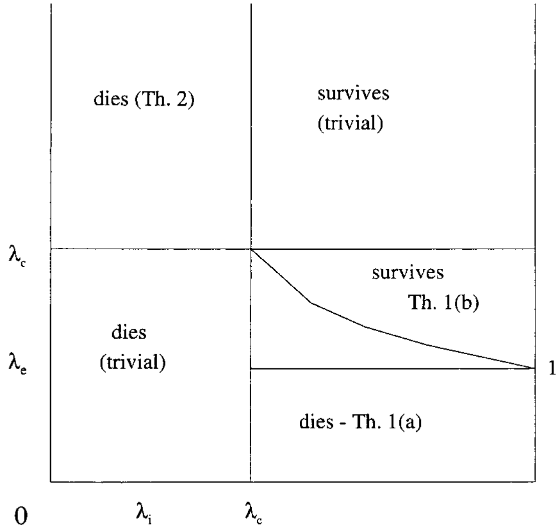


Fig. 1. Phase diagram for boundary contact process, $d=1$.

*Proof of (a).* The interval $[l_t, r_t]$ of occupied sites gets longer by 1 at rate at rate $2\lambda_e$, and gets shorter by at least 1 at rate 2. Comparison with a random walk then shows that the system dies out with probability one if $\lambda_e \leqslant 1$. □

To prove part (b) we start with the fact that when $\lambda_i = \infty$ the interval $[l_t, r_t]$ is always fully occupied. Thus, it gets longer by 1 at rate $2\lambda_e$, gets shorter by 1 at rate 2, and the process survives with positive probability for $\lambda_e > 1$. Combining this idea with a block construction, we can easily prove the second conclusion. We give the details of the proof in Section 2.

Our next result covers the upper left corner, $\lambda_i < \lambda_c$, $\lambda_e > \lambda_c$. In words, it says that a supercritical rate on the outside can't save a process that is subcritical on the inside.

**Theorem 2.** For $\lambda_i < \lambda_c$ the boundary contact process dies for all $\lambda_e > 0$.

This is not as easy to prove as it might seem at first glance. To succeed, we compared with a version of the boundary contact process in which there were no deaths at the boundary and then exploited the fact that the bonus births were occurring on a monotone increasing path.

Theorems 1 and 2 complete our description of the phase diagram for the modified one dimensional model. We do not know how to give a good definition of an "external particle" in two dimensions, so we will instead consider an epidemic in a population where susceptibles who have never had the disease acquire it at rate $\lambda_e$, while those who have had the disease previously acquire it at rate $\lambda_i$. To record the medical history of the susceptibles in the state of the system we write $\eta_t: \mathbf{Z}^d \to \{-1, 0, 1\}$ where, $1 =$ infected, $-1 =$ never infected, $0 =$ susceptible who has been previously infected. We call $\eta_t$ the *three state contact process*. Letting $n_1(x)$ be the number of nearest neighbors of $x$ that are in state 1, we can write its flip rates as:

$$\begin{aligned} -1 &\to 1 \qquad \text{at rate } \lambda_e n_1(x) \\ 0 &\to 1 \qquad \text{at rate } \lambda_i n_1(x) \\ 1 &\to 0 \qquad \text{at rate } 1 \end{aligned}$$

Let $\eta_t^0$ denote the three state contact process started from a single 1 at the origin in a sea of $-1$'s. Let $|\eta_t^0|$ be the number of 1's sites at time $t$, and let $\Omega_0 = \{|\eta_t^0| > 0 \text{ for all } t \geqslant 0\}$ be the event that the infection does not die out. If $P_{\lambda_i, \lambda_e}(\Omega_0) > 0$ we say that the three state contact process *survives*. It is trivial that if both $\lambda_i$ and $\lambda_e$ are larger than the critical value for the contact process then the system will survive, while if both are smaller than $\lambda_c$ it will die out. So again, we have two quadrants of parameters to explore.

To facilitate comparison with Theorem 2, we will start this time with the upper left quadrant: $\lambda_i < \lambda_c$, $\lambda_e > \lambda_c$. In the extreme case $\lambda_i = 0$ our model is the standard spatial epidemic or forest fire model (see Kulasmaa[9]), so it follows from known result that

**Proposition 2.** If $\lambda_e > \lambda_f$, the critical value for the forest fire, and $\lambda_i \geqslant 0$ then the three state contact process survives.

The last result is vacuous in $d = 1$ since $\lambda_f = \infty$ there. The next result which has a proof very similar to that of Theorem 2 gives a good reason for this.

**Theorem 3.** In dimension 1, if $\lambda_i < \lambda_c$ then the infection dies out in the three state contact process for any $\lambda_e \geqslant 0$.

Turning to the lower right corner, $\lambda_i > \lambda_c$, $\lambda_e < \lambda_c$ we encounter a behavior totally different from the result for the one dimensional boundary contact process given in Theorem 1. To be able to contrast our next result with the survival in Proposition 2, we formulate a notion stronger than mere survival. We say that the infection *persists* if:

$$\liminf_{t \to \infty} P_{\lambda_i, \lambda_e}(\eta_t^0(x) = 1) > 0$$

**Theorem 4.** For the three state contact process, if $\lambda_i > \lambda_c$ then the infection persists for any $\lambda_e > 0$.

Note that the result we have stated in Theorem 1(b) implies persistence but in contrast only holds when $\lambda_e > 1$ and $\lambda_i > \Lambda(\lambda_e)$.

Combining Theorems 3 and 4 we see that in one dimension, we have the trivial behavior

| | |
|---|---|
| $\lambda_i < \lambda_c$ | infection dies out |
| $\lambda_i > \lambda_c$ | infection persist |

Figure 2 gives a diagram of the situation in $d > 1$. To complete the picture, we have

**Theorem 5.** Persistence is impossible when $\lambda_i < \lambda_c$, for the three state contact process.

Putting together Proposition 2 and Theorem 5, we see that if $\lambda_e > \lambda_f$ and $\lambda_i < \lambda_c$ then the three state contact process survives but does not persist. In other words, there is a positive probability that there is always at least one 1 somewhere in the graph but the probability of having a 1 at a fixed site at time $t$ goes to 0 as $t$ goes to infinity.

