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Chutes and Ladders in Markov Chains

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We investigate how the stationary distribution of a Markov chain changes when transitions from a single state are modified. In particular, adding a single directed edge to nearest neighbor random walk on a finite discrete torus in dimensions one, two, or three changes the stationary distribution linearly, logarithmically, or only locally. Related results are derived for birth and death chains approximating Bessel diffusions and for random walk on the Sierpinski gasket.

KEY WORDS: Markov chains; stationary distribution; Bessel diffusions; Sierspinski gasket.

1. INTRODUCTION

Given an irreducible Markov chain X_n with finite state space X and transition probability $p(x, y)$, pick a state w and define a new Markov chain $\bar{X}_n$ with transition probability $\bar{p}(x, y) = p(x, y)$ when $x \neq w$. The other row $\bar{p}(w, \cdot)$ is for the moment arbitrary, but we will usually take $\bar{p}(w, z) = 1$ for some z. In this case, as in the children's game *Chutes and Ladders*, a marker that lands at w in on the next step transported to z.

The main question we will adress here is: how much can the stationary distribution of our Markov chain be changed by altering one row of the transition probability? The answer is simple if we have random walk on a graph and add one new two-way edge connecting sites w and z. If the

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original degrees of the vertices x of the graph were $d(x)$ and we define the degrees for the new graph by

$$\bar{d}(x) = \begin{cases} d(x) & x \neq z, w \\ d(x)+1 & x \in \{z, w\} \end{cases}$$

and let $D = \sum d(x)$, then the new stationary distribution is $\bar{\pi}(x) = \bar{d}(x)/(D+2)$. When $x \neq w, z$, $\bar{\pi}(x) = \pi(x) \cdot D/(D+2)$. When $x \in \{z, w\}$, $\pi(x) = \frac{d(x)+1}{(D+2)} \leqslant \pi(x) + \frac{1}{D+2}$.

As we will see, adding a one way edge to a random walk on the torus can lead to a large change in the stationary distribution. To begin to develop our results we let $T_w = \inf\{n \geqslant 1 : X_n = w\}$ be the usual hitting time of w, let $V_x = \inf\{n \geqslant 0 : X_n = x\}$ be the time of the first visit to x, and let $\bar{T}_w = \inf\{n \geqslant 1 : \bar{X}_n = w\}$. By Markov chain theory (see e.g., (4.3) on p. 303 in Durrett[(3)]) the following formulas define (unnormalized) stationary measures for the chains X_n and $\bar{X}_n$

$$\mu_w(x) = \sum_{n=0}^{\infty} P_w(X_n = x, T_w > n)$$

$$\bar{\mu}_w(x) = \sum_{n=0}^{\infty} P_w(\bar{X}_n = x, \bar{T}_w > n)$$

The basic equation which underlies most of our results is the following fact proved in the Appendix.

Perturbation Formula. If $x \neq w$

$$\frac{\bar{\mu}_w(x)}{\mu_w(x)} = \sum_z \bar{p}(w, z) \frac{P_z(V_x < T_w)}{P_w(V_x < T_w)}$$

If $x = w$ then $\bar{\mu}_w(w)/\mu_w(w) = 1/1$. When $p(w, z) = 1$ the formula reduces to

$$\frac{\bar{\mu}_w(x)}{\mu_w(x)} = \frac{P_z(V_x < T_w)}{P_w(V_x < T_w)} \quad \text{when} \quad x \neq w \tag{$\star$}$$

To see that changing one row in the transition probability can have a significant effect on the stationary distribution, we begin with a simple example that will be treated in detail in Section 2.

Example 1 (Random walk on $[0, L]$ with reflecting boundary conditions). Here $p(x, x+1) = p(x, x-1) = 1/2$ for $0 < x < L$ while $p(0, 0) = p(0, 1) = p(L, L-1) = p(L, L) = 1/2$. With these boundary conditions the walk is doubly stochastic, so the stationary distribution is uniform. Let $w = 0$ and $z = [L/2]$. In words when the walk hits the left boundary it is transported back to the center of the state space.

Using simple martingale arguments, we can evaluate the numerator of (★)

$$P_z(V_x < T_w) = \begin{cases} 1 & w < x \leqslant z \\ z/x & z < x \leqslant L \end{cases}$$

and the denominator

$$P_w(V_x < T_w) = \frac{1}{2} \cdot \frac{1}{x}$$

The fact that the stationary distribution is uniform and $\mu_w(w) = 1$ implies $\mu_w(x) \equiv 1$, so plugging into (★) gives

$$\bar{\mu}_w(x) = \begin{cases} 2x & 0 < x \leqslant z \\ 2z & z < x \leqslant L \end{cases}$$

In words, the stationary distribution is linear on $[1, z]$ and constant on $[z, L]$. Rates of convergence for the modified chain appear in Diaconis and Saloff-Coste,[2] Example 2F. A detailed study of stationary distribution and rates of convergence for chains obtained by making similar small perturbations of nice chains appears in Wilmer.[6]

From the explicit formula for $\bar{\mu}_w$, it is easy to understand its limiting behavior as $L \to \infty$. Rescaling $\{0, 1, 2,..., L\}$ to become $\{0, 1/L,..., 1\}$ and normalizing this to make a probability distribution we have

$$\pi_L(x) \Rightarrow \tfrac{4}{3}(2x \wedge 1) \qquad \text{where} \Rightarrow \text{denotes weak convergence}$$

To prepare for our later results, we will take a more complicated approach to reach the same conclusion. Let X_t^L = simple random walk on $\{0, 1/L, 2/L,..., 1\}$ with jumps at times k/L^2, and let $\bar{X}_t^L$ be our modification with $w = 0$, $z = [L/2]/L$, and $\bar{p}(w, z) = 1$. It is easy to see, but tedious to prove rigorously that

Claim. As $L \to \infty$, $\bar{X}_t^L \Rightarrow \bar{B}_t$ where $\bar{B}_t$ is Brownian motion with reflection at 1 and a jump to 1/2 when the process hits 0. $\bar{B}_t$ has stationary distribution $\pi(x) = \frac{4}{3}(2x \wedge 1)$.

Sketch of Proof. It is easy to see that $\bar{B}_t$ will behave like Brownian motion before it hits 0 and at that time will be transported to 1/2. Weak convergence then follows by checking tightness, etc. To prove the second claim we note that the stationary distributions for the approximating chains converge to the proposed limit, so it must be a stationary distribution for the limit. Uniqueness of the stationary distribution for $\bar{B}_t$, which follows from Doeblin's theory of Markov chains, then completes the proof. □

The reader will see the reason for our interest in this claim when we come to Example 4, and will encounter there a problem more interesting than filling in the details of the argument above.

Example 2 (Random walk on the torus $\Lambda(L) = (\mathbf{Z} \bmod L)^d$ in dimensions $d \geqslant 2$). Skipping over the problematic borderline case of $d = 2$ we begin with $d \geqslant 3$. Suppose for simplicity that the distance from w to z on the torus $\rho(w, z) \to \infty$. Here and in what follows, w and z may depend on L but we suppress that dependence from the notation. To approximate the numerator in (★) we will prove

$$P_z^L(V_x > T_w) \approx \tfrac{1}{2} P_z(V_x = \infty) \tag{1.1}$$

where by $\approx$ we mean that $\sup_x$ error $\to 0$ as $L \to \infty$. Here P_z^L refers to the random walk on the torus, and P_z to the random walk on $\mathbf{Z}^d$. The intuition behind this result is that if the random walk gets a distance $L/2$ from its starting point without hitting x then is is unlikely to hit either point before its distribution becomes uniform on the torus, and after that each point will be hit first with probability $1/2$. This will be proved in Section 3.

Similar reasoning leads to the relationship

$$P_w^L(V_x < T_w) \approx P_w(V_x < T_w) + \tfrac{1}{2} P_w(V_x = \infty, T_w = \infty) \tag{1.2}$$

Using (1.1) and (1.2) in (★) it follows that for $x \neq w$

$$\frac{\bar{\mu}_w(x)}{\mu_w(x)} \approx \frac{1 - \frac{1}{2} P_z(V_x = \infty)}{P_w(V_x < T_w) + \frac{1}{2} P_w(V_x, T_w = \infty)} \tag{1.3}$$

If $\rho(x, w), \rho(x, z) \to \infty$ then

$$\frac{\bar{\mu}_w(x)}{\mu_w(x)} \to \frac{1}{P_w(T_w = \infty)} \tag{1.4}$$

In words, in dimension $d \geqslant 3$ the effect of the change is confined to the sites within $O(1)$ of w and z.

