Stochastic Processes and their Applications 9 (1979) 117–135

MAXIMA OF BRANCHING RANDOM WALKS vs. INDEPENDENT RANDOM WALKS*

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Received 25 October 1978
Revised 14 June 1979

In recent years several authors have obtained limit theorems for the location of the right most particle in a supercritical branching random walk. In this paper we will consider analogous problems for an exponentially growing number of independent random walks. A comparison of our results with the known results of branching random walk then identifies the limit behaviors which are due to the number of particles and those which are determined by the branching structure.

AMS (1970) Subj. Class.: Primary 60F05;
Secondary 60F10, 60J15, 60J80
Random walks — branching random walks
large deviations — regular variation

1. Introduction

Recently several authors have obtained limit theorems for the growth of a supercritical branching random walk. The first results were due to Hammersley [10], Kingman [14] and Biggins [3] who proved the following 'law of large numbers'.

Theorem 1. *Let L_n be the position of the right most particle in a supercritical branching random walk with displacement distribution F and suppose that*

$$\phi(\theta)=\int e^{\theta t}\,dF(t)<\infty \quad \textit{for some } \theta>0.$$

As $n\to\infty$, L_n/n converges almost surely to $\gamma=\sup\{a:\Phi(a)\geq 1\}$. Here $\Phi(a)=\inf(e^{-\theta a}m\phi(\theta):\theta>0)$ and $m>1$ is the mean of the offspring distribution.

In the result above L_n/n converges to a constant so the next problem is to define sequences of normalizing constants c_n and a_n so that $(L_n-c_n)/a_n$ converges to a nondegenerate limit distribution. Bramson [4] has done this for branching Brownian motions.

Theorem 2. *There is a sequence of constants $c_n=2^{1/2}n-3(2)^{-3/2}\log n+O(1)$ so that L_n-c_n converges weakly to a continuous limit distribution.*

* The research for this paper was supported by funds from NSF grant MCS 77-02121.

Since we do not expect the normal distribution or the continuous time branching to be special it seems reasonable to conjecture that similar results will hold for branching random walks (under appropriate moment assumptions). Bramson [5] has analyzed the case in which the distribution F is bounded above. If $\sup\{x : F(x) < 1\} = A < \infty$ and $m(1-F(A-)) > 1$, then the subprocess of individuals who are displaced by exactly A is supercritical. From this we get that $\{L_n - nA \equiv 0\}$ has positive probability and it is easy to show that conditioned on non-extinction $L_n - nA$ converges weakly (and a.s.?).

If $(1-F(A-))m = 1$, then the limit behavior changes. In this case the subprocess is critical so $L_n - nA \to -\infty$ and we have to choose a new sequence of norming constants. Bramson [5] has given a complete solution of this problem. In the case $F(\{0\}) = p$ and $F(\{-1\}) = 1-p$ his result may be stated as

Theorem 3. *In the offspring distribution has $p \sum_i ip_i = 1$ and $\sum_i i^{2+\delta} p_i < \infty$ for some $\delta > 0$, then conditioned on nonextinction there is a random variable V and a sequence ε_n of random variables so that $\varepsilon_n \to 0$ almost surely and*

$$L_n + \left\lceil \frac{\log\log n - \log(V+\varepsilon_n)}{\log 2} \right\rceil \to 0 \quad a.s.$$

where $\lceil x \rceil$ is the least integer $\geq x$.

This result says that along almost every sample path $L_n \to \infty$ like the (deterministic) function $\lceil(\log\log n - \log V)/\log 2\rceil$, i.e. L_n increases through the integers 0, 1, 2 ... in a very slow and predictable manner. The variables ε_n are needed in the statement of the result to adjust for the fluctuations in the transitions between successive integers.

All the results above are for distributions with $\int e^{\theta x}\, dF(x) < \infty$ for some $\theta > 0$. If the distributions have larger upper tails the limit behavior is quite different. Durrett [6] has shown

Theorem 4. *Suppose that there is a slowly varying function L so that $1-F(x) \sim x^{-q}L(x)$ as $x \to \infty$ and suppose that $\log(-x)F(x) \to 0$ as $x \to -\infty$. Then there is a sequence of constants $a_n \to \infty$ so that for all $x > 0$*

$$\mathbf{P}(L_n \leq a_n x) \to \int_{[0,\infty]} \mathbf{P}\left(\frac{mW}{m-1} \in dy\right) \exp(-ryx^{-q}).$$

Here W is the a.s. limit of Z_n/c_n which appears in Seneta's well known result (see [1, p. 30]) and the sequence a_n is chosen so that $c_n(1-F(a_n)) \to 1$. For the result above if $\sum_i (i \log i)p_i < \infty$ we can take $c_n = m^n$ and if $1-F(x) \sim x^{-q}$ we can let $a_n = c_n^{1/q}$. If both conditions hold we have $a_n = m^{n/q}$ so $L_n \to \infty$ very rapidly. This result is in sharp contrast to the linear growth observed when $\int e^{\theta x}\, dF(x) < \infty$ for some $\theta > 0$.

At this point we have surveyed the known results for branching random walks and we will begin to describe our results for the maxima of independent random walks. For each $i \geq 1$ let $\{S_n^i, n \geq 0\}$ be an independent random walk generated by F. Let $\beta > 0$ and let $M_n = \max\{S_n^i : 1 \leq i \leq \exp(\eta\beta)\}$. If we let $\beta = \log m$, then the number of random walks is comparable to the number of particles in the branching process so if we compare the results above for L_n with the results below for M_n we can see how the dependence in the branching chain effects the location of the maximum. Our first result is the analogue of Theorem 1.

Theorem 5. *If $\phi(\theta) = \int e^{\theta x}\, dF(x) < \infty$ for some $\theta > 0$, then M_n/n converges almost surely to $\gamma = \sup\{y : \Psi(y) \geq 1\}$ where $\Psi(y) = \inf(e^{-\theta y + \beta}\phi(\theta) : \theta > 0)$.*

Comparing this result with Theorem 1 shows that L_n and M_n grow at the same rate so the effect of the branching is $o(n)$. Differences between the branching and independent cases begin to appear when we consider the fluctuations in L_n and M_n. Our result for independent random walks is:

Theorem 6. *Suppose F is a nonlattice distribution with $\phi(\theta) = \int e^{\theta x}\, dF(x) < \infty$ for some $\theta > 0$ and suppose there is an $h < \theta$ so that $\phi'(h)/\phi(h) = \gamma$, the constant defined in Theorem 5. Then there is a sequence of constants $c_n = n\gamma - (2h)^{-1}\log n - K$ so that*

$$M_n - c_n \Rightarrow \exp(-\exp(-xh)).$$

Although the notation here is different, the asymptotic formula given in Theorem 6 is very close to Bramson's result for branching Brownian motions. If F is a normal distribution with mean zero and variance one, then $\phi(\theta) = \exp(\frac{1}{2}\theta^2)$ so $\phi'(\theta)/\phi(\theta) = \theta$ and $h = \gamma$. In Bramson's situation $\gamma = 2^{1/2}$ so applying Theorem 6 gives that for independent Brownian motions $c_n = 2^{1/2} - 2^{-3/2}\log n + K$. Comparing this result with Bramson's theorem shows that the dependence in the branching random walk makes L_n smaller than M_n by $2^{-1/2}\log n$.