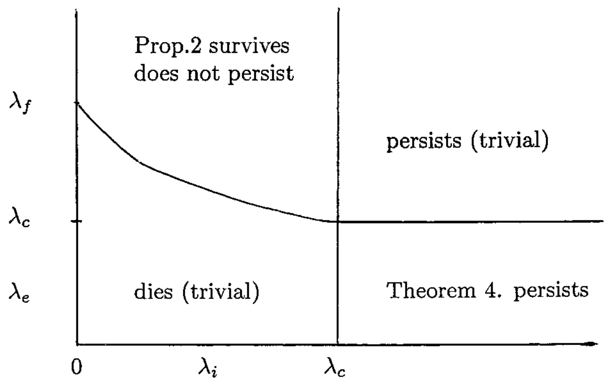


Fig. 2. Phase diagram for three state contact process.

The remainder of the paper is devoted to proofs. Proposition 1 and Theorem 1 are proved in Section 2, Theorems 2 and 3 are proved in Section 3, Theorem 4 in Section 4, and Theorem 5 in Section 5.

## 2. THE CONSTRUCTION AND THE PROOFS OF PROPOSITION 1 AND THEOREM 1

We begin by presenting a construction of the boundary contact process from independent Poisson processes $\{T_n^{x,k}, n \geqslant 1\}$ $x \in \mathbf{Z}$ and $0 \leqslant k \leqslant 4$.

The processes $\{T_n^{x,0}, n \geqslant 1\}$ have rate 1. At their arrival times we kill the particle at $x$ if it is occupied.

The processes $\{T_n^k, n \geqslant 1\}$ for $k = 1, 2$ (note no superscript $x$) have rate $\lambda_e$. At the arrival times $T_n^1$ a birth to the left comes from the left-most particle. Likewise, at the $T_n^2$, a birth to the right will occur from the right-most particle.

The processes $\{T_n^{x,k}, n \geqslant 1\}$ for $k = 3, 4$ have rate $\lambda_i$. At the arrival times $T_n^{x,3}$, a particle at $x$ will send an offspring to $x-1$, but only if the particle at $x$ is not the left-most particle, since those births are taken care of by $T_n^1$. Likewise, at times $T_n^{x,4}$ then there will be a birth to $x+1$ provided the particle at $x$ is not the right-most particle.

Although there are infinitely many Poisson processes involved, and hence no first arrival, it is not hard to show that:

**(2.1) Lemma.** The rules described above specify a unique process.

Harris's (1972) argument covers this. You can read about the details, for example, in Section 2 of Durrett.[6]

*Proof of Proposition 1.* Let $(\lambda_i, \lambda_e)$ and $(\lambda_i', \lambda_e')$ be such that $\lambda_i \leqslant \lambda_i'$ and $\lambda_e \leqslant \lambda_e'$. Consider a finite set $A = \{x_1, x_2,..., x_n\}$ with $x_1 < x_2 < \cdots < x_n$. We say that a finite set $B$ is *more spread out than* $A$ if there is a function $\phi$: $A \to B$ so that

$$|\phi(x_i) - \phi(x_j)| \geqslant |x_i - x_j| \qquad \text{for all} \quad x_i, x_j \in A$$

and

$$\min B = y_1 = \phi(x_1) < \cdots < \phi(x_n) = y_n = \max B$$

We say that the configuration $\xi_t'$ is *more spread out than* $\xi_t$ if $\{x$: $\xi_0'(x) = 1\}$ is more spread out than $\{x$: $\xi_0(x) = 1\}$.

**(2.2) Lemma.** If $\xi_0'$ is more spread out than $\xi_0$ then the two process can be constructed on the same space so that $\xi_t'$ is more spread out than $\xi_t$ at all $t \geqslant 0$.

The existence of this coupling is enough to prove Proposition 1, so it suffices to prove the lemma. The ideas here are very similar to those in the proof of Theorem 1.9, Chap. VI in Liggett.[10]

*Proof.* To do this we will have to use a construction slightly different from the graphical representation. Again all the Poisson processes used are independent. Let $x_1 < x_2 < \cdots < x_n$ be all the sites occupied by $\xi_t$. Let $y_1 < y_2 < \cdots < y_n$ be the corresponding sites occupied in $\xi'_t$ with $y_1$ and $y_n$ being the leftmost and the rightmost sites occupied by $\xi'_t$, respectively.

(i) *Death Events* for the particles in $x_m$ and $y_m$ are derived from the same Poisson process $\{\bar{T}^{m,0}_n, n \geqslant 1\}$ with rate 1.

(ii) *External Birth Events* are performed simultaneously for processes $\xi_t$ and $\xi'_t$ by using the processes $T^1_n$ and $T^2_n$ (they both have rate $\lambda_e$) introduced above. To take care of the extra births in $\xi'_t$, we introduce two Poisson processes $S^1_n$ and $S^2_n$ with rates $\lambda'_e - \lambda_e$. At the arrival times of $S^1_n$ and $S^2_n$ the leftmost and rightmost particles of $\xi'_t$ give birth to their left and right, respectively.

(iii) *Internal Birth Events* are performed simultaneously for the particles at $x_m$ and $y_m$ by using the processes $\bar{T}^{m,k}_n$ with rate $\lambda_i$, to the left ($k=3$) and to the right ($k=4$). Again, we introduce Poisson processes $S^{m,k}_n$ with rate $\lambda'_i - \lambda_i$ (as we did in (ii)) to take care of the extra births for the particles at $y_m$.