To obtain quantitative results we have simulated the process on a $100 \times 100 \times 100$ torus with $w = (25, 25, 0)$ and $z = (75, 75, 0)$ for 10^9 steps.

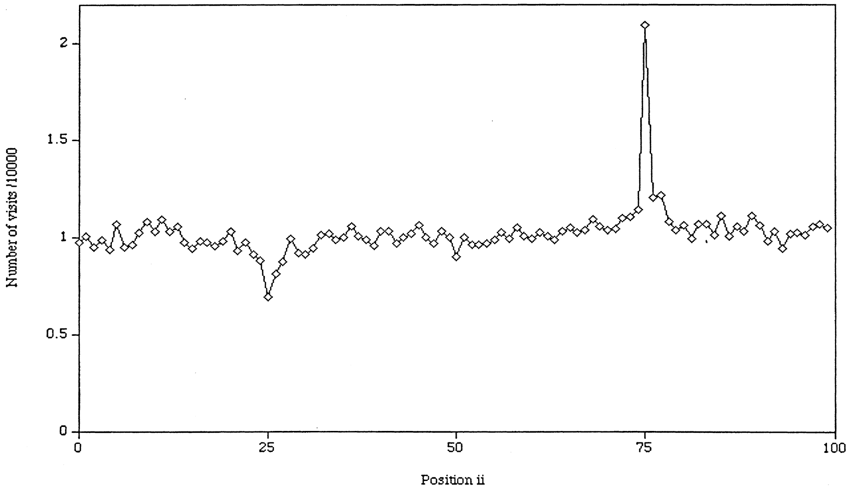

Figure 1

Figure 1 shows the number of visits to $(i, i, 0)$ for $0 \leqslant i < 100$. Now $\mu_w(x) \equiv 1$ for all x, and (1.4) implies the total mass $\bar{\mu}(\Lambda(L)) \approx L^3/P_w(T_w = \infty)$ so using $P_z(V_z = \infty) = 0$ and $P_w(V_z < T_w) \approx 0$ in (1.3) we have that the stationary distribution $\bar{\pi}(z) \approx 2/L^3$ in agreement with the simulation. At $x = w$, $\bar{\mu}_w(w) = \mu_w(w) = 1$, so $\bar{\pi}(z) \approx P_w(T_w = \infty)/L^3$. The numerical value $P_w(T_w = \infty) \approx 0.6595$ (see, e.g., Durrett,[3] p. 196) again agrees well with the simulation. Note that except for i within 1 or 2 of 25 or 75, there is very little departure from the uniform distribution.

Turning now to the borderline case $d = 2$, it is well known that there is a constant c_1 so that

$$P_0(T_0 > t) \sim c_1/\log t \qquad \text{as} \quad t \to \infty \tag{1.5}$$

It will be shown in Section 4 that

$$P_x(V_0 > t) \approx \left\{ \frac{\log^+ |x|^2}{\log t} \wedge 1 \right\} \tag{1.6}$$

where by $\approx$ we mean that $\sup_x$ error $\to 0$ as $t \to \infty$. Suppose for simplicity that

$$\liminf_{L \to \infty} \rho(w, z)/L > 0$$

In this case we will show that

$$P_z^L(V_x > T_w) \approx \tfrac{1}{2} \cdot P_z(V_x > L^2) \tag{1.7}$$

Combining this with (1.6) it follows that the numerator in (★)

$$P_z^L(V_x < T_w) \approx 1 - \frac{1}{2} \cdot \frac{\log^+ |z-x|^2}{\log(L^2)} \tag{1.8}$$

Similar reasoning shows that the denominator

$$P_w^L(V_x < T_w) \approx \tfrac{1}{2} P_w(T_w > |x-w|^2) \tag{1.9}$$

Using (1.8), (1.9), and (1.5) in (★) leads to the result

$$\frac{c_1}{2 \log L} \cdot \frac{\bar{\mu}_w(x)}{\mu_w(x)} \approx \left(2 - \frac{\log^+ |z-x|}{\log L}\right) \cdot \left(\frac{\log^+ |x-w|}{\log L}\right) \tag{1.10}$$

Again, the $\approx$ means $\lim_{L \to \infty} \sup_x$ error $\to 0$.

To see what (1.10) says let $p < 1$ and note that if $|z-x| = L^p$ (and hence $|x-w| = O(L)$) then the answer is $\approx (2-p) \cdot 1$, while if $|w-x| = L^p$, the answer $\approx 1 \cdot p$. In words, the effect of the change is confined to within $o(L)$ of w and z, while the size of the perturbation is linear in the logarithm of the distance from $\{w, z\}$. The last result implies that as $L \to \infty$ the stationary distribution on the rescaled torus $\Lambda(L)/L$ approaches Lebesgue measure but it also indicates that the convergence of the stationary distribution of the uniform limit is slow. Figure 2 shows the occupation time distribution for a 100×100 torus with $w = (25, 25)$ and $z = (75, 75)$ run for 10^9 steps. To compare the asymptotic formula in (1.10) with the stationary distribution estimated from simulation, we have compared the values of the two at points of the form (i, i) in Fig. 3. The qualitative shape is the same but there are some important quantitative diferences.

To further estimate the dependence of our answers on the dimension, we will consider objects with non-integral dimension. In Section 2, we will consider birth and death chains that approximate Bessel processes. A more interesting possibility is

Example 3 (Random walk on the Sierpinski gasket). To construct a sequence of graphs G_n that approximate the Sierpinski gasket, we start with the six vertex graph G_1 given in Fig. 4 and then for $n \geqslant 2$ replace the three outer triangles v_i, m_j, m_k with $\{i, j, k\} = \{1, 2, 3\}$ by copies of G_{n-1}.