Although the branching structure has a mild effect on the location of the maximum, it has a drastic effect on the limit distribution. For independent random walks the limit is always a double exponential distribution. For a branching random walk the limit is a complicated function of the offspring and displacement distributions and we do not know if the limit exists under the hypotheses of Theorem 6. The reader should observe that even for independent random walks some technical condition is needed to rule out the bounded case for in this case there is behavior similar to, but much simpler than that observed by Bramson [5] for branching random walks. If $\sup\{x : F(x) < 1\} = A < \infty$ and $e^{\beta}(1 - F(A-)) > 1$, then $\mathbf{P}(M_n = nA) \to 1$. If $e^{\beta}(1 - F(A-)) = 1$, then $\mathbf{P}(M_n = nA) \to 1 - e^{-1}$ and for $x < A$

$$\mathbf{P}(M_n - (n-1)A < x) \to e^{-1}(F(x)/F(A-)).$$

If $e^{\beta}(1 - F(A-)) < 1$, then $\mathbf{P}(M_n = nA) \to 0$ and we can show that the conclusion of Theorem 6 holds in this case.

All the results above are for distributions with $\int e^{\theta x}\, dF(x) < \infty$ for some $\theta > 0$. In Sections 3 and 4 we consider what happens when $1-F(x) \approx x^{-q}L(x)$. Section 3 is devoted to technicalities. In this section we prove some large deviations results which consolidate and generalize results due to Nagaev [16] for $q \geq 2$ and Heyde [11–13] for $0 \leq q \leq 2$, $q \neq 1$. The theorems which we have obtained are the following:

Theorem 7. *If $x_n \uparrow \infty$ is such that S_n/x_n converges to 0 in probability, then for all $\varepsilon > 0$*

$$\liminf_{n\to\infty} \frac{P(S_n \geq x_n)}{n\, P(S_1 \geq (1+\varepsilon)x_n)} \geq 1.$$

Theorem 8. *Let $0 \leq q \leq \infty$ and suppose $1-F(x) \approx x^{-q}L(x)$ as $X \to \infty$. If some $\delta > 0$ $x_n/n^{(1+\delta)/(q\wedge 1)} \to \infty$, then*

$$\limsup_{n\to\infty} \frac{P(S_n \geq x_n)}{n\, P(S_1 \geq x_n)} \leq 1.$$

Combining Theorems 7 and 8 shows that if $1-F(x) \approx x^{-q}L(x)$ and $x_n \to \infty$ fast enough we have $P(S_n \geq x_n) \approx n P(S_1 \geq x_n)$. Using this fact it is easy to prove the following result:

Theorem 9. *Let F be a distribution such that*

(a) $1-F(x) \approx x^{-q}L(x)$ *as $x \to \infty$ and*

(b) $(\log(-x))F(x) \to 0$ *as $x \to -\infty$.*

If we pick c_n so that $n\, e^{\beta n}(1-F(c_n)) \to 1$, then

$$P\left(\frac{M_n}{c_n} \leq x\right) \to \exp(-x^{-q}) \quad \text{for all } x.$$

The reader should observe that if $1-F(x) \sim x^{-q}$, *then* $c_n \sim (n\, e^{\beta n})^{1/q}$ so M_n goes to ∞ very rapidly.

If we let $\beta = \log m$ then $c_n = (nm^n)^{1/q}$ which compares with $c_n = m^{n/q}$ in Theorem 4. This difference is easy to explain. The location of M_n is determined in each case by one large jump. In the case of independent random walks there are nm^n opportunities. In branching random walks there are

$$r \sum_{k=0}^{n} m^k \approx m^n\left(\frac{m}{m-1}\right).$$

2.

In this section we will obtain limit theorems for M_n when the underlying distribution F satisfies the following condition

$$\phi(\theta) = \int \exp(\theta x)\, dF(x) < \infty \quad \text{for some } \theta > 0.$$

The first result we will prove is the law of large numbers for M_n given in the introduction

Theorem 5. *As $n \to \infty$, M_n/n converges almost surely to $\gamma = \sup\{y : \Psi(y) \geq 1\}$ where $\Psi(y) = \inf(e^{-\theta y + \beta}\phi(\theta) : \theta > 0)$.*

Proof. We start by observing that for all $\theta > 0$

$$P(M_n \geq ny) \leq \text{expected number of particles in } [ny, \infty) = e^{\beta n} P(S_n \geq ny)$$
$$\leq e^{\beta n} e^{-\theta n y}\phi(\theta)^n$$

so if $\Psi(y) < 1$, $\sum P(M_n \geq ny) < \infty$ and from the Borel–Cantelli lemma we have

$$\limsup_{n\to\infty} \frac{M_n}{n} \leq \gamma_1 \quad \text{almost surely} \tag{1}$$

where $\gamma_1 = \inf\{y : \Psi(y) < 1\}$.

The last computation gives an upper bound for the growth of M_n. The next step in the proof is to establish the following lower bound

$$\liminf_{n\to\infty} \frac{M_n}{n} \geq \gamma_2 = \sup\{y : \Psi(y) > 1\}. \tag{2}$$

After we have done this we will show $\gamma_2 = \gamma = \gamma_1$ to complete the proof.

To prove (2) we will start by considering the case in which F is bounded above – i.e. $\sup\{x : F(x) < 1\} = A < \infty$. In this case $\phi(\theta) < \infty$ for all $\theta > 0$ so for any $\theta > 0$ we can define a distribution F^θ by the formula

$$F^\theta(y) = \phi(\theta)^{-1} \int_{-\infty}^{y} e^{\theta x}\, dF(x).$$

The distribution has mean

$$m(\theta) = \phi(\theta)^{-1} \int_{-\infty}^{\infty} x\, e^{\theta x}\, dF(x) = \frac{\phi'(\theta)}{\phi(\theta)}$$

and variance

$$\sigma^2(\theta) = \phi(\theta)^{-1} \int_{-\infty}^{\infty} x^2 e^{\theta x}\, dF(x) - m(\theta)^2 = \frac{\phi''(\theta)}{\phi(\theta)} - \left(\frac{\phi'(\theta)}{\phi(\theta)}\right)^2 = m'(\theta).$$

Since we have assumed F is nondegenerate each F^θ has positive variance and so $m(\theta)$ is strictly increasing. $m(0) = \mu = \int x\, dF(x) \in [-\infty, \infty)$ and the dominated convergence theorem implies $m(\theta)$ is continuous so it follows from this that there is a unique positive solution of $m(\theta) = y$ for all

$$\mu < y < \lim_{\theta\uparrow\infty} m(\theta) = A.$$

When $m(\theta)=y$ has a positive solution we can use Theorems 1 and 6 of [19] to obtain the following asymptotic formula for $\mathbf{P}(S_n > yn)$.