(iv) *Additional flips for the particles of $\xi'_t$.* In addition to the $y_m$ there may be other sites (between $y_1$ and $y_n$) that are occupied by particles in the process $\xi'_t$. In order to take care of these flips we use independent Poisson processes $\{T^{y,0}_n, n \geqslant 1\}$ with rate 1, $\{T^{y,3}_n, n \geqslant 1\}$ and $\{T^{y,4}_n, n \geqslant 1\}$ with rate $\lambda'_i$. At the arrival times of $T^{y,k}_n$ we do nothing if $y$ is one of the $y_m$. However, if $y$ is not one of the $y_m$ then at the arrival times of $T^{y,0}_n$ we kill the particle (if any) at $y$, at the arrival times of $T^{y,3}_n$ and $T^{y,4}_n$ the particle at $y$ (if any) will give birth on $y-1$ and $y+1$, respectively.

The processes $\xi_t$ and $\xi'_t$ are constructed simultaneously by using (i) through (iv). To prove Lemma 2.2 it suffices to check that no transition in any of the Poisson processes can destroy the comparison. To do this is useful to have a concrete example to keep in mind:

$$\begin{array}{cccccccccccccccc}
 & & & y_1 & & y_2 & & & & & y_3 & & & y_4 & & \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & x_1 & x_2 & & & & x_3 & x_4 & & & & &
\end{array}$$

Deaths at internal sites clearly preserve the comparison. After a death at the left edge, the new left-most particle will be at $x_2$ in $\xi_t$ and at a point $\leqslant y_2$ in $\xi'_t$, so things are more spread out than before. External birth events are easy to accommodate. One to the right from the right end leads to new particles $x_{n+1}$ and $y_{n+1}$ that are paired up. At internal birth events, several things can happen: (i) the birth may be possible in $\xi'_t$ but not in $\xi_t$ (e.g., at time $\bar{T}_n^{1,4}$ for the example above), or (ii) a birth in $\xi_t$ may be matched by a birth onto an already occupied site in $\xi'_t$ (e.g., $\bar{T}_n^{3,3}$).

*Proof of Theorem 1.* Part (a) was proved in the introduction, so we only establish (b). We start by introducing some notation. Let

$$\mathscr{L} = \{(m, n) \in \mathbf{Z}^2 : m+n \text{ is even}\}$$

$$B = (-4L, 4L) \times [0, T] \qquad B_{m,n} = (2mL, nT) + B$$

$$I = [-L, L] \qquad I_m = 2mL + I$$

where $L$ and $T$ are parameters to be chosen later.

We declare $(m, n) \in \mathscr{L}$ to be "wet" if for the process starting with every site in $I_m$ in state 1 at time $nT$, and not allowed to give birth outside of the space-time box $B_{m,n}$, every site of $I_{m-1}$ and every site of $I_{m+1}$ has a 1 at time $(n+1)T$. We impose the spatial cutoff so that the events $\{(m, n)$ is wet$\}$ and $\{(j, k)$ is wet$\}$ are independent if $(j, k)$ and $(m, n)$ are not nearest neighbors in $\mathscr{L}$. In other words, the events that the various sites are open are 1-dependent.

We fix $\lambda_e > 1$. We are going to show that when $\lambda_i = \infty$, for any $\varepsilon > 0$ there are $L$ and $T$ such that

$$P((m, n) \text{ is wet}) \geqslant 1 - \varepsilon$$

By translation invariance we may consider the site $(0, 0)$ in $\mathscr{L}$. Assume that at time 0 each site of the interval $I$ is occupied by a 1. Let $\ell_t$ and $r_t$ be the leftmost and rightmost sites occupied by a 1, respectively. If $\ell_t < r_t$ we have

$$r_t \to r_t + 1 \qquad \text{at rate } \lambda_e$$

$$r_t \to r_t - 1 \qquad \text{at rate } 1$$

The process $r_t$ does not move more than one step to the left since the occupied sites from a block with no holes on $\mathbf{Z}$. It is easy to see that by taking $L$ large enough

$$P(r_t > \ell_t, \text{ for all } t \geqslant 0) > 1 - \varepsilon$$

By the strong law of large numbers we have almost surely on $\{r_t > \ell_t, \forall t > 0\}$

$$\lim_{t \to \infty} r_t/t = \lambda_e - 1$$

Taking $L$ large enough and $T = 5L/2(\lambda_e - 1)$ (recall we are assuming that $\lambda_e > 1$) we get that

$$P_{\infty, \lambda_e}(r_T \geqslant 3\lambda) \geqslant 1 - \varepsilon$$

The leftmost site occupied by a 1, $\ell_t$, behaves in a symmetric way and all the sites between the leftmost 1 and the rightmost 1 are occupied by 1's. Thus,

$$P((m, n) \text{ is wet}) \geqslant 1 - \varepsilon \qquad \text{if} \quad \lambda_i = \infty$$

Since $B$ is a finite box we have that as $\lambda_i \to \infty$ the process restricted to $B$ with parameters $(\lambda_i, \lambda_e)$ converges in distribution to the process restricted to $B$ with parameters $(\infty, \lambda_e)$. So there is $\Lambda$ depending only on $\varepsilon$ and $L$ such that

$$P((m, n) \text{ is wet}) \geqslant 1 - 2\varepsilon \qquad \text{if} \quad \lambda_i > \Lambda$$

If we assume that $\lambda_e \leqslant \lambda_i$ the basic coupling shows that the system restricted to the finite boxes $B_{m,n}$ has less particles than the unrestricted infinite system. We may assume that $\lambda_e \leqslant \lambda_i$ with no loss of generality here. So the boundary contact process dominates a 1-dependent oriented percolation system. For $\varepsilon > 0$ small enough it is known that there is percolation with positive probability (see Durrett (1984)). This proves that 1's survive forever with positive probability and the proof of Theorem 1(b) is complete.

## 3. PROOFS OF THEOREMS 2 AND 3

We begin with the harder task.

*Proof of Theorem 2.* We want to show that any system with $\lambda_i < \lambda_c$ dies out. It is enough to prove that

**(3.1) Lemma.** There is a constant $C$ such that for any time $t \geqslant 0$, $E\,|\xi_t^0| \leqslant C$.

For if the system survives the number of particles goes to infinity as $t$ goes to infinity and Fatou's lemma implies that $E\,|\xi_t^0| \to \infty$.