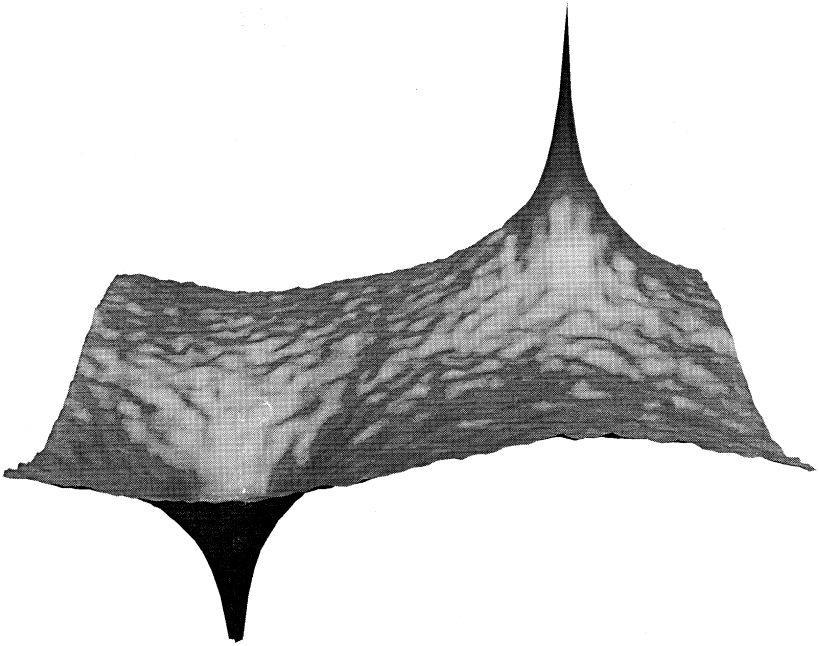

Figure 2

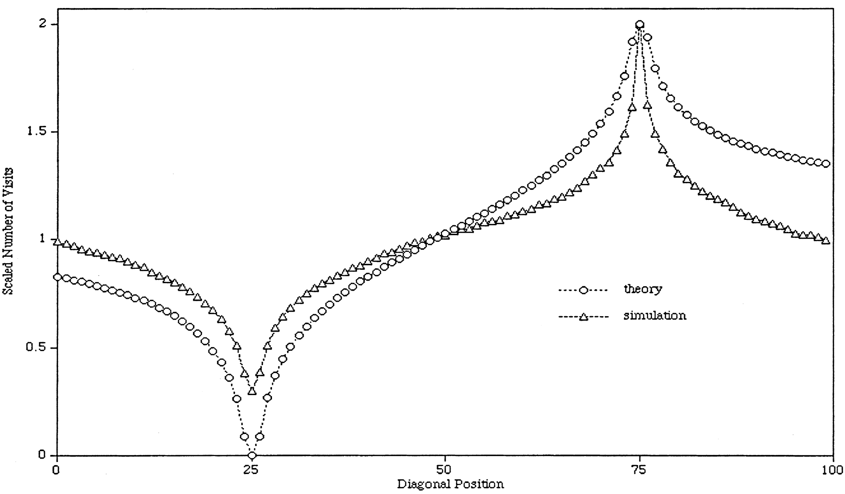

Figure 3

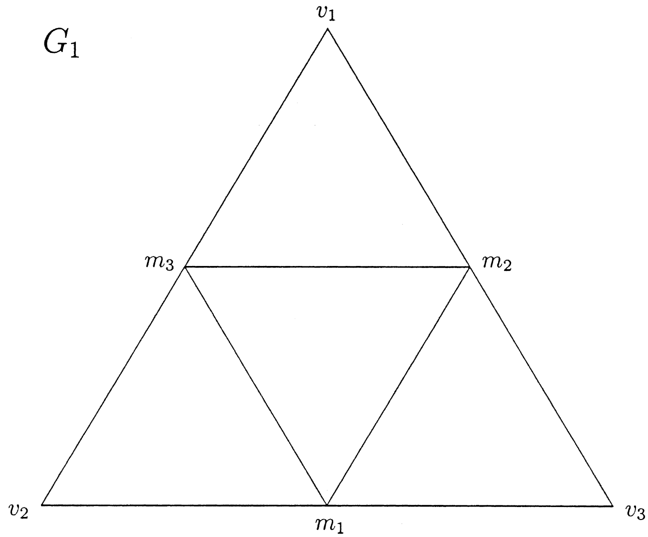

Figure 4

A picture of G_3 is given in Fig. 5. Let X^n be the random walk that at points with four neighbors jumps to each with probability 1/4, and does not move with probability 1/2. As in Example 1, with this choice of boundary behavior the chain is doubly stochastic and has a uniform stationary distribution.

To perturb the chain we let w and z be two corners of G_n and let $\bar{p}(w, z) = 1$. The key to our analysis of this example is

Decimation Invariance. Let $X_0^M \in G_m$, $m \leqslant M$ and define hitting times by $S_0 = 0$ and for $k \geqslant 1$, $S_k = \inf\{n > S_{k-1} : X_n^M \in G_m\}$. Then $\{X^M(S_k), k \geqslant 0\}$ and $\{X_k^m, k \geqslant 0\}$ have the same distribution.

For this property and much more on the Sierpinski gasket, the reader can consult Barlow and Perkins.[1]

From decimation invariance it follows immediately that if $x \in G_m$ and $m \leqslant M$ then

$$P_z^M(V_x < T_w) = P_z^m(V_x < T_w) \tag{1.11}$$

and hence $\lim_{M \to \infty} P_z^M(V_x < T_w)$ exists.

To begin to analyze the denominator in ($\star$) suppose w is the top vertex and let x_k, y_k be the points on the left and right side of the outer

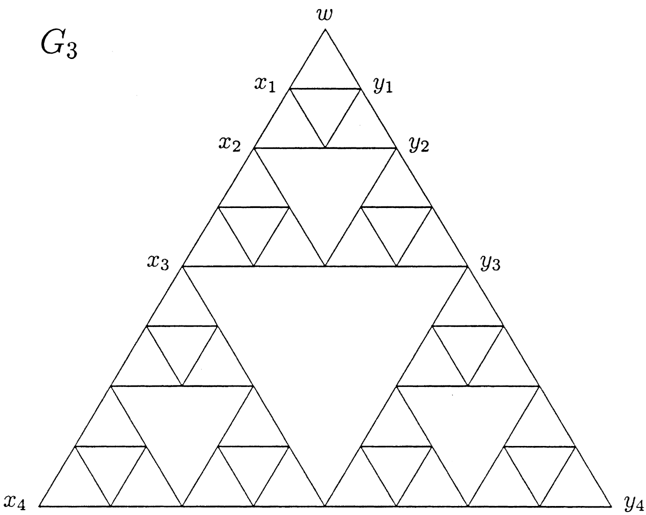

Figure 5

triangle at distance 2^{k-1} from w (see Fig. 5), and let $S_k = \inf\{n: X_n \in \{x_k, y_k\}\}$. Using symmetry and a simple computation for the random walk on G_1 one can show

$$P_w(S_k < T_w) = \tfrac{1}{2} \cdot (3/5)^{k-1} \tag{1.12}$$

Proof. Since this is the only fact we need to prove to complete the treatment of Example 3, we give the details here. Using the notation introduced in Fig. 4, let $h(x)$ be the probability that starting from x we visit v_1 before v_2 or v_3. Clearly $h(v_1) = 1$, $h(v_2) = h(v_3) = 0$, and symmetry implies $h(m_2) = h(m_3)$. Let $a = h(m_2) = h(m_3)$ and $b = h(m_1)$. By considering what happens on one step

$$b = a/2 \qquad a = \tfrac{1}{4}(a + b + 1)$$

Plugging the first equation into the second we have $a = 3a/8 + 1/4$ and solving gives $a = 2/5$, $b = 1/5$. Using decimation invariance and symmetry now we see that starting from x_k or y_k the probability of hitting $\{x_{k+1}, y_{k+1}\}$ before w is $3/5$. To get the induction started we note $P_w(S_1 < T_w) = 1/2$ and the desired result follows. □

From this we see that if $x \in G_m$, then $(5/3)^M \cdot P_w^M(V_x < T_w)$ is constant for $m \geqslant M$ and hence

$$\lim_{M \to \infty} (5/3)^M \cdot P_w^M(V_x < T_w) \quad \text{exists} \tag{1.13}$$

Combining the results in (1.11) and (1.13) it follows that if $x \neq w$, then there is a function $\psi(x)$ so that

$$\frac{3^M}{5^M} \cdot \frac{P_z^M(V_x < T_w)}{P_w^M(V_x < T_w)} \to \psi(x) \tag{1.14}$$

Thus as in one dimension the perturbation makes a significant change in the stationary distribution.

It does not seem possible to get an explicit formula for ψ. Figure 6 shows a simulation of the process on G_7 run for 10 million times steps. Here w is the lower left corner, z is the top vertex, and the equilateral triangles have been replaced by right triangles to facilitate storage in a rectangular array in the computer. Figure 7 takes a closer look at the stationary distribution on the boundary of the triangle. Notice the lack of monotonicity on the bottom edge.