Lemma 1. *Let F be a nondegenerate distribution with $\phi(\theta)<\infty$ for some $\theta>0$. If $h<\theta$ and $m(h)=y$, then*

$$\log \mathbf{P}(S_n > ny) = n \log \phi(h) - nhy - \tfrac{1}{2}\log n + O(1). \tag{3}$$

From this result we can compute the asymptotic behavior of M_n. To do this we observe that

$$\mathbf{P}(M_n \leq ny) = (1-\mathbf{P}(S_n > ny))^{\exp(\beta n)}$$

and for Lemma 1 we have

$$\log(\exp(\beta n)\mathbf{P}(S_n > ny)) = n(\beta + \log \varphi(h) - hy)\tfrac{1}{2}\log n + O(1)$$

so

$$\mathbf{P}(M_n \leq ny) \to \begin{cases} 0, & \text{if } \beta + \log\phi(h) - hy > 0, \\ 1, & \text{if } \beta + \log\phi(h) - hy \leq 0. \end{cases}$$

In the first case we can sharpen the result to say $\sum \mathbf{P}(M_n \geq ny) < \infty$. To do this we observe that $1-x \leq e^x$ so

$$(1-\mathbf{P}(S_n \geq ny))^{\exp(\beta n)} \leq \exp(-\exp(\beta n)\mathbf{P}(S_n \geq ny)).$$

To bound the right-hand side note that if $\beta + \log\phi(h) - hy = a > 0$, then there is an $\varepsilon > 0$ so that

$$\exp(\beta n)\mathbf{P}(S_n \geq ny) \geq \varepsilon\, e^{na/2}.$$

Combining the last two results shows

$$\sum_{n=1}^{\infty} \mathbf{P}(M_n \geq ny) \leq \sum_{n=1}^{\infty} \exp(-\varepsilon\, e^{na/2}) < \infty$$

and applying the Borel-Cantelli lemma gives that

$$\liminf_{n\to\infty} \frac{M_n}{n} \geq \gamma_3 = \sup\{y : \beta + \log\phi(h) - hy \geq 0\}.$$

To transform the last result into the one given in (2) we have to show that $\gamma_2 = \gamma_3$. To do this we observe that

$$\frac{d}{d\theta} e^{-\theta y+\beta}\phi(\theta) = e^{-\theta y+\beta}\phi(\theta)\left(-y + \frac{\phi'(\theta)}{\phi(\theta)}\right)$$

so if $y \in (\mu, A)$ the infinum of $e^{-\theta y+\beta}\phi(\theta)$ on $(0,\infty)$ is attained at the $\theta = \theta(y)$ which solves $\phi'(\theta)/\phi(\theta) = y$. The value of the function at this point is

$$\exp(\beta + \log\phi(\theta(y)) - y\theta(y)).$$

This number is >1 or ≤ 1 accordingly as $\beta + \log \phi(\theta(y)) - y\theta(y) > 0$ or ≤ 0 so $\gamma_2 = \gamma_3$.

At this point we have proved (2) under the assumption that $\sup\{x : F(x) < 1\} < \infty$. To prove (2) for unbounded F we will proceed by truncation. For each $K < \infty$ we can define a distribution by

$$\bar{F}_K(x) = \begin{cases} F(x) & \text{for } x < K, \\ 1 & \text{for } x \geq K. \end{cases}$$

If we let M_n^K be the maximum of $e^{\beta n}$ independent random walks generated by $\bar{F}_K$, then we have from the results above that

$$\liminf_{n\to\infty} \frac{M_n^K}{n} \geq \gamma_2^K = \sup\{y : \Psi^K(y) > 1\} \tag{4}$$

where $\Psi^K(y) = \inf(e^{-\theta y + \beta}\phi^K(\theta) : \theta > 0)$ and $\phi^K(\theta) = \int e^{\theta x}\, dF^K(x)$.

From the last result it is easy to prove (2). To do this we observe that $\bar{F}_1 \geq \bar{F}_2 \geq \cdots \geq F$ so if we use the recipe $\bar{F}_K^{-1}(U)$, U uniform on $(0, 1)$ to define the displacements we can define $M_n^1, M_n^2, \ldots, M_n$ on the same probability space so that

$$M_n^1 \leq M_n^2 \leq \cdots \leq M_n.$$

Combining this result with (4) shows that

$$\liminf_{n\to\infty} \frac{M_n}{n} \geq \lim_{K\uparrow\infty} \gamma_2^K$$

so it remains to check that $\lim_{K\uparrow\infty} \gamma_2^K \geq \gamma_2$. To do this we observe that as $K\uparrow\infty$, $\phi^K(\theta)\uparrow\phi(\theta)$ so

$$\Psi^K(y) = \left(\inf_{\theta\geq 0} e^{-\theta y + \beta}\phi^K(\theta) : \theta \geq 0\right)\uparrow$$

and

$$\liminf_{K\to\infty} \Psi^K(y) \leq \Psi(y)$$

(strict inequality is possible if $\phi(\theta) = \infty$ for some $\theta > 0$). From this it is easy to see that $\lim_{K\uparrow\infty} \gamma_2^K \geq \gamma_2$ which completes the proof of (2).

The last detail in proving Theorem 5 is to show that $\gamma_1 = \gamma_2$. To do this we use

Lemma 2. *$\psi(y)$ is strictly decreasing on $\{y : y \geq \mu : \Psi(y) > 0\}$.*

Proof. As we have observed before, if $y < A_0 = \sup(\phi'(\theta)/\phi(\theta) : \phi(\theta) < \infty)$, the infimum is attained at the $\theta(y)$ which solves $\phi'(\theta)/\phi(\theta) = y$ and the value at this point is $\exp(-y\theta(y) + \beta)\phi(\theta(y))$. Differentiating with respect to y gives

$$\exp(-y\theta(y) + \beta)\phi(\theta(y))\left[-\theta(y) - y\theta'(y) + \frac{\phi'(\theta(y))}{\phi(\theta(y))}\theta'(y)\right]$$

which is <0 (since $\phi'(\theta(y))/\phi(\theta(y)) = y$) so Ψ is strictly decreasing on $[\mu, A_0]$.

If $y > A_0$, then the infimum is approached as $\theta \to \theta_0 = \sup(\theta : \phi(\theta) < \infty)$. If $\Psi(y) > 0$, then for any $\varepsilon > 0$ we can pick a θ_1 so that if $\theta \in (\theta_1, \theta_0)$

$$e^{-\theta y+\beta}\phi(\theta) \leq (1+\varepsilon)M(y).$$

If $z > y$ and $\theta \in (\theta_1, \theta_0)$, then

$$M(z) \leq e^{-\theta z+\beta}\phi(\theta) \leq (1+\varepsilon)\, e^{-\theta(z-y)}M(y).$$

Since this holds for all $\varepsilon > 0$

$$M(z) \leq e^{-\theta(z-y)}M(y) < M(y)$$

so Ψ is strictly decreasing on $[A_0, \infty) \cap \{y : \Psi(y) > 0\}$ and the proof of the lemma is complete.

