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ON THE GROWTH OF ONE DIMENSIONAL CONTACT PROCESSES[1]

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In this paper we will study the number of particles alive at time t in a one dimensional contact process ξ_t^0 which starts with one particle at 0 at time 0. In the case of a nearest neighbor interaction we will show that if $|\xi_t^0|$ is the number of particles and r_t, l_t are the positions of the rightmost and leftmost particles (with $r_t = l_t = 0$ if $|\xi_t^0| = 0$) then there are constants γ, α, and β so that $|\xi_t^0|/t$, r_t/t, and l_t/t converge in L^1 to $\gamma 1_\Lambda$, $\alpha 1_\Lambda$ and $\beta 1_\Lambda$ where $\Lambda = \{|\xi_t^0| > 0$ for all $t\}$. The constant $\gamma = \rho(\alpha - \beta)^+$ where ρ is the density of the "upper invariant measure" ξ_∞^Z.

1. Introduction. In (1974) Harris introduced a class of Markov processes with state space $\{0, 1\}^Z$ which he called contact processes. If we interpret the 1's as occupied sites and the 0's as vacant sites then the evolution of the system may be informally described as follows: (i) 1's die (i.e., become 0's) at rate one (independent of the configuration) while (ii) there is a finite set $N \subset Z$ so that births occur at an unoccupied state x at a rate which depends upon the number of neighbors $y \in x + N$ which are occupied with (iii) the birth rate is 0 if none of the neighbors are occupied. The last assumption explains why these processes are called contact processes—1's can only spread to adjacent sites where adjacency is defined by the neighborhood set N.

Let ξ_t be the state of the contact process at time t and let $|\xi_t|$ be the number of ones in the configuration. Assumption (iii) implies that the state $\xi \equiv 0$ is absorbing so if we start from a state with $|\xi_0| < \infty$ there is a positive probability that the system will be absorbed in the all 0's state. It is easy to show that if the birth rates are too small (e.g., all the birth rates are less than $1/|N|$) then this probability is 1 whenever $|\xi_0| < \infty$. It is much more difficult to show that the other alternative can occur. Harris (1974) has shown that if $\lambda_i \not\equiv i = 1, \cdots, |N|$ are the birth rates when the number of occupied neighbors is i and the λ_i are sufficiently large then starting from any $\xi \not\equiv 0$ there is a positive probability of not being absorbed in the all zeros state.

It is easy to show that if $|\xi_t| \nrightarrow 0$ then $|\xi_t| \to \infty$. This suggests the problem of determining the rate at which $|\xi_t|$ grows. Harris (1978, Theorem 13.5) has shown that

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THEOREM 1.1. *Suppose* $N \supset \{-1, 1\}$. *If* $\lambda_1, \cdots, \lambda_{|N|}$ *are sufficiently large then starting from any* ξ_0 *with* $|\xi_0| > 0$

$$P\left(\inf_{t>0} \frac{|\xi_t|}{t} > 0 \middle| |\xi_t| \nrightarrow 0\right) = 1.$$

It is easy to show, by comparing with a system with no deaths, that $|\xi_t|$ can grow at most linearly so on the basis of Harris' result it seems natural to conjecture that if $|\xi_0| = 1$ then as $t \to \infty |\xi_t|/t$ will converge a.s. to a limit W. In this paper we will obtain a partial solution of this problem. We will show

THEOREM 1.2. *If* ξ_t *is a contact process with* $\lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_{|N|}$ *and* $N \subset \{-1, 1\}$ *then there is a constant* γ *so that if* $|\xi_0| = 1$ *and* $= \{|\xi_t^0| > 0$ *for all* $t\}$ *then*

$$|\xi_t|/t \to \gamma 1_\lambda \quad \text{in } L^1.$$

The assumption that $\lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_{|N|}$ is natural and has been used by many authors. It says that the birth rate is an increasing function of the number of occupied neighbors and it implies that if we have initial configurations which have $\xi_0(x) \leqslant \xi_0'(x)$ for all x then we can construct a joint realization of the contact process starting from these initial configurations which has the property that $\xi_t(x) \leqslant \xi_t'(x)$ for all x, t. (This is called the basic coupling, see Liggett (1976), Section 2.2 for details.)

The assumption that $N \subset \{-1, 1\}$ is a technicality which is required by our proof. At several points in Section 3 we use a coupling result (Lemma 3.1) which is only valid in the nearest neighbor case. To explain where and how this difficulty occurs we have to describe our method of proof.

To prove Theorem 1.2 we begin by studying $\bar{r}_t$ the position of the rightmost particle when the initial configuration is $1_{(-\infty,0]}$. The processes $\{\bar{r}_t, t \geqslant 0\}$ while they do not have stationary increments are in many ways similar to the subadditive processes of Hammersley and Kingman. In Section 2 using some ideas from Kingman's proof of the subadditive ergodic theorem and several applications of the basic coupling we show that

THEOREM 1.3. *There is a constant* α *so that* $\bar{r}_t/t \to \alpha$ a.s. *If* $\alpha > -\infty$ *the convergence also occurs in* L^1.