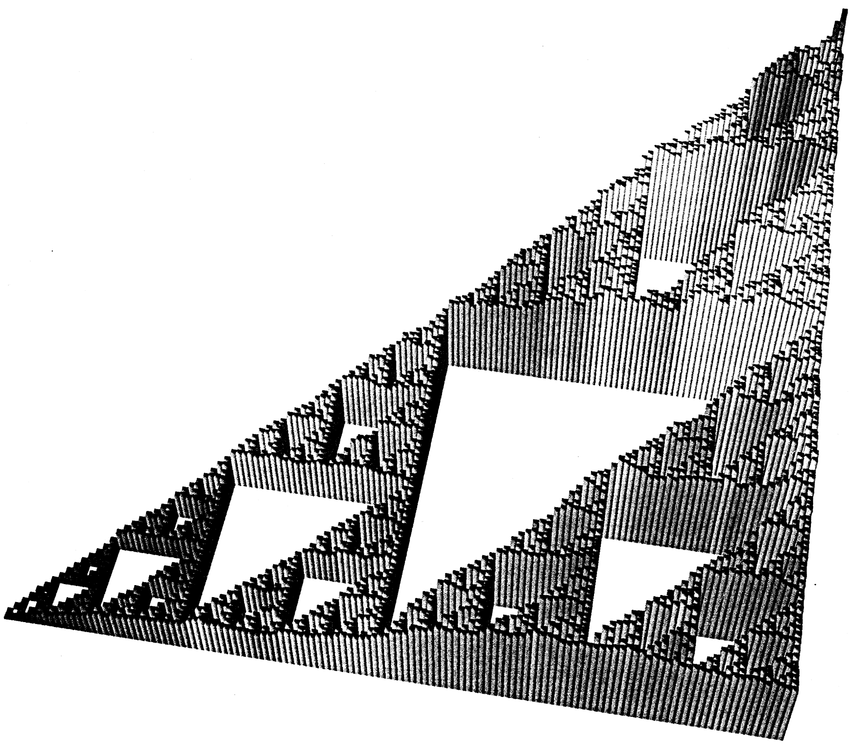

Figure 6

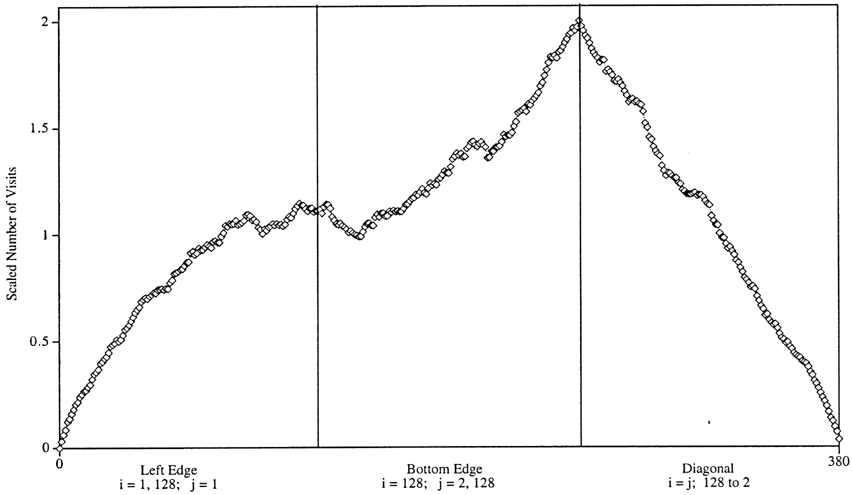

Figure 7

By analogy with the result in one dimension we

Conjecture. $\psi(x)$ is a stationary measure for Brownian motion on the Sierpinski gasket with a jump to z when it hits w.

We leave this problem and the generalization of our results to Lindstrom's[5] nested fractals for someone skilled in diffusions on fractals.

To try to bring out the patterns in the results we have found, let $\alpha_L = E_w T_w / E_z T_w$, the total mass $\bar{\mu}_w(S) = E_z T_w$ and $\mu_w(S) = E_w T_w = 1/\pi(w)$. In our examples this is as follows:

	$\mu_w(S)$	$\bar{\mu}_w(S)$
$d=1$	L	cL^2
$d=2$	L^2	$cL^2 \log L$
$d>2$	L^d	cL^d
gasket	3^n	5^n

From this one can see that all of our results have the form:

$$\alpha_L \cdot \frac{\bar{\mu}_w(x)}{\mu_w(x)} \approx \psi_L(x)$$

Random walks in $d=1$ or on the Sierpinski Gasket when rescaled converge to diffusions Y_t that hit points, so

$$\psi_L(x) \to \psi(x)$$

where $\psi(x)$ is the stationary distribution of Y_t modified by a jump from the limit of the w's to the limit of the z's. On the other hand, random walks in $d \geqslant 2$: converge to a limiting Brownian motion Y_t that does not hit points, so $\psi_L(x) \to 1$ when $x \neq z, w$. It would be interesting to prove a general result that encapsulates the dichotomy described in the last two sentences.

The remainder of the paper is devoted to proofs. In Section 2 we analyze Example 1 and random walks approximating Bessel diffusions. Section 3 considers random walks on the torus in $d>3$ with the more difficult borderline case $d=2$ in Section 4. Before turning to these tasks we should note that, as Robert Israel and Oscar Rothaus have pointed out to us, one can approach the "rank one" perturbation we have studied by matrix theory. Specifically, one has the following (see the Appendix for the proof):

Matrix Perturbation Formula. Let u be a column vector, v be a row vector with $\sum_y v(y)=0$, and define the perturbed matrix

$$\bar{p}(x, y) = p(x, y) + u(x)\, v(y)$$

Let $L^0 = \{f : \sum_x f(x) = 0\}$. Define the row vector α by $\alpha(I-p) = v$ and the number γ by $\pi u/(1-\alpha u)$. The stationary distribution for $\bar{p}$ is

$$\bar{\pi}(x) = \pi(x) + \gamma\alpha(x)$$

To recover a version of our perturbation formula we can take $u(x)=1$ if $x=w$ and 0 otherwise, and define $v(y) = \bar{p}(w, y) - p(w, y)$. The two formulas are equal since they compute the same thing but they are not identical. The new formula is in terms of the Green's function $(I-p)^{-1}$ while the former is in terms of the hitting probabilities. In our main example, random walk on the torus, one can use Fourier analysis to compute $(I-p)^{-1}$ explicitly. Having already solved our problems once with our perturbation formula we leave it to others to use the matrix perturbation formula given here to prove our results.

2. ONE DIMENSIONAL RESULTS

Let X_n be an irreducible birth and death chain on $\{0, 1, 2, \ldots, r\}$. Imitating the concept of the natural scale of a diffusion, we pick an increasing

ϕ with $\phi(0)=0$ that makes $\phi(X_n)$ a martingale, and call ϕ the *natural scale*. If the birth rates are p_j and the death rates are q_j we can write ϕ explicity (see Durrett,[3] Example 3.4, p. 395) as

$$\phi(n)=\sum_{m=0}^{n-1}\prod_{j=1}^{m}\frac{q_j}{p_j}$$

Let $w=0$, and set $\bar{p}(w,z)=1$. Standard martingale arguments tell us that for $z>0$

$$P_z(V_x<T_0)=\begin{cases}1 & x\in[1,z]\\ \phi(z)/\phi(x) & x\in[z,L]\end{cases}$$

while for $z=0$ we have

$$P_0(V_x<T_0)=\frac{p(0,1)\,\phi(1)}{\phi(x)}$$

Plugging these results into (★) we see that for $x>0$

$$\frac{\bar{\mu}_0(x)}{\mu_0(x)}=\frac{1}{p(0,1)\,\phi(1)}\cdot\phi(x\wedge z) \tag{2.1}$$

From this we can immediately get results for Example 1. In that case we have $\phi(x)=x$ and $\mu_0(x)=1$ for all x so

$$\bar{\mu}_0(x)=\begin{cases}1 & x=0\\ 2(x\wedge z) & x>0\end{cases} \tag{2.2}$$

Example 4 (Bessel random walks on $[0,r]$). Let $\gamma\geqslant 0$ and define

$$p(m,m+1)=\frac{1}{2}\left(\frac{m+3\gamma}{m+2\gamma}\right)\qquad 0\leqslant m<r$$

$$p(m,m-1)=\frac{1}{2}\left(\frac{m+\gamma}{m+2\gamma}\right)\qquad 0<m\leqslant r$$

with $p(0,0)=1-p(0,1)$ and $p(r,r)=1-p(r,r-1)$.

To explain the name of this chain note that if $r=\infty$ then computing infinitesimal means and variances and using a theorem of Stroock and

Varadhan (see e.g., Durrett[4] Section 8.6) we have $X_{[n^2 t]}/n$ converges weakly to Z_t where Z_t has infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} - \frac{\gamma}{x}\frac{d}{dx}$$

Comparing with the generator of the radial part of d-dimensional Brownian motion shows that $\gamma = (d-1)/2$.