From Lemma 2 it is trivial to see that $\gamma_1 = \gamma_2$. Combining this observation with (1) and (2) proves that M_n/n converges to γ almost surely. While this result is some accomplishment it is only the first step in determining the asymptotic behavior of M_n, the next step is to consider the convergence of $(M_n - c_n)/a_n$. It is easy to determine what the norming constants should be. If there are constants a_n and c_n so that $(M_n - c_n)/a_n$ converges weakly to a continuous distribution, then these constants can be chosen by picking $0 < p < q < 1$ and defining c_n and a_n by $\mathbf{P}(M_n < c_n) \to p$, $\mathbf{P}(M_n \leq c_n + a_n) \to q$ or equivalently

$$\exp(\beta n)\mathbf{P}(S_n > c_n) \to -\log p, \tag{5}$$

$$\frac{\mathbf{P}(S_n > c_n + a_n)}{\mathbf{P}(S_n > c_n)} \to \frac{\log q}{\log p}. \tag{6}$$

To compute the norming constants from (5) and (6) we need a formula for $\mathbf{P}(S_n > ny)$ which is more accurate than (3). The result which we will use is Theorem 2 of [19].

Lemma 3. *Suppose F has a nonlattice distribution and $\phi(\theta) = \int e^{\theta x}\, dF(x) < \infty$ for some $\theta > 0$. If there is an $h > 0$ so that $\phi'(h)/\phi(h) = \gamma$, then for any sequence $\delta_n \to 0$*

$$\mathbf{P}(S_n \geq n(\gamma+\delta_n)) \sim \frac{\exp(n[\log\phi(h) - h(\gamma+\delta_n) - \delta_n^2(1+O(\delta_n))/2\sigma^2(h)])}{h(2\pi n\sigma^2(h))^{1/2}}. \tag{7}$$

Now from the definition of γ we have $\beta + \log\phi(h) - h\gamma = 0$ so from the formula above we have

$$e^{\beta n}\mathbf{P}(S_n \geq n(\gamma+\delta)) \sim \frac{\exp(n[-h\delta_n - \delta_n^2(1+O(\delta_n))/2\sigma^2(h)])}{h(2\pi n\sigma^2(h))^{1/2}}. \tag{8}$$

From this it follows that if we let $c_n^* = n(\gamma+\delta_n^*)$ where $\delta_n^* = -\log(h(2\pi n\sigma^2(h))^{1/2})/hn$, then $n(\delta_n^*)^2 \to 0$ so

$$e^{\beta n}\mathbf{P}(S_n \geq c_n^*) \to 1$$

and if we let $p = e^{-1}$ in (5) we can pick

$$c_n = n\gamma - \log(h(2\pi n\sigma^2(h))^{1/2})/h = n\gamma - (2h)^{-1}\log n - h^{-1}\log(h(2\pi\sigma^2(h))^{1/2}). \quad (9)$$

The next step is to determine the scaling constants a_n. To do this we observe that from (8) and the definition of c_n

$$\begin{aligned} e^{\beta n}\mathbf{P}(S_n \geq c_n + a_n x) &= e^{\beta n}\mathbf{P}(S_n \geq n(\gamma+\delta_n^*) + a_n x) \\ &\sim \frac{\exp(n[-h(\delta_n^* + a_n x/n) - \frac{(\delta_n^* + a_n x/n)^2}{2\sigma^2(h)}(1+O(\delta_n^* + a_n x/n))])}{h(2\pi n\sigma^2(h))^{1/2}} \\ &= \exp\left(-ha_n x - n\frac{(\delta_n^* + a_n x/n)^2}{2\sigma^2(h)}(1+O(\delta_n^* + a_n x/n))\right). \end{aligned}$$

Since $\delta_n^* \sim -\log n/2hn$ it follows from this that if we take $a_n \equiv 1$

$$e^{\beta n}\mathbf{P}(S_n \geq n(\gamma+\delta_n^*) + x) \to e^{-hx}.$$

and so

$$\mathbf{P}(M_n \leq n(\gamma+\delta_n^*) + x) \to \exp(-e^{-hx}).$$

We can restate the last result as

Theorem 6. *Suppose F is a nonlattice distribution with $\phi(\theta) = \int \exp(\theta x)\,dF(x) < \infty$ for some $\theta > 0$ and suppose there is an $h < \theta$ so that $\phi'(h)/\phi(h) = \gamma$ the constant defined in Theorem 5. If we define c_n by (9) we have*

$$M_n - c_n \Rightarrow \exp(-\exp(-xh)).$$

The result above cannot be applied if $\phi'(\cdot)/\phi(\cdot) = \gamma$ has no solution. This can happen if

(1) $\sup\{x : F(x) < 1\} = A$ and $1 - F(A-) > 0$ or if

(2) $\sup\{\theta : \phi(\theta) < \infty\}$ and $\sup\{\theta : \phi'(\theta)/\phi(\theta)\}$ are both $< \infty$.

In the first case we can determine the asymptotic behavior of M_n. If $e^{\beta}\mathbf{P}(S_1 = A) > 1$ the problem is trivial: $e^{\beta n}\mathbf{P}(S_n = nA) \to \infty$ so $\mathbf{P}(M_n = nA) \to 1$. If $e^{\beta}\mathbf{P}(S_1 = A) < 1$, then we can pick an $\varepsilon > 0$ so that $e^{\beta}\mathbf{P}(S_1 \geq A - \varepsilon) < 1$ and we have $e^{\beta n}\mathbf{P}(S_n \geq n(A-\varepsilon)) \to 0$. This implies $\mathbf{P}(M_n \leq n(A-\varepsilon)) \to 1$ so $\gamma \in (u, A-\varepsilon]$. From computations in the proof of Theorem 5 it follows that there is an $h > 0$ so that $\phi'(h)/\phi(h) = \gamma$ and Theorem 6 can be applied.

If $e^{\beta}\mathbf{P}(S_1 = A) = 1$, then we are in a critical case. $e^{\beta n}\mathbf{P}(S_n = nA) = 1$ for all n so we have

$$\mathbf{P}(M_n = nA) = 1 - (\mathbf{P}(S_n < nA_0))^{\exp(\beta n)} \to 1 - e^{-1}.$$

To determine the distribution of M_n on $\{M_n < nA\}$ we will consider the limit of

$$J_n = \min_{1 \leq i \leq \exp(\beta n)} I_n^i$$

where

$$I_n^i = |\{k : S_k^i - S_{k-1}^i \neq A, 1 \leq k \leq n\}|.$$

From the calculations above we have $\mathbf{P}(J_n = 0) \to 1 - e^{-1}$. This implies $\mathbf{P}(J_n \geq 1) \to e^{-1}$. To compute the limit of $\mathbf{P}(J_n \geq 2)$ we observe

$$\mathbf{P}(J_n \geq 2) = \mathbf{P}(I_n^1 \geq 2)^{\exp(\beta n)} = (1 - \mathbf{P}(I_n^1 \leq 1))^{\exp(\beta n)},$$

$$\mathbf{P}(I_n^1 \leq 1) = \mathbf{P}(I_n^1 = 0) + \mathbf{P}(I_n^1 = 1) = \mathbf{P}(S_1 = A)^n + n\mathbf{P}(S_1 = A)^{n-1}\mathbf{P}(S_1 < A).$$

When $e^{\beta}\mathbf{P}(S_1 = A) = 1$

$$e^{\beta n}\mathbf{P}(I_n^1 \leq 1) = 1 + n\, e^{\beta}\mathbf{P}(S_1 < A).$$

Since $\mathbf{P}(S_1 = A) = e^{-\beta} < 1$ this $\to \infty$ as $n \to \infty$ so $\mathbf{P}(J_n \geq 2) \to 0$. In the critical case then J_n converges weakly to J where $\mathbf{P}(J = 0) = 1 - e^{-1}$ and $\mathbf{P}(J = 1) = e^{-1}$. From this we see that

$$\mathbf{P}(M_n - (n-1)A = A) \to 1 - e^{-1}$$

and

$$\mathbf{P}(M_n - (n-1)A < x) \to e^{-1}(F(x)/F(A-))$$

for $x < A$.