(The disclaimer in the second line is necessary. D. Griffeath has shown that $\alpha = -\infty$ for $\lambda < \lambda_{cr}$. For this and other related results the reader is referred to [3].)

The last result of Theorem 1.3 does not require the assumption that $N \subset \{-1, 1\}$. In fact it is valid for any system which is attractive (i.e., joint realizations can be constructed using the basic coupling), has $\xi \equiv 0$ as an absorbing state, and is described by a set of flip rates which are translation invariant and have finite range (hereafter these systems will be called growth models).

Let ξ_t^R be a realization of the growth model with $\xi_t^R = 1_{(-\infty,0]}$ and let $\bar{r}_t = \sup\{y : \xi_t^R(y) = 1\}$. If we use the basic coupling to construct ξ_t^0 and ξ_t^R on the same space then $\xi_t^R \geqslant \xi_t^0$ so $r_t \leqslant \bar{r}_t$ and we can use Theorem 1.3 to get upper bounds on the rate of growth of r_t. We conjecture that these upper bounds give the exact rate of growth in all cases. We have only been able to prove this conjecture in the nearest neighbor case. In this case if we construct ξ_t^R and ξ_t^0 on the same space using the basic coupling then it follows (from Lemma 3.1) that $\xi_t^0 = \xi_t^R$ for all $x \geqslant l_t = \inf\{y : \xi_t^0(y) = 1\}$ so $r_t = \bar{r}_t$ on $\Omega_t = \{|\xi_t^0| > 0\}$. Using this observation Theorem 1.3 can be translated into the following result about r_t.

THEOREM 1.4. *If ξ_t^0 is a nearest neighbor growth model then there is a constant α so that $r_t/t \to \alpha$ a.s. on $\Omega_\infty = \{|\xi_t^0| > 0$ for all $t\}$ and*

$$(r_t/t)\, 1_{\Omega_t} \to \alpha 1_{\Omega_\infty} \text{ in } L^1.$$

Applying Theorem 1.4 to $\tilde{\xi}_t(x) = \xi_t(-x)$ gives that there is a constant β so that $(l_t/t)\, 1_{\Omega_t} \to \beta 1_{\Omega_\infty}$ a.s. and in L^1. By using this result and Theorem 1.4 it is easy to prove Theorem 1.2. To do this we observe that if we let ξ_t^Z be a realization of the contact process with initial configuration $\xi_0^Z \equiv 1$ and we construct (ξ_t^Z and ξ_t^0 on the same space using the basic coupling then) $\xi_t^Z(x) = \xi_t^0(x)$ for $l_t \leqslant x \leqslant r_t$. Now it is a known fact about attractive processes that as $t \to \infty, \xi_t^Z$ converges weakly to a limit ξ_∞^Z with $P(\xi_\infty^Z(0) = 1) = \rho$ and we will show that $\{\xi_\infty^Z(x), x \in Z\}$ is an ergodic stationary sequence so it follows from the last three results that Theorem 1.2 holds with $\gamma = \rho(\alpha - \beta)^+$. The reader should note that while we know r_t/t and l_t/t converge a.s. we have not been able to show that $|\xi_t^0|/t$ converges a.s.

In order to claim that Theorem 1.2 gives the exact rate of growth for contact processes we need to show that $\gamma > 0$. It is easy to show that if $\alpha - \beta < 0$ then $P(\Omega_\infty) = 0$ but it is difficult to rule out the possibility that $\alpha - \beta = 0$ or $\rho = 0$ when $P(\Omega_\infty) > 0$. In Section 4 we solve this problem for the one-sided ($N = \{-1\}, \lambda_1 = \lambda$) and basic two-sided ($N = \{-1, 1\}, \lambda_1 = \lambda, \lambda_2 = 2\lambda$) contact processes and we will show that in each case $\gamma > 0$ if

$$\lambda > \lambda_{cr} = \sup\{\lambda : P\{|\xi_t^0| \to 0\} = 1\}.$$

In the two-sided case this result can be applied to prove the following convergence theorem (which is essentially due to Griffeath (1978)).

THEOREM 1.5. *If $\lambda > \lambda_{cr}$ and ξ_0 is an arbitrary initial distribution then as $t \to \infty$ ξ_t converges weakly to*

$$P(|\xi_t| \to 0)\, \nu_0 + P(|\xi_t| \nrightarrow 0)\, \nu_1$$

where ν_0 is the pointmass on $\xi \equiv 0$ and ν_1 is the limit distribution when the initial configuration is $\xi_0^Z \equiv 1$.