*Proof of (3.1).* The difficulty in analyzing $|\xi_t^0|$ comes from the fact the edge processes may move inwards by an unbounded number of steps. To avoid this problem, we now introduce a system whose edges behave more nicely. Let $\bar{\xi}_t^0$ be a Markov process with the same birth and death rates as $\xi_t$ except that the leftmost and rightmost particles do not die. We construct the processes $\xi_t$ and $\bar{\xi}_t$ by using the construction described in the proof of Proposition 1, with the result that if both processes start with a single particle at the origin then $|\bar{\xi}_t^0| \geqslant |\xi_t^0|$. Thus, if we are able to show that

$$E\,|\bar{\xi}_t^0| \leqslant C \tag{3.2}$$

we will be done.

The right edge of $\bar{\xi}_t^0$ moves to the right only and the left edge moves to the left only, one step at the time. For a site $x$ to be occupied in $\bar{\xi}_t^0$, it must be a descendent of a site on rightmost or leftmost paths. If we let $N_t$ be the number of sites at time $t$ that are a descendant of a site on the path of the rightmost particle then by symmetry we have

$$E(|\bar{\xi}_t|) \leqslant 2E(N_t) \tag{3.3}$$

Let $H(x, r, s)$ be the event that the *basic* contact process with birth rate $\lambda_i$ starting from $x$ has a path to $x-r$ or $x+r$ or survives until times $s$. Bezuidenhout and Grimmett[1] have proved that for $\lambda_i < \lambda_c$ there are constants $0 < \gamma$, $C < \infty$ so that

$$P(H(x, r, s)) \leqslant C(e^{-\gamma r} + e^{-\gamma s}) \tag{3.4}$$

Conditioning on the path of the rightmost particle of $\bar{\xi}_t$ we have

$$E(N_t \mid \text{rightmost path}) \leqslant \sum_{y \geqslant 1} P(H(r_t - y, [y/2], T_{r_t - [y/2]})) \tag{3.5}$$

where $T_y$ is the first time the right edge reaches $y \geqslant 1$. Using (3.4) and (3.5) now, then taking expected value gives

$$\begin{aligned} E(N_t) &\leqslant \sum_{y \geqslant 1} C(e^{-\gamma[y/2]} + Ee^{-\gamma T_{[y/2]}}) \\ &\leqslant 2C \sum_{m=0}^{\infty} e^{-\gamma m} + (Ee^{-\gamma T_1})^m \end{aligned}$$

since thanks to the condition of no deaths at the boundary $T_m - T_{m-1}$ are independent and identically distributed. Since $Ee^{-\gamma T_1} < 1$ the sum converges and we have

$$E(N_t) \leqslant C' \tag{3.6}$$

where $C'$ is a constant independent of $t$. (3.6) and (3.3) show that (3.2) holds and completes the proof of Theorem 2. □

*Proof of Theorem 3.* Again we begin by comparing with a process with no deaths at the edge and conclude that it is enough to show (3.2). Since the three state contact process with no deaths at the edge behaves as the boundary contact process with no deaths at the edge, the desired result follows. □

## 4. PROOF OF THEOREM 4

We will first prove the result for very super-critical 1-dependent three state oriented two dimensional site percolation on our favorite lattice $\mathscr{L} = \{(x, n) \in Z^2 : m + n \text{ is even}\}$. The result will then be extended to the three state contact process on $\mathbf{Z}^d$ by using the work of Bezuidenhout and Grimmett.[1]

Before we can do the first step, we must first define the three state oriented percolation process precisely. Given are variables $\omega(x, n) \in \{0, 1\}$ for $(x, n) \in \mathscr{L}$ that are 1-dependent and have density at least $p$. That is, whenever $(x_i, n_i)$, $1 \leqslant i \leqslant I$ have $\|(x_i, n_i) - (x_j, n_j)\|_\infty > 1$ if $i \neq j$ we have

$$P(\omega(x_i, n_i) = 0 \text{ for } 1 \leqslant i \leqslant I) \leqslant (1-p)^I$$

We think of the $\omega(x, n)$ as representing the ordinary part of our three state process, i.e., the process that would result if there were no $-1$'s.

To take care of the sites that are in state $-1$ we will add an independent set of random variables $\omega'(x, n) \in \{0, 1\}$ that are themselves independent and have $P(\omega'(x, n) = 1) = \mu$. If $x$ was in state $-1$ at time $n-2$ these variables are used instead of the $\omega(x, n)$ to see if the site will be open at time $n$. It is easy to see that if we are given $\xi(x, m)$ for all $(x, m) \in \mathscr{L}$ with $m = 0$ or $m = -1$ (the latter to be able to compute the fate at time 1 of sites $x$ with $\xi(x, -1) = -1$) then we can, by induction, compute $\xi(x, m)$ from the $\omega(x, n)$ and $\omega'(x, n)$. In the last sentence and in what follows, having to look back in time one or two steps to compute the next state is annoying. To fix this problem we will extend our process from $\mathscr{L}$ to $\mathbf{Z}^2$ by setting $\xi(x, n) = \xi(x, n-1)$ when $(x, n) \in \mathbf{Z}^2 - \mathscr{L}$.

To prove Theorem 4 for oriented percolation, we will apply the block construction as explained in Section 4 of Durrett.[6] To this end we use the notation of that reference. Lest the authors become confused, we should note that in this part of the proof we are only concerned with proving the result for a very supercritical two dimensional percolation process, so we let

$$B = (-4L, 4L) \times [0, T] \qquad B_{m,n} = (2Lm, Tn) + B$$

$$I = [-L, L] \qquad I_m = 2Lm + I$$

Here $L$ and $T$ are parameters to be chosen later. The primary ingredient in the block construction is a set of happy configurations $H$ determined by the values of the process on $[-L, L]$. The one we will choose has a two part description:

(H1) There are no $-1$'s in $I$, at time 0.

(H2) There are at least $\sqrt{L}$ 1's in $I$ at time 0.