This completes the consideration of case (1) and leaves us with case (2). In this case there is not much to say. Let $A_0 = \sup\{\phi'(\theta)/\phi(\theta) : \phi(\theta) < \infty\}$. If $\gamma < A_0$, then there is an $h > 0$ which has $\phi'(h)/\phi(h) = \gamma$ and Theorem 6 can be applied. If β is too large $\gamma \geq A_0$ and we need new methods to determine the limit behavior of M_n. We conjecture that as we observed in the bounded case, the limit behavior changes drastically from that given in Theorem 6 but we have not been able to solve the problem in this case.

3.

In this section we will consider large deviations probabilities for distributions F which have $1 - F(x) \sim x^{-q}L(x)$ for some $q > 0$ and slowly varying function L. Our aim will be to show that if $x_n \to \infty$ sufficiently rapidly, then $\mathbf{P}(S_n > x_n)/n\mathbf{P}(S_1 > x_n) \to 1$. The first step in doing this is to obtain a lower bound for $\mathbf{P}(S_n > x_n)$. This result does not require the assumption that $1 - F(x) \sim x^{-q}L(x)$.

Theorem 7. *If $x_n \uparrow \infty$ is such that $x_n^{-1}S_n$ converges to 0 in probability, then for all $\varepsilon > 0$*

$$\liminf_{n \to \infty} \frac{\mathbf{P}(S_n > x_n)}{n\mathbf{P}(S_1 > (1+\varepsilon)x_n)} \geq 1.$$

Proof. The proof follows the same lines as a proof of a result due to Heyde [12, p. 1576]. Let $A_i=\{X_i>(1+\varepsilon)x_n\}$ and $B_i=\{S_n-X_i>-\varepsilon x_n\}$, then

$$\begin{aligned}\mathbf{P}(S_n\geq x_n)&\geq\mathbf{P}\Big(\bigcup_{i=1}^n A_i\cap B_i\Big)=\sum_{i=1}^n\mathbf{P}\Big((A_i\cap B_i)\cap\Big(\bigcup_{j=1}^{i-1}A_j\cap B_j\Big)^c\Big)\\&\geq\sum_{i=1}^n\mathbf{P}\Big((A_i\cap B_i)\cap\Big(\bigcup_{j=1}^{i-1}A_j\Big)^c\Big)\geq\sum_{i=1}^n\Big(\mathbf{P}(A_i\cap B_i)-\mathbf{P}\Big(\bigcup_{j=1}^{i-1}A_j\cap A_i\Big)\Big)\\&\geq n\mathbf{P}(A_1)\mathbf{P}(B_1)-n^2\mathbf{P}(A_1)^2\end{aligned}$$

The hypothesis implies $\mathbf{P}(B_1)\to 1$ so to complete the proof it suffices to show $n\mathbf{P}(A_1)\to 0$ but this is a consequence of the degenerate convergence criterion given below (see [9, p, 134]). If $c_n\uparrow\infty$, then $c_n^{-1}S_n\xrightarrow{\mathbf{P}}0$ if and only if

(i) $n\mathbf{P}(|S_1|>\varepsilon c_n)\to 0$ for all $\varepsilon>0$,
(ii) $nc_n^{-2}\int_{|x|\geq c_n}x^2\,\mathrm{d}F(x)\to 0$ and
(iii) $nc_n^{-1}\int_{|x|\leq c_n}x\,\mathrm{d}F(x)\to 0$.

The next result gives an upper for $\mathbf{P}(S_n>x_n)$.

Theorem 8. *Let* $0<q<\infty$ *and suppose* $1-F(x)\sim x^{-q}L(x)$ *as* $x\to\infty$. *If for some* $\delta>0$, $x_n/n^{(1+\delta)/(q\wedge 1)}\to\infty$, *then*

$$\liminf_{n\to\infty}\frac{\mathbf{P}(S_n>x_n)}{n\mathbf{P}(S_1>x_n)}\leq 1.$$

Proof. The proof of this fact is based on the following lemma which is a generalization of a result of Nagaev [16].

Lemma 3. *Suppose*
(a) $p\geq 1$, $\mathbf{E}(S_1^+)^p<\infty$, $\mathbf{E}(S_1^-)^2<\infty$ *and* $\mathbf{E}S_1=0$ *or*
(b) $0<p<1$ *and* $\mathbf{E}(S_1^+)^p<\infty$.
If $x_n/(n^{1/(p\wedge 2)}\log n)\to\infty$ *and* $y_n=(1-\varepsilon)x_n$, *then there is a constant* K_p *such that for all n sufficiently large*

$$\mathbf{P}(S_n>x_n)\leq n\mathbf{P}(S_1>y_n)+3\Big(\frac{nK_p}{y_n^p}\Big)^{1/(1-\varepsilon)}$$

Proof. Let $F^y(x)=F(x)\wedge F(y)$. Let F_n and F_n^y be the nth convolutions of F and F^y. From the definition of F_y it follows that we have

$$F_n^y(x)=\mathbf{P}(S_n\leq x,\ x_j\leq y\text{ for }1\leq j\leq n)\leq F_n(x)$$

so

$$1-F_n(x)\leq 1-F_n^y(x)=1-F_n^y(\infty)+F_n^y(\infty)-F_n^y(x).$$

From the first equation we see $1-F_n^y(\infty)\leq n(1-F(y))$. To estimate the other term we let $0<h<1$ and define

$$F_n^{y,h}(x)=\int_{(-\infty,x]} \mathrm{e}^{hu}\,\mathrm{d}F_n^y(u),$$

$$R(y,h)=\int_{(-\infty,y]} \mathrm{e}^{hu}\,\mathrm{d}F^y(u)$$

and

$$\bar{F}^{y,h}(x)=F_n^{y,h}(x)/R(y,h)^n.$$

Using these definitions we can write

$$F_n^y(\infty)-F_n^y(x)=\int_{(x,\infty)} \mathrm{e}^{-hu}\,\mathrm{d}F_n^{y,h}(u)=R(y,h)^n\int_{(x,\infty)} \mathrm{e}^{-hu}\,\mathrm{d}\bar{F}_n^{y,h}(u). \quad (10)$$

Since $\bar{F}_n^{y,h}$ is a probability distribution and $h>0$ we have

$$\int_{(x,\infty)} \mathrm{e}^{-hu}\,\mathrm{d}\bar{F}_n^{y,h}(u)\leq \mathrm{e}^{-hx}.$$