At this point the result is an easy consequence of the coupling of ξ_t^Z and ξ_t^0 and the limit theorems for l_t and r_t. The details are given in Section 5, following the outline of Griffeath (1979) with a slightly different ending. Our contribution has been to show that $l_t \to -\infty$ and $r_t \to \infty$ almost surely on Ω_∞ for all $\lambda > \lambda_{cr}$ (he was only able to show this when $\lambda \geqslant 4 > \lambda_{cr}$).

2. Limit theorems for $\bar{r}_t/t$. Suppose ξ_t is a growth model with a finite range interaction. Let ξ_t^R be a realization of the process with initial configuration $\xi_0^R = 1_{(-\infty,0]}$ and let $\bar{r}_t = \max\{y : \xi_t^R(y) = 1\}$. In this section we will show

THEOREM 2.1. *There is a constant α so that $\bar{r}_t/t \to \alpha$ almost surely. If $\alpha > -\infty$ then the convergence also occurs in L^1.*

PROOF. To analyze the growth of $\bar{r}_t$ we introduce a family of "reset" approximations $\xi_t^{R,M}$, $M \in (0,\infty)$ which start from ξ_0^R and evolve according to the following rules.

(i) On the time intervals $[0, M)$, $[M, 2M), \cdots$ the process evolves according to the rules of the growth model.

(ii) At times $M, 2M, \cdots$ we "reset to one" the values at all the sites to the left of the rightmost one, that is, if $k \geqslant 1$ we let $\xi_{kM}^{R,M}(y) = 1$ if there is an $x \geqslant y$ for which $\xi_{kM-}^{R,M}(x) = 1$.

Since the growth model is attractive ξ_t^R and $\xi_t^{R,M}$ can be constructed on the same space in such a way that $\xi_t^R \leqslant \xi_t^{R,M}$ for all t so if we let $\bar{r}_t^M = \max\{y : \xi_t^{R,M}(y) = 1\}$ then $\bar{r}_t^M \geqslant \bar{r}_t$. The growth rate of $\bar{r}_t^M$ is easy to determine. Because of the resetting $\bar{r}_{kM}^M - \bar{r}_{(k-1)M}^M$, $k \geqslant 1$ is a sequence of independent and identically distributed random variables, so $k^{-1}\bar{r}_{kM}^M \to E\bar{r}_M^M = E\bar{r}_M$ almost surely. To extend the convergence to times which are not multiples of M we need a bound on how fast $\bar{r}_t$ can grow. Let L be the range of the interaction and Λ the maximum birth rate. If we consider a system with no deaths and suppose that each $x \in [kL+1, (k+1)L]$ becomes occupied at rate Λ when some site in $[(k-1)L+1, kL]$ is occupied then we get a process which grows at a faster rate than the growth model in the sense that if we let ξ_t^* be a realization of the process with initial configuration $1_{(-\infty,0]}$ then we can construct ξ_t^* and ξ_t^R on the same space in such a way that $\xi_t^* \geqslant \xi_t^R$ for all t. Let $r_t^* = \max\{y : \xi_t^*(y) = 1\}$. It is easy to see that

$$P((k-1)L < r_t^* \leqslant kL) = e^{-\Lambda L t}(\Lambda L t)^k/k!$$

so we have

$$P(r_t^* > kL) = e^{-\Lambda L t}\Sigma_{j=k+1}^{\infty}(\Lambda L t)^j/j! \tag{1}$$

and

$$E(\max_{0<t<M} r_t^*) = Er_M^* \leqslant L\Sigma_{k=0}^{\infty}P(r_M^* > kL) \leqslant \Lambda L^2 M.$$

From the definition of $\xi_t^{R,M}$, the coupling, and the last equation it follows that if $\varepsilon > 0$

$$\Sigma_{k=1}^{\infty}P\big(\max_{(k-1)M<t<kM}\bar{r}_t^M - \bar{r}_{(k-1)M}^M > (k-1)M\varepsilon\big)$$
$$\leqslant \Sigma_{k=1}^{\infty}P(r_M^* > (k-1)M\varepsilon) \leqslant \frac{Er_M^*}{M\varepsilon} < \infty$$

so applying the Borel-Cantelli lemma we have

$$\limsup_{t\to\infty}\bar{r}_t^M/t \leqslant E\bar{r}_M/M$$

and it follows from the coupling that

$$\limsup_{t\to\infty}\bar{r}_t/t \leqslant E\bar{r}_M/M.$$

Since this result holds for all M we have

(2) $$\limsup_{t\to\infty}\bar{r}_t/t \leqslant \alpha = \inf_{M>0}E\bar{r}_M/M.$$

At this point we have proved half of the almost sure convergence. If $\alpha = -\infty$ we are done so for the rest of the proof we will suppose $\alpha > -\infty$. Our next step is to prove the L^1 convergence. To do this we observe that $|x| = 2x^+ - x$ so