Let $\sigma_y \xi$ be the operator that translates the configuration by $-y$. That is, $\sigma_y \xi(x) = \xi(x+y)$. To check the interpretation note that $\sigma_y \xi(0) = \xi(y)$, i.e., the value that used to be at $y$ has been moved to 0. Our goal is prove the following result:

**(4.1) Proposition.** Let $\varepsilon > 0$. If $p$ is sufficiently close to 1 and $L$ is large enough then for any starting state $\xi \in H$ for the oriented percolation process in which no births are allowed outside $(-4L, 4L)$ there is $T > 0$ such that we have $\sigma_{2L} \xi_T \in H$ and $\sigma_{-2L} \xi_T \in H$ with probability $\geqslant 1 - \varepsilon$.

*Proof of Theorem 4.* Once (4.1) is established, we can use Theorem 4.2 in Durrett[6] to establish a comparison between our process and a 1-dependent oriented percolation. The result in Theorem 4 then follows from Theorem A.2 of Durrett.[6] □

*Proof of (4.1).* We will proceed in two steps that parallel those in the definition of the happy configurations, $H$.

(i) If we start with $\sqrt{L}$ 1's in $[-L, L]$ and no $-1$'s in $[-L, L]$ then with probability $\geqslant 1 - \varepsilon/10$ we will have no $-1$'s in $(-4L, 4L)$ by time $2L^2$.

(ii) When (i) occurs, with probability $\geqslant 1 - \varepsilon/10$ we will have at least $\sqrt{L}$ particles in $[L, 3L]$ and in $[-3L, -L]$ at time $T = 3L^2$.

At an intuitive level, these statements are easy to believe.

(a) From the basic facts about very supercritical oriented percolation in a strip of finite width (see Durrett and Liu,[7] and Durrett and Schonmann[8]), there is a $\gamma > 0$ so that if $L$ is large then the process on $(-4L, 4L)$ starting with $\sqrt{L}$ individuals will with high probability survive until time $e^{\gamma L}$. Suppose in the worst case that the site $L+1$ is in state $-1$. Since the right edge of supercritical oriented percolation has positive drift (see Durrett[3]), we expect that the mean time between returns of the process to $L$ will be finite. Each occupancy gives the process an opportunity to kill the $-1$ at $L+1$, so it is very unlikely that the $-1$ can survive up to time $L/3$. Repeating this argument $3L$ times, we get (i).

(b) Once the interval $(-4L, 4L)$ is cleared of $-1$'s we have ordinary oriented percolation in a strip of finite width. In this situation, well known techniques (see the papers cited in (i) and Durrett[3]) will allow us to control the number of 1's in the interval at time $T$.

Turning the intuition behind (a) into a proof requires some care. To keep control of the ordinary oriented percolation that comes from only considering the variables $\omega(x, n)$, we will build a chain link fence of paths with mesh $K = L^{0.1}$ in the space time set $(-4L, 4L) \times [0, 2L^2]$. See Fig. 3. Here, the exact value 0.1 for the exponent is not important. This could be any small positive number, but at this point the Greek alphabet beyond its breaking point.

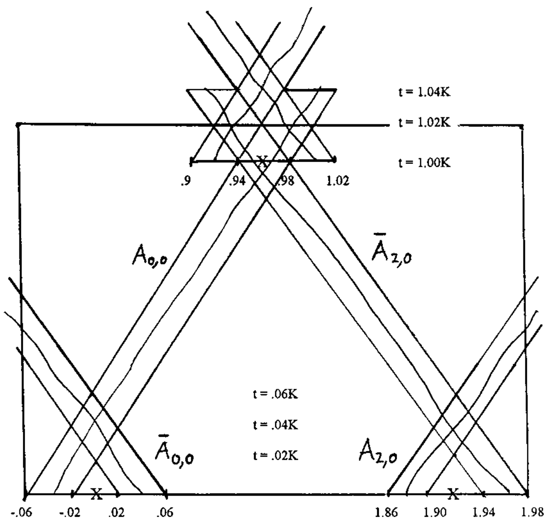


Fig. 3. Block Construction.

The chain link fence is a potential set of connections that is realized in the part of the interval that is free from $-1$'s. We build this network first then use it to break down the $-1$'s. Once the $-1$ are gone, the percolation process can fully express itself and fill up a reasonably dense subset of the interval.

Since $p$ is close to 1, we take $\alpha=1$ and $\delta=0.04$ in Section 11 of Durrett[3] to arrive at the definition of $A$ as the parallelogram with vertices

$$\begin{array}{cccc} & & (0.98K, 1.04K) & (1.02K, 1.04K) \\ (-0.06K, 0) & (-0.2K, 0) & & \end{array}$$

Reflecting around the axis we can define $\bar{A}=-A$. To extend these definitions to the other points $(m, n)\in\mathscr{L}$ we let

$$A_{m,n}=(0.96Km, Kn)+A \qquad \bar{A}_{m,n}=(0.96Km, Kn)+\bar{A}$$

To see the reason for translation by multiples of $0.96K$, note that our choice of 0.96 makes the slice of $A_{0,0}$ at time $K$ fit in the space between the tubes $A_{1,1}$ and $\bar{A}_{1,1}$. See Fig. 3.

We say that $A_{0,0}$ is occupied if there is a path in the oriented percolation inside $A_{0,0}$ that starts in $(-0.05K, -0.03K)\times\{0\}$ and ends in $(0.99K, 1.01K)\times\{1.04K\}$. Our next goal is to show

**(4.2) Lemma.** If $1-p\leqslant 3^{-3601}$ then $P(A_{0,0}\text{ is occupied})\geqslant 1-Ce^{-\gamma K}$.

Here and in what follows $C$ and $\gamma$ are strictly positive finite constants whose values may change from line to line. To inspire the reader for the details of the proof of (4.2), we note that $K=L^{0.1}$ and there are fewer than $(8L+11)\cdot 2L^2$ trapezoids $A_{i,j}$ that touch our space time box $(-4L, 4L)\times[0, 2L^2]$. Thus if $L$ is large, then with probability $\geqslant 1-\varepsilon/10$ all the trapezoids that touch the box are occupied. When this good event, $G_1$, occurs we say that we have a chain link fence of paths. Once the fence is touched by a wet site in oriented percolation (i.e., one that can be reached from the initial wet sites) the wet region expands at a rate almost one to cover the entire network. Since ALL the trapezoids that make up the fence are occupied, we have a network of paths that bangs into the boundary at predictable intervals and automatically leads to a large number of occupied sites at the final time.