Returning to the Markov chain itself, we will introduce a perturbation with $w = 0$ and $\bar{p}(0, z) = 1$ so our first task is to compute $\mu_0(x)$. By definition $\mu_0(0) = 1$. The detailed balance condition implies $\mu_0(x) = \mu_0(x-1)\, p(x-1, x)/p(x, x-1)$ so we have

$$\mu_0(x) = \prod_{m=1}^{x} \frac{p(m-1, m)}{p(m, m-1)} = \prod_{m=1}^{x} \frac{m-1+3\gamma}{m-1+2\gamma} \cdot \frac{m+2\gamma}{m+\gamma}$$

The mth term in the product is $1 + 2\gamma/m + O(1/m^2)$, so

$$\mu_0(x) \sim C x^{2\gamma} \qquad \text{as} \quad x \to \infty \tag{2.3}$$

To check this recall the connection with the radial part of d-dimensional Brownian motion and note that $2\gamma = d-1$.

To compute the natural scale ϕ we begin with the definition

$$\phi(k) = p(k, k+1)\, \phi(k+1) + p(k, k-1)\, \phi(k-1)$$

and rearrange to get

$$\phi(k+1) - \phi(k) = \frac{p(k, k-1)}{p(k, k+1)} [\phi(k) - \phi(k-1)]$$

Taking $\phi(0) = 0$, $\phi(1) = 1$, and iterating we have for $m \geqslant 1$

$$\phi(m+1) - \phi(m) = \prod_{k=1}^{m} \frac{p(k, k-1)}{p(k, k+1)} = \prod_{k=1}^{m} \frac{k+\gamma}{k+3\gamma}$$

The kth term in the product is $1 - 2\gamma/k + O(1/k^2)$ so

$$\phi(m+1) - \phi(m) \sim C m^{-2\gamma} \tag{2.4}$$

From this we see that

$$\begin{aligned} &\text{if} \quad \gamma > 1/2 \ \text{(i.e., } d > 2\text{)} && \text{then} \quad \phi(m) \to \phi(\infty) \\ &\text{if} \quad \gamma = 1/2 \ \text{(i.e., } d = 2\text{)} && \text{then} \quad \phi(m) \sim C \log m \\ &\text{if} \quad \gamma < 1/2 \ \text{(i.e., } d < 2\text{)} && \text{then} \quad \phi(m) \sim C m^{1-2\gamma} \end{aligned} \tag{2.5}$$

where the C's here are not the same as those in (2.4) and will change from line to line in what follows. Plugging into (2.2) now we see that for $x > 0$

$$\frac{\bar{\mu}_0(x)}{\mu_0(x)} = C\phi(x \wedge z) \tag{2.6}$$

Consider now a limit in which $r \to \infty$ and $z/r \to b \in (0, 1]$. If $d > 2$ then (2.5) and (2.6) imply that the effect of the perturbation is confined to points close to 0. In $d = 2$ we have (for large x)

$$\frac{\bar{\mu}_0(x)}{\mu_0(x)} \approx C \log(x \wedge z) \tag{2.7}$$

(2.3) tells us that $\mu_0(x) \sim Cx$ so dividing by $r \log r$ and rearranging we have (again for large x) that

$$\frac{\bar{\mu}_0(x)}{r \log r} \approx \frac{C \log(x \wedge z)}{\log r} \cdot \frac{x}{r} \tag{2.8}$$

In $d \leqslant 2$ we have (for large x)

$$\frac{\bar{\mu}_0(x)}{\mu_0(x)} \approx C(x \wedge z)^{1-2\gamma} \tag{2.9}$$

(2.3) tells us that $\mu_0(x) \sim Cx^{1-2\gamma}$ so dividing by r and rearranging we have (again for large x) that

$$\frac{\bar{\mu}_0(x)}{r} \approx \frac{C(x \wedge z)^{1-2\gamma} \cdot x^{2\gamma}}{r} \tag{2.10}$$

Note that when $d = 1$ and hence $\gamma = 0$ this reduces to our previous formula.

3. RANDOM WALKS ON TORI

Let $\Lambda(L) = (\mathbf{Z} \bmod L)^d$ be the d-dimensional torus. In this section and the next, all differences $x - y$ of points in $\Lambda(L)$ are computed modulo

L with the representatives of the equivalence classes chosen from $-L/2+1, \ldots, L/2$. For example, when we define the transition probability for simple random walk on $\Lambda(L)$ we set

$$p(x, y) = 1/(2d+1) \qquad \text{if} \quad |x-y| \leqslant 1$$

Here our choice of $x-y \bmod L$ is needed so that $|x-y|$ is the usual distance between points on $\Lambda(L)$. We have chosen $p(x, x) > 0$ to make our discrete time walk aperiodic.

We will modify random walk on the torus by setting $p(w, z) = 1$. The points w and z chosen depend on L even though this will not be recorded in the notation. To compute the asymptotic behavior of the quantities that appear in the perturbation formula, we need several preliminary results about random walk. Here and in what follows plain P_x's refer to random walk on $\mathbf{Z}^d$ while a superscript L indicates we are considering random walk on $\Lambda(L)$. For Lemmas 3.1–3.4 we will consider a general d. After that we will specialize to $d > 2$.

Lemma 3.1. Let $\|y\|_\infty = \max_i |y_i|$. If $p > 1$ there is a constant $0 < C_{p,d} < \infty$ which only depends on p and the dimension d so that

$$P_x(\max_{m \leqslant n} \|X_m - x\|_\infty \geqslant r) \leqslant C_{p,d}(r/\sqrt{n})^{-p}$$

Proof. The L^p maximal inequality for discrete time martingales, M_n says

$$E(\max_{m \leqslant n} |M_m|^p) \leqslant A_p E\, |M_n|^p$$

Applying this to $M_n = X^i_n - x^i$ where the X^i_n are the components of the random walk, noting that in this case $E\,|X^i_n - x^i|^p \leqslant B_p n^{p/2}$, and then using Chebyshev's inequality,

$$P_x(\max_{m \leqslant n} |X^i_m - x^i| \geqslant r) \leqslant r^{-p} E(\max_{m \leqslant n} |X^i_m - x^i|^p)$$
$$\leqslant r^{-p} \cdot A_p B_p n^{p/2} \qquad \square$$

Lemma 3.2. There is a constant $0 < C < \infty$ so that for $t \geqslant 1$

$$\sup_y p_t(x, y) \leqslant C/t^{d/2}$$

Proof. This follows from standard estimates. $\square$

Lemma 3.3. There are constants $0 < \eta < 1$, $0 < C < \infty$ (which depend on the dimension but not on L) so that for any integer k

$$\sup_{x,\, y} |L^d P_x^L(X_{kL^2} = y) - 1| \leqslant C(1-\eta)^k$$

Proof. The result for $k = 1$ follows from the central limit theorem. Iteration and the stationarity of the uniform distribution then does the rest. Of course Fourier analysis on the torus also gives the result. □

Lemma 3.4. If π is the uniform probability distribution on $\Lambda(L)$ then for $x \neq y$

$$P_\pi^L(V_x > V_y) = 1/2$$

Proof. By translation invariance and reflection symmetry

$$\begin{aligned} P_\pi^L(V_x > V_y) &= P_\pi^L(V_{x-y} > V_0) \\ &= P_\pi^L(V_{y-x} > V_0) = P_\pi^L(V_y > V_x) \end{aligned}$$

Since $P_\pi^L(V_x = V_y) = 0$ the desired result follows. □

Numerator. In $d > 2$ if $|z - w| \to \infty$ then

$$\sup_{x \in \Lambda(L) - \{w\}} |P_z^L(V_x > T_w) - \tfrac{1}{2} P_{z-x}(V_0 = \infty)| \to 0$$

Proof. We begin by decomposing

$$P_z^L(V_x > T_w) = P_z^L(V_x > T_w, T_w \leqslant \sqrt{L}) + P_z^L(V_x > T_w > \sqrt{L}) \tag{3.1}$$

and writing the second terms as

$$P_z^L(V_x > T_w \mid V_x, T_w > \sqrt{L}) \cdot P_z^L(V_x, T_w > \sqrt{L}) \tag{3.2}$$

A simple comparison and translation imply

$$\begin{aligned} |P_z^L(V_y > \sqrt{L}) - P_{z-y}(V_0 > \sqrt{L})| &\leqslant P_0(\max_{m \leqslant \sqrt{L}} \|X_m\|_\infty \geqslant L/2) \\ &\leqslant C_2(\sqrt{L}/2)^{-2} \to 0 \end{aligned} \tag{3.3}$$

by Lemma 3.1 with $p=2$, $r=L/2$, and $n=\sqrt{L}$. To replace $\sqrt{L}$ by ∞ in the second probability we note that Lemma 3.2 implies

$$\sup_y P_z(\sqrt{L} \leqslant V_y < \infty) \leqslant \sum_{s=\sqrt{L}}^{\infty} \frac{c}{s^{d/2}} \to 0 \tag{3.4}$$

Our assumption that $|z-w| \to \infty$ implies $P_{z-w}(V_0 = \infty) \to 1$, so it follows from (3.3) that

$$P_z^L(T_w > \sqrt{L}) \to 1 \tag{3.5}$$

This shows that the first term in (3.1) tends to 0.