To estimate $R(y,h)$ we write (assuming $1/h\leq y$)

$$R(y,h)=\int_{(-\infty,1/h]} \mathrm{e}^{hu}\,\mathrm{d}F(u)+\int_{(1/h,y]} \mathrm{e}^{hu}\,\mathrm{d}F(u)\equiv R_1(h)+R_2(y,h).$$

To obtain an estimate for $R_2(y,h)$ we observe that integrating by parts gives

$$R_2(y,h)=(F(y)-1)\,\mathrm{e}^{hy}-(F(1/h)-1)\,\mathrm{e}-\int_{1/h}^{y}(F(u)-1)h\,\mathrm{e}^{hu}\,\mathrm{d}u.$$

Since $c_p=\mathbf{E}(S_1^+)^p<\infty$ we have $1-F(1/h)\leq c_p h^p$ and

$$\int_{1/h}^{y}(1-F(u))\,\mathrm{e}^{hu}\,\mathrm{d}u\leq c_p\int_{1/h}^{y}u^{-p}\,\mathrm{e}^{hu}\,\mathrm{d}u=c_p h^{p-1}\int_{1}^{yh}u^{-p}\,\mathrm{e}^{u}\,\mathrm{d}u.$$

Integrating by parts gives

$$\int_1^{yh}u^{-p}\,\mathrm{e}^u\,\mathrm{d}u=u^{-p}\,\mathrm{e}^u\Big|_1^{yh}+p\int_1^{yh}u^{-p-1}\,\mathrm{e}^u\,\mathrm{d}u.$$

Now $u^{-p-1}\,\mathrm{e}^u$ is decreasing on $[1,p+1]$ and increasing on $[p+1,\infty)$ so

$$\int_1^{yh}u^{-p}\,\mathrm{e}^u\,\mathrm{d}u\leq(yh)^{-p}\,\mathrm{e}^{yh}+pyh\max((yh)^{-p-1}\,\mathrm{e}^{yh},\mathrm{e})=$$

$$=(yh)^{-p}\,\mathrm{e}^{yh}(1+p\max(1,(yh)^{p+1}/\mathrm{e}^{yh-1})).$$

Since x^{p+1}/e^{x-1} is continuous and $\to 0$ as $x\to\infty$ there is a constant K_p so that

$$\int_1^{yh}u^{-p}\,\mathrm{e}^u\,\mathrm{d}u\leq K_p(yh)^{-p}\,\mathrm{e}^{yh}.$$

Using this inequality with the results above gives

$$R_2(y,h)\leqslant \mathrm{e}c_p h^p + K_p(yh)^{-p}\,\mathrm{e}^{yh}.$$

The next step is to obtain an estimate for $R_1(h)$. To do this we will consider several cases. We start by supposing $p\geqslant 2$. In this case we observe that if u is any real number we can write

$$\mathrm{e}^{hu}=1+hu+\frac{(hu)^2}{2}\,\mathrm{e}^{\theta(hu)}\quad\text{where } 0\leqslant\frac{\theta(hu)}{hu}\leqslant 1.$$

Using this formula and observing $\mathbf{E}S_1=0$ we have

$$\begin{aligned}
R_1(h)&=\int_{-\infty}^{1/h}\mathrm{d}F(u)+\int_{-\infty}^{1/h}hu\,\mathrm{d}F(u)+\int_{-\infty}^{1/h}\frac{(hu)^2}{2}\,\mathrm{e}^{\theta(hu)}\,\mathrm{d}F(u)\\
&=1-\int_{1/h}^{\infty}\mathrm{d}F(u)-\int_{1/h}^{\infty}hu\,\mathrm{d}F(u)+\frac{h^2}{2}\int_{-\infty}^{1/h}u^2\,\mathrm{e}^{\theta(hu)}\,\mathrm{d}F(u)\\
&\leqslant 1+\frac{h^2}{2}\int_{-\infty}^{1/h}u^2\,\mathrm{e}^{\theta(hu)}\,\mathrm{d}F(u).
\end{aligned}$$

To estimate this expression we observe

$$\int_{-\infty}^{1/h}u^2\,\mathrm{e}^{\theta(hu)}\,\mathrm{d}F(u)\leqslant\int_{-\infty}^{0}u^2\,\mathrm{d}F(u)+\mathrm{e}\int_{0}^{1/h}u^2\,\mathrm{d}F(u)\leqslant\mathbf{E}(S_1^-)^2+\mathrm{e}c_2$$

so

$$R_1(h)\leqslant 1+\frac{h^2}{2}(\mathbf{E}(S_1^-)^2+\mathrm{e}c_2).$$

If $1\leqslant p<2$ we still have $\mathbf{E}S_1=0$ but c_2 may be infinite so we need to use a different bound for the last integral in the expansion. In this case we observe that $u^p(1-F(u))\leqslant c_p$ so

$$\begin{aligned}
\int_{0}^{1/h}u^2\,\mathrm{d}F(u)&=\mathbf{E}[(S_1^+)^2;S_1^+\leqslant 1/h]\leqslant\mathbf{E}[(S_1^+\wedge h^{-1})^2]=\\
&=\int_0^{1/h}2u(1-F(u))\,\mathrm{d}u\leqslant 1+\int_1^{1/h}2c_pu^{1-p}\,\mathrm{d}u\leqslant 1+\frac{2c_ph^{p-2}}{2-p}.
\end{aligned}$$

Combining this with other estimates from the case $p\geqslant 2$ gives

$$\int_{-\infty}^{1/h}u^2\,\mathrm{e}^{\theta(hu)}\,\mathrm{d}u\leqslant\mathbf{E}(S_1^-)^2+\mathrm{e}+\frac{2\mathrm{e}c_ph^{p-2}}{2-p}$$

and

$$R_1(h)\leqslant 1+\left(\frac{\mathrm{e}c_p}{2-p}\right)h^p+\left(\frac{\mathbf{E}(S_1^-)^2+\mathrm{e}}{2}\right)h^2.$$

If $0<p<1$, then the mean of S_1 does not exist so the arguments given above do not apply. In this case we will expand only to first order in hu. To do this we observe that if u is any real number we can write

$$e^{hu}=1+hu\,e^{\theta'(hu)} \quad \text{where } 0\leq\frac{\theta'(hu)}{hu}\leq 1.$$

Using this formula we have

$$\begin{aligned}R_1(h)&=1-\int_{1/h}^{\infty} dF(u)+h\int_{-\infty}^{1/h} u\,e^{\theta'(hu)}\,dF(u)\\&\leq 1+h\,e\int_0^{1/h} u\,dF(u).\end{aligned}$$

Again we observe $u^p(1-F(u))\leq c_p$ so

$$\begin{aligned}\int_0^{1/h} u\,dF(u)&=E[S_1^+;S_1^+\leq 1/h]\leq E[S_1^+\wedge h^{-1}]\\&=\int_0^{1/h} 1-F(u)\,du\leq 1+\int_1^{1/h} c_pu^{-p}\,du\leq 1+\frac{c_ph^{p-1}}{1-p}\end{aligned}$$

and

$$R_1(h)\leq 1+\frac{ec_ph^p}{1-p}+eh.$$

Combining the estimates above and using the fact that $h\leq 1$ we have

$$R_1(h)\leq 1+K_p'h^{(p\wedge 2)} \quad \text{for } 0<p<\infty$$

where

$$K_p'=\begin{cases}\dfrac{E(S_1^-)^2+ec_2}{2}, & \text{if } p\geq 2,\\[2ex] \dfrac{ecp}{2-p}+\dfrac{E(S_1^-)^2+e}{2}, & \text{if } 1\leq p<2,\\[2ex] \dfrac{ecp}{1-p}+e, & \text{if } 0<p<1.\end{cases}$$