*Proof of (4.2).* We begin with a known result for the right edge of oriented percolation starting from a half line of occupied sites. Using a small modification of the notation of Section 12 of Durrett[3] we let $\mathscr{C} = \{y\text{: for some } m \leqslant 0, (m, 0) \to y\}$ and let $s_n = \sup\{m\text{: } (m, n) \in \mathscr{C}\}$. With this notation, (2) of Section 11 of Durrett[3] can be stated as

**(4.3) Lemma.** If $0 < q < 1$ and $(1-p) < 3^{-36/(1-q)}$ then

$$P(s_n \leqslant nq) \leqslant Ce^{-\gamma n}$$

Using (4.3) with $n = 1.04K$ and $q = 1.02/1.04 < 0.99$ we see that

$$P(-0.03K + s_{1.04K} \leqslant 0.99K) \leqslant Ce^{-\gamma K}$$

and hence with high probability the right edge of the process starting with $(-\infty, -0.03K]$ occupied at time 0 will be $\geqslant 0.99K$ at time $1.04K$.

The next step is to show that this event guarantees the existence of the desired path. In words, we can say that since we are in discrete time and have chosen $\alpha = 1$, the edges of the parallelogram move at the speed of light, this is trivial and furthermore the probability of failure is 0. To explain this more slowly, we begin by noting that the right edge of the process starting with $(-\infty, -0.03K]$ occupied at time 0 cannot be to the right of $1.01K$ at time $1.04K$. Likewise a path that ends up in $[0.99K, 1.01K]$ at time $1.04K$ must start from a point $\geqslant -0.05K$ at time 0 and cannot leave the parallelogram on the left side. For if it did, then it could never reach the target interval at time $1.04K$. $\square$

Having built our chain link fence, we are three short steps from the end of the proof of Theorem 4 for highly supercritical oriented percolation. For a given $\varepsilon > 0$ one can choose $L$ so large that:

(4.4) Let $G_2 = \{$at least one 1 in the initial configuration makes a connection with the chain link fence$\}$. On $G_1$, the failure of event $G_2$ has probability $\leqslant \varepsilon/100$.

(4.5) Let $G_3 = \{$no $-1$'s in $(-4L, 4L)$ at time $2L^2\}$. On $G_1 \cap G_2$, the event $G_3$ fails with probability $\leqslant \varepsilon/100$.

(4.6) On $G_1 \cap G_2 \cap G_3$ we always have at least $L^{0.9} - 2$ occupied sites in the intervals $[L, 3L]$ and in $[-3L, -L]$, at time $3L^2$.

Clearly (4.4)–(4.6) gives us (i) and (ii), so the proof of (4.1) will be complete once we establish these three results.

*Proof of (4.4).* To estimate the probability in (4.4) we note that if there at least $L^{0.5}$ sites occupied by a 1 at time then since $K = L^{0.1}$, there must be at least $L^{0.4}/2$ intervals $[1.92Kj, 1.92K(j+1))$ with $-L^{0.9}/1.92 < j < L^{0.9}/1.92$ that contain at least one 1. For otherwise, even if there intervals were completely filled, there would not be enough 1's. To achieve independence, we don't want our intervals to touch, so we observe that there must have at least $L^{0.4}/4$ intervals with odd values of $j$ or at least $L^{0.4}/4$ intervals with even values of $j$. We will suppose without loss of generality that the first alternative occurs.

If a starting 1 in $[1.92Kj, 1.92K(j+1))$ survives up to time $1.02K$ or exits from the interval $((j-0.06)K, (j+1.98)K)$ in the unrestricted version of the oriented percolation then it must touch the chain link fence. (Again see Fig. 3.) Thanks to our use of only odd $j$ these events are independent for different values of $j$. All these events have a probability that is bounded below by

$$\rho = 1 - \sum_{m=4}^{\infty} 3^{n-1}(1-p)^{m/36}$$

the contour argument lower bound (see e.g., (2) on p. 1027 of Durrett[3]) for the probability of percolation. Thus the probability all of our attempts will fail is at most

$$(1-\rho)^{L^{0.4}/4} = \exp(-\gamma L^{0.4})$$

The right hand side converges to 0 as $L \to \infty$ and we have established (4.4). □

*Proof of (4.5).* The first thing we have to do is to wait at most $8L$ units of time, which is $< L^2$ if $L$ is large, for the wet region (containing 1's) to spread along the chain link fence to reach both end points of the interval $[-L, L]$. So by time $L^2$ we have expanded to visit the end of $[-L, L]$ and we are ready to knock on the door at $L+1$. Our mesh in $L^{0.1}$ so we have about $L^{0.9}/3$ independent chances to kill the possible $-1$ at $L+1$ by time $L^2 + L/3$. Most of the time, i.e., with probability at least $1 - C\exp(-\gamma L^{0.9})$ we will succeed on the first step. Then we have $L/3$ tries at $L+2$ and so on... The probability we will fail on one of our $6L$ quests (considering both sides) is at most $6LC\exp(-\gamma L^{0.9})$. □

*Proof of (4.6).* On $G_1 \cap G_2 \cap G_3$ the initial configuration makes contact with the fence and drives all the $-1$'s out of $(-4L, 4L)$ by time $2L^2$. Since the distance between tubes is $1.92L^{0.1}$ and each of its arcs contains a path the desired result follows. Here we have subtracted 2 to take care of the partial intervals at the end where the particle our construction produces might end up outside the interval of interest. □

(4.6) is the last piece of the proof of (4.1), so we have established Theorem 4 for highly supercritical 1-dependent oriented percolation on $\mathscr{L}$. Our next step is to use another block argument to extend the result to the three state contact process in dimensions $d \geqslant 1$ with $\lambda_i > \lambda_c$.