(3) $$E|\bar{r}_t/t - \alpha| = 2E(\bar{r}_t/t - \alpha)^+ - E(\bar{r}_t/t - \alpha).$$

From (2) it follows that $(\bar{r}_t/t - \alpha)^+ \to 0$ almost surely. To show that $E(\bar{r}_t/t - \alpha)^+ \to 0$ we observe that

$$(\bar{r}_t/t - \alpha)^+ \leqslant |\alpha| + r_t^+/t \leqslant |\alpha| + r_t^*/t$$

where r_t^* is the upper bound we defined earlier. From (1) it follows that r_t^*/t is uniformly integrable so $E(\bar{r}_t/t - \alpha)^+ \to 0$. Using this in (3) gives

$$\limsup_{t\to\infty}E|\bar{r}_t/t - \alpha| \leqslant \limsup_{t\to\infty}(\alpha - E\bar{r}_t/t) \leqslant 0$$

since $\alpha = \inf_{M>0}E\bar{r}_M/M$.

The last inequality proves the L^1 convergence $\bar{r}_t/t$ to α so to complete the proof of Theorem 2.1 we need to show

(4) $$\limsup_{t\to\infty}\bar{r}_t/t \geqslant \alpha.$$

To do this we will (i) construct a stationary sequence $\{Y_k, k \geqslant 1\}$ so that $\bar{r}_n \geqslant \Sigma_{k=1}^n Y_k$ a.s. and then (ii) show that $\liminf n^{-1}\Sigma_{k=1}^n Y_k \geqslant \alpha$.

To accomplish (i) we will take a limit of the reset processes $\bar{r}_k^n$. The increments $X_k^n = \bar{r}_k^n - \bar{r}_{k-1}^n$ of these processes are not stationary but they are periodic so if we introduce an independent random variable U_n with $P(U_n = k) = 1/n$ for $0 \leqslant k < n$ the shifted increments $Y_k^n = X_{k+U_n}^n$ are a stationary sequence with

$$EY_1^n = n^{-1}\Sigma_{k=1}^n EX_k^n = n^{-1}E\bar{r}_k^n \geqslant \alpha.$$

By considering what happens in a contact process without deaths and using (1) we see that $E(Y_1^n)^+ \leqslant \Lambda L^2$. Since $EY_1^n \geqslant \alpha$ it follows that $E(Y_1^n)^- \leqslant \Lambda L^2 - \alpha$ so $E|Y_1^n| \leqslant 2\Lambda L^2 - \alpha$. From the last inequality we see the sequence of processes $\{Y_k^n, k \geqslant 1\}$ is tight (as a sequence of random elements of $R^{\{0,1,\cdots\}}$) so we can find a subsequence which converges weakly to a limit $\{Y_k, k \geqslant 1\}$ (see Billingsley (1968), page 19).

Having constructed the sequence we want for (i), the next step is to show that $\bar{r}_n$, $n \geqslant 1$ and Y_k^∞, $k \geqslant 1$ can be constructed on the same probability space in such a way that

(5) $$\Sigma_{k=1}^n Y_k^\infty \leqslant \bar{r}_n \quad \text{for all} \quad n \geqslant 1.$$

To do this we will use a coupling argument. Let $\xi_t^1 = \xi_t^R$. Let ξ_t^2 be a process which starts at time $-U_n$ with initial configuration ξ_0^R and runs from time $-U_n$ to time 0 according to the rules of the contact process. At time 0 we translate the process so that the rightmost particle is at 0. Since the translated configuration is $\leqslant \xi_0^R$ it follows that if we let ξ_t^2, $t \geqslant 0$ be the process which results from running the contact process from this initial distribution with resetting at times $kn - U_n$, $k \geqslant 1$ we can use the basic coupling to construct ξ_t^1 and ξ_t^2 on the same space in such a way that for $t \geqslant 0$, $\xi_t^1 \geqslant \xi_t^2$ on $\{t \leqslant n - U_n\}$. From the last inequality it follows that

$$\bar{r}_k \geqslant \Sigma_{j=1}^k Y_j^n \qquad \text{on} \quad \{k \leqslant n - U_n\}.$$

For any fixed K, $P\{U_n \leqslant n - K\} \to 1$ as $n \to \infty$ so it follows from the last inequality that we can construct $\bar{r}_k$ and Y_j on the same space in such a way that

$$\bar{r}_k \geqslant \Sigma_{j=1}^k Y_j \qquad \text{for all} \quad 1 \leqslant k \leqslant K.$$

Since K is arbitrary this proves (5).