Combining this with the estimate for $R_2(y,h)$ gives

$$R(y,h)\leq 1+(K_p'+ec_p)h^{p\wedge 2}+K_p(yh)^{-p}\,e^{yh}.$$

Now if $\theta\geq 0$, then $(1+\theta)^n\leq e^{n\theta}=1+n\theta+(n^2/n)\theta+\cdots$ so from this it follows that

$$R(y,h)^n\leq\exp(n[K_p'+ec_p)h^{p\wedge 2}+K_p(yh)^{-p}\,e^{yh}]).$$

Substituting $h=-\log(nK_p y^{-p})/y$ in the last expression and noticing that $0<h<1$ and $1/h\leq y$ if $y>p\log y>1+\log nK_p$ gives that

$$R(y,h)^n\leq \exp\left(1+n(K'+ec_p)\left(\frac{p\log y-\log nK_p}{y}\right)^{p\wedge 2}\right)$$

and

$$F_n^y(\infty)-F_n^y(x)\leq e^{-hx}R(y,h)^n$$
$$\leq\left(\frac{nK_p}{y^p}\right)^{x/y}\exp\left(1+n(K_p'+ec_p)\left(\frac{p\log y-\log nK_p}{y}\right)^{p\wedge 2}\right)$$

whenever $y>p\log y>1+\log nK_p$.

Now for the purposes of the lemma we are interested in the value of the expression when $x_n/(n^{1/(p\wedge 2)}\log n)\to\infty$ and $y_n=(1-\varepsilon)x_n$. In this case we will have $y_n>p\log y_n>1+\log nK_p$ when n is sufficiently large and we have that the exponent in the right-hand side of the inequality converges to 1. From this it follows that for n sufficiently large we have

$$F_n^y(\infty)-F_n^y(x_n)\leq 3\left(\frac{nK_p}{y_n^p}\right)^{1/1-\varepsilon}.$$

Combining this inequality with the fact that $1-F_n^y(\infty)\leq n(1-F(y))$ proves the lemma.

Proof of Theorem 8. Having proved Lemma 3 it is easy to prove the theorem. We will first consider the case $0<q\leq 1$. Let $\varepsilon>0$ and $y_n=(1-\varepsilon)x_n$

$$\frac{1-F_n(x_n)}{n(1-F(x_n))}=\frac{1-F_n(x_n)}{n(1-F(y_n))}\,\frac{1-F(y_n)}{1-F(x_n)}.$$

Since $1-F(x)\sim x^{-q}L(x)$

$$\frac{1-F_n(y_n)}{1-F(x_n)}\to(1-\varepsilon)^{-q}.$$

To estimate the other term observe that $\mathbf{E}(S_1^+)^p<\infty$ for all $p<q$ and if we pick $p>q/(1+\delta)$, then $x_n/(n^{1/p}\log n)\to\infty$ so Lemma 3 can be applied to give

$$\limsup_{n\to\infty}\frac{1-F_n(x_n)}{n(1-F(y_n))}\leq 1+3\limsup_{n\to\infty}\frac{(nK_p)^{1/1-\varepsilon}}{ny_n^{p/1-\varepsilon}(1-F(y_n))}.$$

The next step is to show that the lim sup on the right-hand side is 0. To do this we pick $\varepsilon'<\varepsilon$ so that $\varepsilon'>q\varepsilon/p(1+\delta)$ and write the fraction as

$$\frac{n^{\varepsilon/1-\varepsilon}K_p^{1/1-\varepsilon}}{y_n^{p\varepsilon'/(1-\varepsilon)}y_n^{p(1-\varepsilon')/(1-\varepsilon)}(1-F(y_n))}. \tag{12}$$

Since $\varepsilon' < \varepsilon$, $y_n^{p(1-\varepsilon')/(1-\varepsilon)}(1-F(y_n)) \to \infty$. (see [8, p. 277]). Since $\varepsilon' > q\varepsilon/p(1+\delta)$ and $y_n/n^{(1+\delta)/q} \to \infty$ we have

$$\frac{n^{\varepsilon/1-\varepsilon}}{y_n^{p\varepsilon'/1-\varepsilon}} \to 0.$$

Combining the last two observations and the computations above gives

$$\limsup_{n\to\infty} \frac{1-F_n(x_n)}{n(1-F(x_n))} \leq (1-\varepsilon)^{-q}.$$

Since ε was arbitrary this proves the desired result when $0 < q \leq 1$.

To prove the result when $1 < q < \infty$ we begin by proving the result when $\mathbf{E}S_1 = 0$ and $\mathbf{E}(S_1^-)^2 < \infty$. Let $\varepsilon > 0$ and $y_n = (1-\varepsilon)x_n$. From arguments above

$$\limsup_{n\to\infty} \frac{1-F_n(x_n)}{n(1-F(x_n))} \leq (1-\varepsilon)^{-q} \limsup_{n\to\infty} \frac{1-F_n(x_n)}{n(1-F(y_n))}.$$

To estimate the right-hand side we observe $x_n/n^{1+\delta} \to 0$ for some $\delta > 0$ so if $1 < p < q$ Lemma 3 can be applied to give

$$\limsup_{n\to\infty} \frac{1-F_n(x_n)}{n(1-F(y_n))} \leq 1 + 3 \limsup_{n\to\infty} \frac{(nK_p)^{1/1-\varepsilon}}{ny_n^{p/1-\varepsilon}(1-F(y_n))}.$$

To show that the lim sup on the right-hand side is 0 we pick $\varepsilon' < \varepsilon$ so that $\varepsilon' > \varepsilon/p$ and write the fraction in the same way as in (12). Since $\varepsilon' < \varepsilon$

$$y_n^{p(1-\varepsilon')/1-\varepsilon}(1-F(y_n)) \to \infty.$$

Since $\varepsilon' > \varepsilon/p$ and $y_n/n^{1+\delta} \to \infty$ for some $\delta > 0$ we have

$$\frac{n^{\varepsilon/1-\varepsilon}}{y_n^{p\varepsilon'/1-\varepsilon}} \to 0.$$

Using these observations in the same way as in the case $p < q \leq 1$ proves the desired inequality when $\mathbf{E}S_1 = 0$ and $\mathbf{E}(S_1^-)^2 < \infty$.