To start to work on (3.2) we notice that combining (3.3)–(3.5) gives

$$\sup_x |P_z^L(V_x, T_w > \sqrt{L}) - P_{z-x}(V_0 = \infty)| \to 0 \tag{3.6}$$

Let $U_y = \inf\{n \geqslant L^2 \log L : X_n = y\}$. Using Lemmas 3.3 and 3.4 now, we have

$$\sup_x |P_z^L(U_x > U_w \mid V_x, T_w > \sqrt{L}) - \tfrac{1}{2}| \to 0 \tag{3.7}$$

as $L \to \infty$. To estimate the probability of hitting x or w at some time in between $\sqrt{L}$ and $L^2 \log L$ we note that using the local central limit theorem for the random walk on $\mathbf{Z}^d$

$$\begin{aligned} &P_z^L(X_m = y \text{ for some } m \in [L^{1/2}, L^2 \log L]) \\ &\qquad \leqslant P_z(\max_{m \leqslant L^2 \log L} |X_m - z| > L \log L) + (2 \log L + 1)^d \sum_{s=\sqrt{L}}^{L^2 \log L} c s^{-d/2} \end{aligned} \tag{3.8}$$

To check this note that when $\max_{m \leqslant L^2 \log L} |X_m - z| \leqslant L \log L$ there are at most $(2 \log L + 1)^d$ points in $\mathbf{Z}^d$ which are possible values at some time m and that map to y on the torus.

The first term on the right in (3.8) tends to 0 by Lemma 3.1, while simple calculation shows that the second tends to 0 when $d > 2$. Random walk in $d > 2$ has

$$\inf_{x \neq z} P_{z-x}(V_0 = \infty) > 0 \tag{3.9}$$

So (3.3) and (3.5) imply that $P_z^L(V_x, V_w > \sqrt{L})$ is bounded away from 0, and it follows that

$$\sup_{x \neq z} |P_z^L(U_x > U_w \mid V_x, T_w > \sqrt{L}) - P_z^L(V_x > V_w \mid V_x, V_w > \sqrt{L})| \to 0 \quad (3.10)$$

Combining (3.7) and (3.10) with (3.6), gives the asymptotic behavior of (3.2) for $x \neq z$. The case $x = z$ is trivial since $P_z^L(V_x > T_w) = 0$, and the proof of the asymptotics for the numerator is complete. □

Denominator. In $d > 2$ if $|z - w| \to \infty$ then

$$\sup_{x \in \Lambda(L) - \{w\}} |P_w^L(V_x < T_w) - \{P_0(V_{x-w} < T_0) + \tfrac{1}{2} P_0(V_{x-w}, T_0 = \infty\}| \to 0$$

Proof. We begin by decomposing

$$P_w^L(V_x < T_w) = P_w^L(V_x < T_w, V_x \leqslant \sqrt{L}) + P_w^L(T_w > V_x > \sqrt{L}) \quad (3.11)$$

and writing the second term as

$$P_w^L(T_w > V_x \mid V_x, T_w > \sqrt{L}) \cdot P_w^L(V_x, T_w > \sqrt{L}) \quad (3.12)$$

Reasoning as in (3.3), we have

$$\begin{aligned} &|P_w^L(V_x < T_w, V_x \leqslant \sqrt{L}) - P_0(V_{x-w} < T_0, V_{x-w} \leqslant \sqrt{L})| \\ &\qquad \leqslant P_0(\max_{m \leqslant \sqrt{L}} \|X_m - z\|_\infty \geqslant L/2) \leqslant C_2(\sqrt{L}/2)^{-2} \to 0 \quad (3.13) \end{aligned}$$

So using (3.4) we have

$$\sup_{x \neq w} |P_w^L(V_x < T_w, V_x \leqslant \sqrt{L}) - P_0(V_{x-w} < T_0)| \to 0 \quad (3.14)$$

A similar argument shows

$$\sup_{x \neq w} |P_w^L(V_x, T_w > \sqrt{L}) - P_0(V_{x-w}, T_0 = \infty)| \to 0 \quad (3.15)$$

It follows from (3.7) and (3.10) that

$$\sup_{x \neq w} |P_w^L(V_x > T_w \mid V_x, T_w > \sqrt{L}) - \tfrac{1}{2}| \to 0 \quad (3.16)$$

as $L \to \infty$. Combining (3.14)–(3.16) with (3.11) and (3.12) gives the desired result. □

4. RANDOM WALK ON TORI, II. $d=2$

We begin with a pair of formulas that are valid for any Markov chain with countable state space $\mathscr{X}$. Define the Green's function by

$$G_t(x, t) = \sum_{s=0}^{t} p_s(x, y)$$

Lemma 4.1. If $a < b$ and $c > 0$

$$\frac{G_b(x, y) - G_a(x, y)}{G_{a-b}(y, y)} \leqslant P_x\{X_n = y \text{ for some } n \in (a, b]\}$$

$$\leqslant \frac{G_{b+c}(x, y) - G_a(x, y)}{G_c(y, y)}$$

Proof. Breaking things down according to $\bar{T}_y = \min\{n > a : X_n = y\}$,

$$G_b(x, y) - G_a(x, y) = \sum_{s=a+1}^{b} P_x(\bar{T}_y = s)\, G_{b-s}(y, y)$$

$$\leqslant G_{b-a}(y, y) \sum_{s=a+1}^{b} P_x(\bar{T}_y = s)$$

To prove the other inequality, we note that

$$G_{b+c}(x, y) - G_a(x, y) \geqslant \sum_{s=a+1}^{b} P_x(\bar{T}_y = s)\, G_{b+c-s}(y, y)$$

$$\geqslant G_c(y, y) \sum_{s=a+1}^{b} P_x(\bar{T}_y = s) \qquad \square$$

From Lemma 4.1, we get the following useful result about hitting times for simple random walk on $\mathbf{Z}^2$. Here $\log^+ x = \max\{\log x, 0\}$.

Lemma 4.2.

$$\lim_{t \to \infty} \sup_x \left| P_x(V_0 > t) - \left\{ \frac{\log^+ |x|^2}{\log t} \wedge 1 \right\} \right| \to 0$$

Proof. It is enough to show that for each $\eta > 0$ there is a finite set $F(\eta)$ so that

$$\limsup_{t \to \infty} \sup_{x \notin F(\eta)} \left| P_x(V_0 \leqslant t) - \left\{ 1 - \frac{\log^+ |x|^2}{\log t} \right\}^+ \right| \leqslant \eta \tag{4.1}$$

The local central limit theorem implies that for each fixed x, $p_t(x, 0) \sim c_0/t$ as $t \to \infty$ so

$$G_t(x, 0) \sim c_0 \log t \tag{4.2}$$

To estimate the numerator on the left-hand side in Lemma 4.1, we note that the local central limit theorem implies that given $\delta > 0$ we can pick K and s_0 so that for $s \geqslant s_0$

$$\inf\{p_s(x, 0) : |x|^2 \leqslant s/K\} \geqslant (c_0 - \delta)/s$$

Summing we see that if $s_0 \leqslant K\,|x|^2 \leqslant t$ then

$$G_t(x, 0) \geqslant \sum_{s = K|x|^2}^{t} \frac{c_0 - \delta}{s}$$

To replace the sum by an integral we note that if $a < b$ are positive integers

$$\sum_{s=a+1}^{b} \frac{1}{s} \leqslant \int_a^b \frac{ds}{s} \leqslant \sum_{s=a}^{b-1} \frac{1}{s} \tag{4.3}$$