At this point we have completed step (i) and it remains to show (ii). The first step is to realize that the sequence Y_j, is stationary and by the computations which showed tightness, has $E|Y_1| < \infty$ so if we let $k \to \infty$ then

$$\frac{1}{k}\Sigma_{j=1}^k Y_j \cdot \to E(Y_1|\mathcal{G}) \tag{6}$$

where $\mathcal{G}$ is the σ-field of shift invariant events. To prove (ii) then it suffices to show

$$E(Y_1|\mathcal{G}) = \alpha. \tag{7}$$

The first step is to show

(a) $E(Y_1|\mathcal{G}) \leqslant \alpha$ almost surely. To do this we observe that since any reset process grows faster than the contact process it follows from (5) that we can construct $\bar{r}_k^n$ and Y_k on the same space in such a way that

$$k^{-1}\bar{r}_k^n \geqslant k^{-1}\Sigma_{j=1}^k Y_i \text{ almost surely.}$$

Letting $k \to \infty$ in the above gives

$$E(Y_1|\mathcal{G}) \leqslant n^{-1}E\bar{r}_n^n.$$

Taking infimums over now proves (a).

Having shown (a) the next step is to show (b) $EY_1 \geqslant \alpha$. To do this let n_i be the subsequence which was chosen to give the limit $\{Y_j, j \geqslant 1\}$. $Y_1^{n_i}$ converges weakly to Y_1 and we have from (1) that

$$P(Y_1^n \geqslant kL) \leqslant e^{-\lambda L}\Sigma_{j=k}^{\infty}(\lambda L)^k/k! \qquad \text{for} \quad k \geqslant 0$$

so it follows from Fatou's lemma that

$$\liminf_{i\to\infty} E(-Y_1^{n_i}) \geqslant E(-Y_1)$$

so we have

$$EY_1 \geqslant \limsup_{i\to\infty} E\,Y_1^{n_i} \geqslant \alpha.$$

Combining (a) and (b) shows $EY_1 = \alpha$ and $E(Y_1|\mathcal{G}) = \alpha$ almost surely, completing the proof of (7). Combining (5), (6), and (7) shows that

$$\liminf_{k\to\infty} \bar{r}_k/k \geqslant \alpha \tag{8}$$

so to complete the proof we need to show that we can replace k by t in the last formula. To do this we observe that if $\varepsilon > 0$ it follows from the Markov property and the monotonicity properties of the contact processes that if

$$A_{k,t} = \{\bar{r}_t < (\alpha - 2\varepsilon)k \quad \text{for some} \quad t \in [k-1,k], \bar{r}_k \geqslant (\alpha - \varepsilon)k\}$$

then

$$P(A_{k,\varepsilon}) \leqslant P(\bar{r}_1^* > k\varepsilon)$$

where $\bar{r}_t^*$ is the position of the rightmost particle in a contact process with no deaths and initial configuration ξ_0^R. Summing on k gives

$$P(A_{k,\varepsilon}\,\text{i.o.}) = 0 \qquad \text{for all} \quad \varepsilon.$$

Combining this with (8) shows

$$\liminf_{t\to\infty} \bar{r}_t/t \geqslant \alpha$$

and completes the proof of Theorem 2.1.

Remark. The reader should note that the first two parts of the argument ($\limsup \bar{r}_t/t \leqslant \alpha$, and $\bar{r}_t/t \to \alpha$ in L^1) above are similar to the first two parts of Kingman's proof of subadditive ergodic theorem. The third part ($\liminf \bar{r}_t/t \geqslant \alpha$) corresponds to the hardest part of Kingman's proof–the subadditive decomposition theorem. In this part the arguments are necessarily different. In Kingman's proof the issue is to show that a general stationary subadditive process dominates an additive process with the same time constant while in our proof we need to show that our particular nonstationary additive process dominates an additive one with the same asymptotic mean.

3. Nearest neighbor growth models. In this section we will assume that the growth model has a nearest neighbor interaction. In this case we have the following useful coupling result.

Lemma 3.1. *Let ξ_t^R and ξ_t^0 be realizations of the growth model with initial configurations $1_{(-\infty,o]}$ and $1_{\{0\}}$. If ξ_t^R and ξ_t^0 are constructed on the same space using the basic coupling then $\xi_t^R(x) = \xi_t^0(x)$ for all $x \geqslant l_t = \inf\{y : \xi_t^0(y) = 1\}$.*

Proof. We will prove this by showing that every transition in the coupled process preserves this property. Suppose the configuration before the flip is ξ_{t-}^0 and the flip occurs at x. If $x > l_{t-}$ then since the flip rates only depend upon the nearest neighbors $\xi^R(x)$ and $\xi^0(x)$ flip together. If $x = l_{t-}$ then $\xi_{t-}^0(l_{t-}) = 1$ so $\xi_t^0(l_{t-}) = 0$ and the new boundary $l_t > l_{t-}$. Since $\xi_{t-}^0(x) = \xi_{t-}^0(x)$ for all $x > l_{t-}$ we have $\xi_t^R(x) = \xi_t^0(x)$ for all $x \geqslant l_t$. Finally we observe that if $x < l_{t-}$ then

$x = l_{t-} - 1$ so $\xi^0_{t-}(x) = 0$ and $\xi^0_t(x) = 1$. Since $\xi^R_t(x) \geqslant \xi^0_t(x)$ it follows that $\xi^R_t(x) = 1$ and we have $\xi^R_t(x) = \xi^0_t(x)$ for all $x \geqslant l_t = l_{t-} - 1$.