To prove the result in general we begin by truncating the distribution. Let

$$F^0(x) = \begin{cases} F(x), & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

If F_n^0 is the nth convolution of F^0, then $1-F_n^0(x) \geq 1-F_n(x)$ and $1-F^0(x) = 1-F(x)$ for $x > 0$ so it suffices to show

$$\limsup_{n\to\infty} \frac{1-F_n^0(x_n)}{n(1-F^0(x_n))} \leq 1.$$

Let a be the mean of F^0. Since $q > 1$, $a < \infty$. If we let $F^1(x) = F^0(x+a)$, then F^1 is a distribution with mean 0 and $\int_{-\infty}^0 x^2 \, dF(x) < \infty$ so using the result from the last part of the proof gives that if $z_n/n^{1+\delta} \to \infty$

$$\limsup_{n\to\infty} \frac{1-F_n^1(z_n)}{n(1-F^1(z_n))} \leq 1$$

or letting $x_n = z_n - an$ we have that if $x_n/n^{1/\delta} \to \infty$

$$\limsup_{n\to\infty} \frac{1-F_n^0(x_n)}{n(1-F^0(x_n+an))} \leq 1.$$

To translate this into the desired result observe that $x_n/(x_n+an) \to 1$ so

$$\lim_{n\to\infty} \frac{1-F^0(x_n)}{1-F^0(x_n+a_n)} = 1.$$

4.

In this section we will obtain a limit theorem for M_n when the underlying distribution has $1-F(x) \sim x^{-q}L(x)$ for some $q>0$ and slowly varying function L. The main result is

Theorem 9. *Let F be a distribution such that*

(a) $1-F(x) \sim x^{-q}L(x)$ *as* $x \to \infty$ *and*

(b) $(\log(-x))F(x) \to 0$ *as* $x \to -\infty$.

If we pick c_n so that $n\,\mathrm{e}^{\beta n}(1-F(c_n)) \to 1$, then for all $x>0$

$$\mathbf{P}\left(\frac{M_n}{c_n} \leq x\right) \to \exp(-x^{-q}).$$

Proof. To apply the results of Section 3 we have to check that $c_n^{-1}S_n$ converges to 0 in probability. By the degenerate convergence criterion (see [4, p. 134]) this happens if and only if

(i) $n\mathbf{P}(|S_1| > \varepsilon c_n) \to 0$ for all $\varepsilon > 0$,

(ii) $nc_n^{-2} \int_{|x|\leq c_n} x^2\, \mathrm{d}F(x) \to 0$ and

(iii) $nc_n^{-1} \int_{|x|\leq c_n} x\, \mathrm{d}F(x) \to 0$.

To check the first condition we observe that if $\delta > 0$ and $r > q$, then from [8, p. 277] we have that $1-F(x) \geq (1+\delta)x^{-r}$ for all x sufficiently large, so if n is large enough we have

$$c_n^{-r} \leq \frac{1-F(c_n)}{(1+\delta)} \leq \frac{1}{n\,\mathrm{e}^{\beta n}}$$

so $c_n \geq n^{1/r}\,\mathrm{e}^{\beta n/r}$. It is easy to see that this implies $n\mathbf{P}(S_1 > \varepsilon c_n) \to 0$. To show that the same result holds for the other tail let $x_n = \mathrm{e}^{\beta n/q}$ and observe that if n is large

$$n\mathbf{P}(S_1 < -\varepsilon c_n) \leq n\mathbf{P}(S_1 < -\varepsilon\, \mathrm{e}^{\beta n/q}) = \frac{q}{\beta} \log x_n \mathbf{P}(S_1 < -\varepsilon x_n)$$

$$= \frac{q}{\beta} \log(\varepsilon x_n)\mathbf{P}(S_1 < -\varepsilon x_n) - \frac{q}{\beta} \log \varepsilon\, \mathbf{P}(S_1 < -\varepsilon x_n)$$

and the last expression $\to 0$ as $n \to \infty$.

To check that conditions (ii) and (iii) are satisfied we begin by estimating the integrals over $[0, c_n]$. For (ii) we observe

$$nc_n^{-2}\int_0^{c_n} x^2\,\mathrm{d}F(x)\le nc_n^{-2}\int_0^{c_n} 2x(1-F(x))\,\mathrm{d}x$$

and if $0<p<q$, then it follows from [8, p. 277] that $1-F(x)\le Kx^{-p}$ for $x\ge 1$ so the expression above

$$\le nc_n^{-2}\left(1+\int_1^{c_n} 2Kx^{1-p}\,\mathrm{d}x\right)\to 0$$

as $n\to\infty$. For (iii) the same estimate works

$$nc_n^{-1}\int_0^{c_n} x\,\mathrm{d}F(x)\le nc_n^{-1}\int_0^{c_n} 1-F(x)\,\mathrm{d}x\le nc_n^{-1}\left(1+\int_1^{c_n} 2Kx^{-p}\,\mathrm{d}x\right)\to 0$$

as $n\to\infty$.

To estimate the integrals over $[-c_n, 0]$ we use a different approach based on assumption (b). We start by observing that

$$nc_n^{-2}\int_{-c_n}^{0} x^2\,\mathrm{d}F(x)\le nc_n^{-2}\int_{-c_n}^{0} (-2x)F(x)\,\mathrm{d}x\le 2nc_n^{-1}\int_{-c_n}^{0} F(x)\,\mathrm{d}x$$

and the last expression is also an upper bound for

$$nc_n^{-1}\int_{-c_n}^{0} (-x)\,\mathrm{d}F(x)$$

so to finish checking (ii) and (iii) it suffices to show that the right-hand side of the last inequality converges to 0. To do this we write the integral as

$$2nc_n^{-1}\int_{-c_n}^{-c_n^{1/2}} F(x)\,\mathrm{d}x+2nc_n^{-1}\int_{-c_n^{1/2}}^{0} F(x)\,\mathrm{d}x\le 2nF(-c_n^{1/2})+2nc_n^{-1/2}.$$

Now if $r>q$, $c_n\ge \mathrm{e}^{\beta n/r}$ so the last expression is

$$\le 2nF(-\mathrm{e}^{\beta n/2r})+2n\,\mathrm{e}^{-\beta n/2r}.$$

Since the last expression converges to 0 as $n\to\infty$ this shows that (ii) and (iii) hold and hence that $c_n^{-1}S_n$ converges to 0 in probability.

The last conclusion shows that the hypothesis of Theorem 7 is satisfied. From assumption (a) we have that the hypothesis of Theorem 8 is satisfied. Combining the conclusions of these two theorems we have that if $x>0$

$$\lim_{n\to\infty}\frac{\mathbb{P}(S_n>xc_n)}{n\,\mathbb{P}(S_1>xc_n)}=1.$$

At this point it is easy to prove the desired limit theorem. To do this we observe that if $x > 0$

$$\mathbf{P}\left(\frac{M_n}{c_n} \leq x\right) = (1 - \mathbf{P}(S_n > xc_n))\,\mathrm{e}^{\beta n}.$$

Now to show that the right-hand side converges to $\exp(-x^{-q})$ it suffices to show

$$\exp(\beta n)\mathbf{P}(S_n > xc_n) \to x^{-q}.$$

To do this we write $\exp(\beta n)\mathbf{P}(S_n > xc_n)$ as

$$n\,\exp(\beta n)\mathbf{P}(S_1 > c_n)\frac{\mathbf{P}(S_1 > xc_n)}{\mathbf{P}(S_1 > c_n)}\,\frac{\mathbf{P}(S_n > xc_n)}{n\,\mathbf{P}(S_1 > xc_n)}$$

and observe that as $n \to \infty$ the first and third terms converge to 1 while the second converges to x^{-q}.

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