Combining the last two inequalities we have

$$G_t(x, 0) \geqslant (c_0 - \delta)\{\log t - \log(K\,|x|^2)\}$$

when $s_0 \leqslant K\,|x|^2 \leqslant t$. Using (4.2) now and the fact that $G_t(x, 0) \geqslant 0$ it follows that if $|x|^2 \geqslant s_0/K$ and $t \geqslant t_0$ then

$$\frac{G_t(x, 0)}{G_t(0, 0)} \geqslant \frac{c_0 - \delta}{c_0 + \delta} \left\{ 1 - \frac{\log(K\,|x|^2)}{\log t} \right\}^+ \tag{4.4}$$

If δ is chosen small enough, (4.4) gives half of (4.1). To prove a result in the other direction, we use the local central limit theorem to conclude that for $t \geqslant t_0$ (here and in what follows, t_0 is a "sufficient large time" that will change from line to line).

$$p_t(x, y) \leqslant (c_0 + \delta)/t \tag{4.5}$$

To upper bound the numerator on the right-hand side of Lemma 4.1, we note that for $t \geqslant t_0$

$$G_{2t}(x, 0) \leqslant \|x\|_\infty^{2(1-\epsilon)} P_x\{T_0 \leqslant \|x\|_\infty^{2(1-\epsilon)}\} + \sum_{s=\|x\|_\infty^{2(1-\epsilon)}+1}^{t} \frac{(c_0+\delta)}{s} \tag{4.6}$$

where the sum is 0 if $t \leqslant \|x\|_\infty^{2(1-\epsilon)}$.

Taking $p = 2(1-\epsilon)/\epsilon$ in Lemma 3.1 which will be > 1 if ϵ is small, then setting $r = \|x\|_\infty$ and $n = r^{2(1-\epsilon)}$ (so $r/\sqrt{n} = r^\epsilon$) we have

$$P_x(\max_{m \leqslant n} \|X_m - x\|_\infty \geqslant r) \leqslant C(r^\epsilon)^{-2(1-\epsilon)/\epsilon} = C\, \|x\|_\infty^{-2(1-\epsilon)}$$

Rearranging it follows that

$$\|x\|_\infty^{2(1-\epsilon)} P_x\{T_0 \leqslant \|x\|_\infty^{2(1-\epsilon)}\} \leqslant C$$

Using this and (4.2) in (4.6) we have that for $t \geqslant t_0$

$$\frac{G_{2t}(x, 0)}{G_t(0, 0)} \leqslant \frac{C + (c_0+\delta)\{\log t - \log(|x|^{2(1-\epsilon)})\}^+}{(c_0 - \delta)\log t} \tag{4.7}$$

The other half of (4.1) follows easily and the proof of Lemma 4.2 is complete. □

From Lemma 4.2 we immediately get a result on the torus for times that are $o(L^2)$.

Lemma 4.3. If $t \to \infty$ and $t/L^2 \to 0$

$$\sup_{x, y \in \Lambda(L)} \left| P_x^L(V_y > t) - \left\{ \frac{\log^+ |x-y|^2}{\log t} \wedge 1 \right\} \right| \to 0$$

Proof. An easy comparison shows

$$P_{x-y}(V_0 > t) \geqslant P_x^L(V_y > t) \geqslant P_{x-y}(V_0 > t) - P_0(\|X_s\|_\infty \geqslant L/2 \text{ for some } s \leqslant t)$$

Lemma 3.1 implies that if $t/L^2 \to 0$ then the last probability tends to 0, and the desired result follows. □

Numerator. If $\liminf |z-w|/L > 0$ then

$$\sup_{x \in \Lambda(L)-\{w\}} \left| P_z^L(V_x > T_w) - \frac{\log^+ |z-x|}{2 \log L} \right| \to 0$$

Proof. Intuitively, our idea is show that

$$P_z^L(V_x > T_w) \approx \tfrac{1}{2} P_z(V_x > L^2) \tag{4.8}$$

in three steps:

(i) Since $\liminf |z-w|/L > 0$, the random walk will with high probability require at least $L^2/\log L$ steps to get from z to w.

(ii) Hitting w or x between time $L^2/\log L$ and $L^2 \sqrt{\log L}$ is unlikely.

(iii) Lemma 3.3 implies that at time $L^2 \sqrt{\log L}$ the distribution of the random walk is almost π. The desired result then follows from Lemma 3.4.

To begin to carry out the plan announced above we note that

$$\begin{aligned} P_z^L(V_x > T_w) = {} & P_z^L(T_w < V_x < L^2/\log L) \\ & + P_z^L(V_x > T_w, V_x \geqslant L^2/\log L) \end{aligned} \tag{4.8}$$

The first term is bounded above by

$$P_z^L(T_w < L^2/\log L) \to 0 \tag{4.9}$$

by Lemma 3.3, since $\liminf_{L \to \infty} |z-w|/L > 0$. The second term can be written as

$$P_z^L(V_x \geqslant L^2/\log L) \cdot P_z^L(V_x > T_w \mid V_x \geqslant L^2/\log L) \tag{4.10}$$

If $|z-x| \leqslant L^\delta$ with δ small then Lemma 4.3 and arithmetic show that both terms in the desired result are small, so we can suppose $|z-x| \geqslant L^\delta$ and it follows from Lemma 4.3 that the conditioning event has a probability bounded away from 0.

Using the Lemma 4.1 now with $a = L^2/\log L$, $b = c = L^2 \sqrt{\log L}$, it follows that

$$P_z^L\{X_s = y \text{ for some } s \in (a, b]\} \leqslant \frac{1}{G_b^L(0,0)} \cdot \sum_{s=a+1}^{2b} p_s^L(z, y) \tag{4.11}$$

To bound the denominator on the right, we note that (4.2) implies that for large L

$$G_b^L(0,0) \geqslant G_b(0,0) \geqslant (c_0/2)\log(L^2) \tag{4.12}$$

To bound the numerator on the right in (4.11) we note that the local central limit theorem result in (4.5) and the convergence to equilibrium result in Lemma 3.3 imply that there is a constant $C < \infty$ so that for all L and $s \geqslant 1$

$$p_s^L(z,y) \leqslant C/(s \wedge L^2) \tag{4.13}$$

and hence using (4.3)

$$\sum_{s=a+1}^{2b} p_s^L(z,y) \leqslant C(\sqrt{\log L} + \log\log L) \leqslant 2C\sqrt{\log L} \tag{4.14}$$

Combining (4.11)–(4.14) we have

$$P_z^L\{X_s = x \text{ or } w \text{ for some } s \in (L^2/\log L, L^2\sqrt{\log L}\,]\} \leqslant \frac{2C\sqrt{\log L}}{c_0 \log L} \to 0 \tag{4.15}$$

For the third and final step in the proof let

$$U_y = \inf\{n \geqslant L^2(\log L)^{1/2} : X_n = y\}$$

and note that (4.15) implies

$$\sup_{x:\,|z-x| \geqslant L^\delta} |P_z^L(V_x > T_w \mid V_x > L^2/\log L) - P_z^L(U_x > U_w \mid V_x > L^2/\log L)| \to 0 \tag{4.16}$$

Using Lemmas 3.3 and 3.4 now we have

$$\sup_x |P_z^L(U_x > U_w \mid V_x > L^2/\log L) - \tfrac{1}{2}| \to 0 \tag{4.17}$$

Combining (4.8)–(4.10) with the last two results we have

$$\sup_{x:\,|z-x| \geqslant L^\delta} |P_z^L(V_x > T_w) - \tfrac{1}{2}P_z^L(V_x \geqslant L^2/\log L)| \to 0$$

Combining this with Lemma 4.3 and our earlier analysis of $|z-x| \leqslant L^\delta$ completes the proof of the asymptotics for the numerator. □