Note if the interaction has range 2 and the configuration is

$$\begin{array}{cccc|ccc|ccc} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

then we can get a 1-0 discrepancy at $r_t - 1$ or get a 1 at $l_t - 2$ which will cause the left boundary to move across a discrepancy.

From Lemma 3.1 it follows that on $\Omega_t = \{|\xi^0_t| > 0\}$ we have $r_t = \bar{r}_t$. Using this observation we can translate Theorem 2.1 into a result about r_t.

THEOREM 3.2. *Let $\Omega_\infty = \{|\xi^0_t| > 0$ for all $t\}$. There is a constant α so that*

$$r_t/t \to \alpha \qquad \text{a.s. on } \Omega_\infty.$$

If $\alpha > -\infty$ then

$$(r_t/t)\, 1_{\Omega_t} \to \alpha 1_{\Omega_\infty} \quad in \quad L^1$$

PROOF. Since $r_t = \bar{r}_t$ on Ω_t, the a.s. convergence is an immediate consequence of Theorem 2.1. To obtain the L^1 convergence observe that

$$|(r_t/t)\, 1_{\Omega_t} - \alpha 1_{\Omega_\infty}| = |(\bar{r}_t/t) - \alpha| 1_{\Omega_t} + \alpha |1_{\Omega_t} - 1_{\Omega_\infty}|$$

and the expectation of the right-hand side converges to 0 as $t \to \infty$.

Applying Theorem 3.2 to $\tilde{\xi}_t(x) = \xi_t(-x)$ shows that if we let $l_t = \inf\{y : \xi^0_t(y) = 1\}$ then there is a constant β so that $l_t/t \to \beta$ a.s. on Ω_∞ and if $\beta < \infty$ then

$$(l_t/t)\, 1_{\Omega_t} \to \beta 1_{\Omega_\infty} \text{ in } L^1.$$

Combining this with Theorem 3.2 shows that the width of the set of occupied sites in ξ^0_t grows like $(\alpha - \beta)^+ t$ on Ω_∞. This suggests the following result.

THEOREM 3.3. *There is a constant γ so that*

$$|\xi^0_t|/t \to \gamma 1_{\Omega_\infty} \quad in \quad L^1.$$

PROOF. To study the convergence of $|\xi^0_t|/t$ we will use the basic coupling to compare ξ^0_t with ξ^Z_t, a version of the growth model starting from $\xi^Z_0(x) \equiv 1$. By imitating the proof of Lemma 3.1 it is easy to show

LEMMA 3.4. *Let $l_t = \inf\{y : \xi^0_t(y) = 1\}$ and $r_t = \sup\{y : \xi^0_t(y) = 1\}$. If ξ^Z_t and ξ^0_t are constructed on the same space by the basic coupling then $\xi^Z_t(x) = \xi^0_t(x)$ for all $l_t \leqslant x \leqslant r_t$.*

Lemma 3.4 says that between l_t and r_t, ξ^0_t looks like ξ^Z_t. From this we get that

$$(1) \qquad |\xi^0_t| = \Sigma^{r_t}_{x=l_t} \xi^0_t(x) = \Sigma^{r_t}_{x=l_t} \xi^Z_t(x) \text{ on } \Omega_t.$$

At this point we have to pause to discard the trivial case. If $P(\Omega_\infty) = 0$ then since $\xi_t^0 \leqslant \xi_t^*$ (the contact process with no deaths) we have that

$$|\xi_t^0|/t \leqslant (|\xi_t^*|/t)\, 1_{\Omega_t} \to 0 \text{ in } L^1$$

since $|\xi_t^*|/t$, $t \geqslant 1$ is uniformly integrable. Having proved the result in the case $P(\Omega_\infty) = 0$ we can for the rest of the proof assume $P(\Omega_\infty) > 0$. If we do this it follows from results in Section 2 that $-\infty < \beta \leqslant \alpha < \infty$ and hence that $r_t/t \to \alpha$ and $l_t/t \to \beta$ in L^1. Using this observation we can replace the l_t and r_t in (1) by βt and αt. Since $|\xi_t^Z(x)| \leqslant 1$ we have

$$|\Sigma_{x=l_t}^{r_t} \xi_t^Z(x) - \Sigma_{x=\beta t}^{\alpha t} \xi_t^Z(x)| \leqslant |r_t - \alpha t| + |l_t - \beta t| \qquad \text{on} \quad \Omega_t$$

so it follows from the results of Section 2 that

$$E\big||\xi_t^0|/t - 1_{\Omega_t}\big(t^{-1}\Sigma_{x=\beta t}^{\alpha t} \xi_t^Z(x)\big)\big| \to 0.$$