*Proof of Theorem 4 for the Three Contact Process.* We will use the results of Bezuidenhout and Grimmett,[1] in the version given in Durrett,[5] to map the $d$-dimensional three state contact process into the two dimensional three state oriented percolation introduced in the first part of the proof. Let

$$\hat{B} = [-2\hat{L}, 2\hat{L}]^d \times [0, \hat{T}] \qquad \hat{B}_{m,n} = (4\hat{L}m, 0,..., 0, 50\hat{T}n) + \hat{B}$$

Let $\zeta_t$ be the basic contact process. Lemma ($\diamond$) on page 13 of Durrett[5] says (after a little editing)

**(4.7) Lemma.** Call a site $(m, n) \in \mathscr{L}$ wet if $\zeta_t \supset x + (-\hat{J}, \hat{J})^d$ for some $(x, t) \in \hat{B}_{m,n}$. If $\hat{J}$, $\hat{L}$, and $\hat{T}$ are chosen appropriately, we can define independent random variables $\omega(m, n)$, $(m, n) \in \mathscr{L}$, with $P(\omega(m, n) = 1) \geqslant 1 - 3^{-3601}$ so that the wet sites contain the occupied sites in the oriented percolation with the corresponding initial condition.

To prove Theorem 4 for the contact process, we need not only the statement in (4.7) but also some of the details of the proof. For this reason we now give a

*Sketch of the Proof of (4.7).* One first shows that a filed copy of the cube $(-\hat{J}, \hat{J})^d$ in the space time box $\hat{B}_{0,0}$ will with high probability give rise to one in the right half of $\hat{B}^{-1,1}$ and in the left half of $\hat{B}_{1,1}$ even when no births are allowed outside $[-2\hat{L}, 2\hat{L}]^d$. Splitting the boxes gives us independence rather than the usual 1-dependence, but for us this is irrelevant, since we have already done the oriented percolation proof in the 1-dependent case.

To create the comparison between the $d$-dimensional contact process and the two dimensional oriented percolation process, we begin by defining the initial condition for the oriented percolation to be the set of even integers $m$ for which $(m, 0)$ is wet in the sense defined in (4.7). For these $m$ we can find the first occupied copy of $(-\hat{J}, \hat{J})^d$ in $B_{m,0}$ and define $\omega(m, 0)$ by seeing if the chosen occupied cube fulfills its mission of having two children in the target areas. For the even integers $m$ so that $(m, 0)$ is not wet, we simply flip an independent coin to determine the state of the site.

Proceeding to time 1, a box $B_{2m+1,1}$ may receive occupied copies of $(-\hat{J}, \hat{J})^d$ from parents in $B_{2m,0}$ or $B_{2m+2,0}$. If it receives two, we take the later one in time. If there is only one then we must use it in the next time period. If there are no occupied cubes from the previous level, we get out our coin and flip it to determine the value of $\omega(2m+1, 0)$. This construction can clearly be repeated at times 2, 3,.... We end up with a process in which some of the $\omega(z)$ are "fake" wet sites that come from coin flips independent of the graphical representation and do not represent an occupied interval in the contact process. The complementary set of wet sides that do correspond to occupied cubes are called "real." With out later proofs in mind, we would like to observe that while some of the wet sites in our paths in our tubes $A_{m,n}$ may be fake, our algorithm implies that any wet site that can be reached from a real wet site is real, i.e., it implies the existence of a nearby fully occupied cube in the underlying contact process. $\square$

To prove theorem 4 now, we will use a block construction defined directly in terms of the underlying $d$-dimensional three state contact process. We use the old notation generalized to higher dimensions and with yet another ornament.

$$\tilde{B} = (-4\tilde{L}, 4\tilde{L}) \times [-2\hat{L}, 2\hat{L}]^{d-1} \times [0, \tilde{T}] \qquad \tilde{B}_{m,n} = (2\tilde{L}m, 0 \cdots 0, \tilde{T}n) + \tilde{B}$$
$$\tilde{I} = [-\tilde{L}, \tilde{L}] \times [-2\hat{L}, 2\hat{L}]^{d-1} \qquad \tilde{I}_m = 2\tilde{L}(m, 0 \cdots 0) + \tilde{I}$$

Our happy configurations $\tilde{H}$ this time have a two part description which is the tilded version of the old one.

($\tilde{H}1$) there are no $-1$'s in $\tilde{I}$ at time 0

($\tilde{H}2$) there are at least $\tilde{L}^{0.6}$ 1's in $\tilde{I}$ at time 0.

By results in Section 4 of Durrett,[6] it is enough to prove.

**(4.8) Proposition.** Let $\varepsilon > 0$. If $\tilde{L}$ is large enough then for any starting state $\xi \in \tilde{H}$ for the three state contact process in which no births are allowed outside $(-4\tilde{L}, 4\tilde{L}) \times [2\hat{L}, 2\hat{L}]^{d-1}$, we have $\sigma_{2\tilde{L}} \xi_{\tilde{T}} \in \tilde{H}$ and $\sigma_{-2\tilde{L}} \xi_{\tilde{T}} \in \tilde{H}$ with probability $\geqslant 1 - \varepsilon$.

As the reader can probably anticipate we will take $\tilde{L} = \hat{L} \cdot L$ where the plain $L$ is the not yet specified length scale for the block construction for the two dimensional oriented percolation. Given an initial configuration for the three state contact process on $\mathbf{Z}^d$, we can run it to time $\hat{T}$ and see how many wet sites (in the sense of (4.7)) we have on the renormalized lattice. Since $\hat{J}$, $\hat{L}$, and $\hat{T}$ are fixed, a completely filled copy of $(-\hat{J}, \hat{J})^d$ has a probability that is bounded away from 0, so for large $L$, (H2) will be satisfied for the oriented percolation with probability $\geqslant 1 - \varepsilon/10$.