Denominator. There is a constant c_1 so that if $\liminf |z-w|/L>0$ then

$$\sup_{x \in \Lambda(L)-\{w\}} \left| \frac{c_1}{2 \log L} \cdot \frac{1}{P^L_w(V_x < T_w)} - \frac{\log^+ |x-w|}{\log L} \right| \to 0$$

Proof. Let $K = \|x-w\|_\infty/2$ where $\|y\|_\infty$ is the L^∞ metric on the torus. Let $B(w, K) = \{y: \|y-w\|_\infty < K\}$ be the square of radius K centered at w, and let $\tau_K = \inf\{t: X_t \notin B(w, K)\}$. This time the heuristic answer is

$$P^L_w(V_x < T_w) \approx \tfrac{1}{2} P_w(T_w > \tau_K)$$

based on the reasoning that:

(i) $T_w > \tau_K$ is a necessary condition for $V_x < T_w$,

(ii) If we let $\partial B(w, K)$ be the boundary of $B(w, K)$, i.e., the set of possible values for X_τ, then for any $\delta > 0$ we can pick K_0 so that for $K_0 \leqslant K \leqslant L/4$ (since $\|x-w\|_\infty \leqslant L/2$)

$$\sup_{y \in \partial B(w, K)} |P^L_y(T_x < T_w) - \tfrac{1}{2}| \leqslant 2\delta \tag{4.18}$$

Once (4.18) is established, the desired result will follow from the well known asymptotic

$$P_w(T_w > \tau_K) \sim c_1/\log K \qquad \text{as} \quad K \to \infty \tag{4.19}$$

To begin to carry out the plan announced in the last paragraph, we recall (i) and note that

$$P^L_w(V_x < T_w) = P^L_w(\tau_K < T_w)\, P^L_w(V_x < T_w \mid \tau_K < T_w) \tag{4.20}$$

Let $b = AK^2$ and let $U_y = \inf\{n \geqslant b : X_n = y\}$. Our first goal is to show that if A is chosen large enough then for $K_0 \leqslant K \leqslant L/4$

$$\sup_{y \in \partial B(w, K)} |P^L_y(U_x < U_w) - \tfrac{1}{2}| \leqslant \delta \tag{4.21}$$

To do this, let $v = (w+x)/2$ be the midpoint of the segment between w and x, let $\hat{u} = 2v - u$ be the reflection of u through v. It is easy to see that $\hat{w} = x$ and $\hat{x} = w$. Using reflection through v to couple random walks starting at u and $\hat{u}$, it follows that

$$P^L_u(V_x < V_w) = P^L_{\hat{u}}(V_w < V_x)$$

and hence

$$P_u^L(V_x < V_w) - \tfrac{1}{2} = P_{\hat{u}}^L(V_w < V_x) - \tfrac{1}{2} = \tfrac{1}{2} - P_{\hat{u}}^L(V_x < V_w) \tag{4.22}$$

Since $\hat{v} = v$, $P_v(V_x < V_w) = 1/2$. Counting all of the other points in the plane twice, then using (4.22) we have

$$\begin{aligned} P_y^L(U_x < U_w) - 1/2 &= \tfrac{1}{2} \sum_{u \neq v} P_y^L(X_b = u)\{P_u^L(V_x < V_w) - 1/2\} \\ &\quad + P_y^L(X_b = \hat{u})\{P_{\hat{u}}^L(V_x < V_w) - 1/2\} \\ &= \tfrac{1}{2} \sum_{u \neq v} \{P_y^L(X_b = u) - P_y^L(X_b = \hat{u})\}\{P_u^L(V_x < V_w) - 1/2\} \end{aligned}$$

Since all probabilities lie between 0 and 1, it follows that

$$|P_y^L(U_x < U_w) - 1/2| \leqslant \tfrac{1}{4} \sum_{u \neq v} |P_y^L(X_b = u) - P_y^L(X_b = \hat{u})| \tag{4.23}$$

The right-hand side is awkward to estimate since the time $b = AK^2$ may or may not be large enough to have a significant chance of going around the torus more than once. To avoid this problem we note

$$\sum_{u \neq v} |P_y^L(X_b = u) - p_y^L(X_b = \hat{u})| \leqslant \sum_{u \neq v} |P_y(X_b = u) - P_y(X_b = \hat{u})| \tag{4.24}$$

Using the local central limit theorem now, it is straighforward though somewhat tedious to show that the right-hand side of (4.24) is $\leqslant \delta$ for all $K \geqslant K_0$. This establishes (4.21).

To pass from (4.21) to (4.18), we let $y \in \partial B$, $a = 0$, $b = c = AK^2$, and use Lemma 3.1 to conclude that

$$P_y^L(T_w \leqslant b) \leqslant \frac{G_{2b}^L(y, w)}{G_b^L(y, y)} \tag{4.25}$$

Reasoning as for (4.12), we conclude that there is a constant $c > 0$ so that

$$G_b^L(w, w) \geqslant \sum_{s=1}^{K^2} \frac{c}{s} \geqslant c \log(K^2) \tag{4.26}$$

where the second inequality follows (4.3). To estimate the numerator on the right in (4.25) we observe that using the reasoning for (4.6), and the result in (4.13)

$$G^L_{2b}(y, w) \leqslant K^{2(1-\epsilon)} P_y\{T_w \leqslant K^{2(1-\epsilon)}\} + \sum_{K^{2(1-\epsilon)}+1}^{K^2} \frac{C}{s} + \sum_{K^2}^{AK^2} \frac{C}{K^2}$$

From the proof of (4.7) we see that the first term on the right is bounded. The middle term can be summed using (4.3) and the third using arithmetic. Plugging the result of this calculation and (4.26) into (4.25) gives

$$P^L_y(T_w \leqslant b) \leqslant \frac{C + C\epsilon \log K + C \cdot A}{c \log K}$$

The same reasoning applies with z in place of w. Thus, if we pick ϵ small then for $K \geqslant K_0$

$$\sup_{y \in \partial B(w, K)} P^L_y\{T_w \wedge T_x \leqslant b\} \leqslant \delta \tag{4.27}$$

This and (4.21) gives (4.18) and the asymptotics for the denominator follow. □

APPENDIX

Here we prove the two perturbation formulas stated in the introduction.

Proof of the Perturbation Formula. If $x \neq w$ then by considering what happens on the first step

$$\bar{\mu}_w(x) = \sum_z \bar{p}(w, z)\, E_z\left(\sum_{j=0}^{T_w-1} 1_{(X_n = x)}\right)$$

Elementary Markov chain theory tells us that

$$E_a\left(\sum_{j=0}^{T_b-1} 1_{(X_n = c)}\right) = \frac{P_a(V_c < T_b)}{P_c(T_c > T_b)}$$

Letting $a = w$, $b = w$, $c = x$, we have

$$\mu_w(x) = E_w\left(\sum_{j=0}^{T_w-1} 1_{(X_n = x)}\right) = \frac{P_w(V_x < T_w)}{P_x(T_x > T_w)}$$

and letting $a=z$, $b=w$, $c=x$,

$$E_z\left(\sum_{j=0}^{T_w-1} 1_{(X_n=x)}\right)=\frac{P_z(V_x<T_w)}{P_x(T_x>T_w)}=\frac{P_z(V_x<T_w)}{P_w(V_x<T_w)}\cdot\mu_w(x)$$

from which the desired result follows easily. □

Proof of the Matrix Perturbation Formula. We want to show that $\bar{p}(x, y)=p(x, y)+u(x)\, v(y)$ has stationary distribution $\bar{\pi}(x)=\pi(x)+\gamma\alpha(x)$ where $\alpha(I-p)=v$ and $\gamma=\pi u/(1-\alpha u)$. To check this we expand out the condition for stationarity $\bar{\pi}\bar{p}=\bar{\pi}$ to get

$$\pi p+\gamma\alpha p+\pi uv+\gamma\alpha uv=\pi+\gamma\alpha$$

Using $\pi p=\pi$ this can be rearranged to get

$$\pi uv=\gamma\alpha(I-p-uv)$$

Using the definition of $\alpha(I-p)=v$ now this becomes

$$(\pi u)=\gamma v-(\gamma\alpha u)\, v$$

where we have introduced parentheses to remind ourselves that the two quantities are numbers. The definition of γ guarantees that this holds, so the proof is complete. □

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