If $\alpha = \beta$ the last expression shows that $|\xi_t^0|/t \to 0$ in L^1 so for the rest of the proof we can assume $\alpha > \beta$. Now

$$|1_{\Omega_t}\big(t^{-1}\Sigma_{x=\beta t}^{\alpha t} \xi_t^Z(x)\big) - \gamma 1_{\Omega_\infty}| = 1_{\Omega_t}|t^{-1}\Sigma_{x=\beta t}^{\alpha t} \xi_t^Z(z) - \gamma| + \gamma|1_{\Omega_t} - 1_{\Omega_\infty}|$$

so to prove Theorem 3.3 in the case $\alpha > \beta$ it suffices to show that

$$t^{-1}\Sigma_{x=\beta t}^{\alpha t} \xi_t^Z(z) \to \gamma \qquad \text{in} \quad L^1.$$

To study the growth of the number of particles in ξ_t^Z in $[\beta t, \alpha t]$ we need three facts from the theory of attractive processes.

I. As $t \to \infty$ the distribution of ξ_t^Z converges to a limit ξ_∞^Z. The distribution of ξ_∞^Z is a stationary distribution for the contact process.

PROOF. This is an easy consequence of the basic coupling (for details see Section 2.2 of Liggett (1976)).

II. If we fix t and consider $\xi_t^Z(x)$, $x \in Z$ as a stationary sequence of random variables, then this is an ergodic sequence.

PROOF. This is true whenever the interaction has finite range (see Holley (1972) page 1967).

III. In $\xi_\infty^Z(x)$, $x \in Z$ is an ergodic stationary sequence.

PROOF. By using the basic coupling it is possible to define the random variables ξ_n^Z, $0 \leqslant n \leqslant \infty$ on the same probability space $(\Omega, \mathfrak{F}, P)$ in such a way that $\xi_0^Z(x) \geqslant \xi_1^Z(x) \geqslant \cdots \geqslant \xi_\infty^Z(x)$ for all x. Let θ_y be the operator which shifts the configuration by y, i.e., $(\theta_y \xi)(x) = \xi(x + y)$. If f is a nondecreasing function then

$$\frac{1}{2n+1}\Sigma_{y=-n}^{n} f\big(\theta_y(\xi_m^Z)\big) \geqslant \frac{1}{2n+1}\Sigma_{y=-n}^{n} f\big(\theta_y(\xi_\infty^Z)\big).$$

From Birkhoff's ergodic theorem and Holley's result it follows that if f is bounded then the left-hand side converges in $L^1(P)$ to $Ef(\xi_m^Z)$ while the right-hand side converges in $L^1(P)$ to $E(f(\xi_\infty^Z)|\mathcal{G})$ where $\mathcal{G}$ is the σ-field of shift invariant events.

Combining the last two results with the inequality above shows that

$$Ef(\xi_m^Z) \geq E(f(\xi_\infty^Z)|\mathcal{G})\ P \text{ a.s.}$$

Letting $m \to \infty$ and using I gives that if f is continuous

$$Ef(\xi_\infty^Z) \geq E(f(\xi_\infty^Z)|\mathcal{G})\ P \text{ a.s.}$$

so we have

$$Ef(\xi_\infty^Z) = E(f(\xi_\infty^Z)|\mathcal{G})\ P \text{ a.s.}$$

for all bounded nondecreasing continuous f.

Now if $\eta \in \{0,1\}^{[-L,L]}$ we have

$$1_{\{\xi=\eta \text{ on } [-L,L]\}} = 1_{\{\xi\geq\eta \text{ on } [-L,L]\}} - 1_{\{\xi\geq\eta \text{ and } \xi\neq\eta \text{ on } [-L,L]\}}.$$

It follows from this that any function which depends upon only finitely many coordinates can be written as a finite linear combination of bounded nondecreasing continuous functions so we have $E(f(\xi_\infty^Z)|\mathcal{G}) = Ef(\xi_\infty^Z)$ for all these functions.

To complete the argument at this point observe that if f, g are bounded functions on $\{0,1\}^Z$

$$E\left|\frac{1}{2n+1}\Sigma_{y=-n}^n f(\theta_y(\xi_\infty^Z)) - \frac{1}{2n+1}\Sigma_{y=-n}^n g(\theta_y\xi_\infty^Z)\right|$$
$$\leq \frac{1}{2n+1}\Sigma_{y=-n}^n E|f(\theta_y\xi_\infty^Z) - g(\theta_y\xi_\infty^Z)| = E|f(\xi_\infty^Z) - g(\xi_\infty^Z)|$$

so it follows now from results proved above that

$$E(f(\xi_\infty^Z)|\mathcal{G}) = Ef(\xi_\infty^Z)$$

for all bounded f on $\{0,1\}^Z$ so $\mathcal{G}$ is trivial for ξ_∞^Z.