The high probability of the existence of a chain link fence, event $G_1$, for the oriented percolation follows from (4.2). It follows from (4.4), by choosing $L$ sufficiently large, one can ensure that on $G_1$ the failure of the event $G_2 = \{$the initial configuration makes a connection with the fence$\}$ has probability $\leqslant \varepsilon/10$. As remarked above, once a real wet site touches the chain link fence, it propagates at a rate almost 1 and each real wet site guarantees the presence of a cube nearby in space and time that is completely filled with 1's.

Since $\hat{L}$ and $\hat{T}$ are fixed, we have a positive probability that an occupied copy of $(-\hat{J}, \hat{J})^d$ will wipe out all of the $-1$'s by time $50\hat{T}$ in the adjacent box: $(4\hat{L}m, 0,..., 0) + [-2\hat{L}, 2\hat{L}]^d$. Given this and the argument for (4.5) it follows that with high probability there are no $-1$'s in $[-4\tilde{L}, 4\tilde{L}] \times [-2\hat{L}, 2\hat{L}]^{d-1}$ at time $2\tilde{L}^2$. Once the $-1$'s are wiped out the chain link fence guarantees there are at least $L^{0.9} - 2$ real wet sites in each of the target boxes.

Each wet site corresponds to an occupied cube $(-\hat{J}, \hat{J})^d$ in a $\hat{B}_{m,n}$. The last renormalized row comes within $50\hat{T}$ of the top of our space-time megabox $\tilde{B}$ so each cube will have a positive probability $\hat{\delta}$ of having at least one site survive to the end in the contact process with no births. Since $\tilde{L} = \hat{L} \cdot L$ and $\hat{L}$ is fixed, the number of successes will be $(\tilde{L})^{0.6}$ with high probability for large $L$. This establishes (4.8) and hence completes the proof of Theorem 4 for the three state contact process. $\square$

## 5. PROOF OF THEOREM 5

Recall that $\eta_t^0$ is the three state contact process starting with a single 1 at the origin in a sea of $-1$'s. Fix a site $x$ in $\mathbf{Z}^d$ and let

$$\Omega_x = \{\eta_t^0(x) = 1 \text{ for infinitely many times } t\}$$

It is clear from set theory that $\lim_{t \to \infty} P_{\lambda_i, \lambda_e}(\eta_t^0(x) = 1; \Omega_x^c) = 0$. We now deal with $\Omega_x$. Given $L$, let $T$ be the first (random) time at which there are no $-1$'s in the cube $x + [-L, L]^d$. On $\Omega_x$ the random time $T$ is almost surely finite, so we have

$$P_{\lambda_i, \lambda_e}(\eta_{2t}^0(x) = 1; \Omega_x) \leqslant P_{\lambda_i, \lambda_e}(t < T < \infty) + P_{\lambda_i, \lambda_e}(\eta_{2t}^0(x) = 1; T < t) \tag{5.1}$$

As $t \to \infty$ the first term tends to 0. To estimate the second term, we now introduce the basic contact process $\zeta_s^{t,L}$, with birth rate $\lambda_i$, that starts at time $t$ with a 1 on each site of $x + [-L, L]^d$ and for which all the sites outside of $x + [-L, L]^d$ are frozen in state 1. Given that $T < t$, there will never be a $-1$ in $x+[-L, L]^d$ after time $t$ and we may couple $\eta_s^0$ and $\zeta_s^{t, L}$ in such a way that

$$\eta_s^0(y) \leqslant \zeta_s^{t, L}(y) \text{ for every } y \in x+[-L, L]^d \text{ and for every } s>t$$

We use the preceding remark in (5.1) to get

$$P_{\lambda_i, \lambda_e}(\eta_{2t}^0(x)=1; T<t) \leqslant P_{\lambda_i}(\zeta_{2t}^{t, L}(x)=1) \tag{5.2}$$

Note that $P_{\lambda_i}(\zeta_{2t}^{t, L}(x)=1)=P_{\lambda_i}(\zeta_t^{0, L}(x)=1)$ and that this is a decreasing function of $t$. So it has a limit at $t$ goes to infinity and the self duality of the contact process we get

$$\lim_{t \to \infty} P_{\lambda_i}(\zeta_t^{0, L}(x)=1)$$
$$=P_{\lambda_i}(\zeta_t^x(y)=1 \text{ for some } y \notin x+[-L, L]^d \text{ and some } t>0)$$

where $\zeta_t^x$ is the unrestricted contact process starting with a 1 at $x$. Since $\lambda_i<\lambda_c$ the last quantity goes to 0 as $L$ goes to infinity and we have proved the desired result. □

## REFERENCES

1. Bezuidenhout, C., and Grimmett, G. (1991). Exponential decay for subcritical contact and percolation processes. *Ann. Probab.* **19**, 984–1009.
2. Bramson, M., and Durrett, R. (1988). A simple proof of the stability criterion of Gray and Griffeath. *Prob. Theo. Related Fields* **80**, 293–298.
3. Durrett, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12**, 999–1040.
4. Durrett, R. (1988). *Lecture Notes on Particle Systems and Percolation*, Wadsworth.
5. Durrett, R. (1991). The contract process, 1974–1979, *Lectures in Applied Mathematics*, Vol. 27, American Mathematical Society, Providence, RI.
6. Durrett, R. (1995). Ten lectures on particle systems, *Lecture Notes in Math.*, Vol. 1608, Springer-Verlag, New York.
7. Durrett, R., and Liu, X. F. (1988). The contact process on a finite set. *Ann. Probab.* **16**, 1158–173.
8. Durrett, R., and Schonmann, R. H. (1988). The contact process on a finite set, II. *Ann. Probab.* **16**, 1570–1583.
9. Kulasmaa, K. (1982). The spatial general epidemic and locally dependent random graphs. *J. Appl. Probab.* **19**, 745–758.
10. Liggett, T. (1985). *Interacting Particle Systems*, Springer-Verlag, New York.