From III we get that as $t \to \infty$, $t^{-1}\Sigma_{x=\beta t}^{\alpha t}\xi_\infty^Z(x)$ converges to $(\alpha-\beta)\,P(\xi_\infty^Z(0)=1)$ in L^1. Using this fact it is easy to prove Theorem 3.3. Let ξ_t^I be a version of the contact process with ξ_0^I having the same distribution as ξ_∞^Z. Using the basic coupling we can construct ξ_t^I and ξ_t^Z on the same probability space in such a way that $\xi_t^I \leq \xi_t^Z$ for all $t \geq 0$. If we do this then

$$E|\Sigma_{x=\beta t}^{\alpha t}\xi_t^Z(x) - \Sigma_{x=\beta t}^{\alpha t}\xi_t^I(x)| = (\Sigma_{x=\beta t}^{\alpha t}1)(E\xi_t^Z(0) - E\xi_\infty^Z(0)).$$

If we divide both sides of the equation by t and let $t \to \infty$ then the right-hand side converges to 0. Combining this with III shows that if $\rho = P(\xi_\infty^Z(0) = 1)$ then

$$t^{-1}\Sigma_{x=\beta t}^{\alpha t}\xi_t^Z(x) \to \rho(\alpha-\beta) \text{ in } L^1.$$

By remarks above this completes the proof of Theorem 3.3.

4. Positivity of γ for the basic contact processes. In this section we will study the one-sided ($N = \{-1\}, \lambda_1 = \lambda$) and the basic two-sided ($N = \{-1,1\}, \lambda_1 = \lambda, \lambda_2 = 2\lambda$) contact processes. Our aim will be to show that in each case if $\lambda > \lambda_{cr} = \sup\{\lambda : P\{|\xi_t^0| \to 0\} = 1\}$ then $\gamma > 0$ so the process really grows at a linear rate. To do this we use the fact that these processes are additive in the sense of Harris (1978). To introduce this concept we need some notation: if $A \subset Z$ let ξ_t^A be a version of the contact process starting from $\xi_0^A = 1_A$.

DEFINITION. A contact process is additive if we can construct all the processes $\{\xi_t^A, t \geqslant 0\}, A \subset Z$ on the same space in such a way that

$$\xi_t^{A \cup B}(x) = \xi_t^A(x) \vee \xi_t^B(x) \qquad \text{for all} \quad x, t. \tag{1}$$

The basic fact we need about additive processes is the following.

LEMMA 4.1. *Suppose ξ_t is an additive process and the processes $\{\xi_t^A, t > 0\}, A \in Z$ have been constructed so that* (1) *holds. Let $r_t^A = \sup\{x : \xi_t^A(x) = 1\}$. If A and B are infinite sets so that $B \subset A \subset (-\infty, 1]$ then for any finite set C*

$$0 \leqslant r_t^{A \cup C} - r_t^A \leqslant r_t^{B \cup C} - r_t^B.$$

PROOF. The assumptions on A, B, and C imply that all the variables in the inequality are finite a.s. From the construction we have

$$\begin{aligned} r_t^{A \cup C} - r_t^A &= \left(r_t^C - r_t^A\right)^+ && \left(\text{since } \xi_t^{A \cup C} = \xi_t^A \vee \xi_t^C\right) \\ r_t^A &\geqslant r_t^B && \left(\text{since } \xi_t^A = \xi_t^B \vee \xi_t^{A-B}\right) \\ \left(r_t^C - r_t^B\right)^+ &= r_t^{B \cup C} - r_t^B && \left(\text{since } \xi_t^{B \cup C} = \xi_t^B \vee \xi_t^C\right). \end{aligned}$$

Since $z \to (r_t^C - z)^+$ is a decreasing function combining the last three equations shows

$$r_t^{A \cup C} - r_t^A \leqslant r_t^{B \cup C} - r_t^B.$$

It follows from the first equation that the left-hand side is nonnegative.

If we let $A = (-\infty, -1]$ and $C = \{0\}$ in Lemma 4.1 we get that for all $B \subset (-\infty, -1]$

$$E\left(r_t^{B \cup \{0\}} - r_t^B\right) \geqslant E\left(r_t^{-\infty, 0]} - {}^{-\infty, -1]}\right) = 1$$

since the system is translation invariant. The last inequality says that if we add a 1 to the right of all the ones in the initial configuration then it increases the expected location of the rightmost one by at least 1. Using this result it is easy to prove

LEMMA 4.2. *Let ξ_t^λ be a one-sided $(N = \{-1\}, \lambda_1 = \lambda)$ or additive two-sided $(N = \{-1, 1\}, \lambda_1 = \lambda, \lambda_2 = \theta\lambda, 1 \leqslant \theta \leqslant 2)$ contact process with initial configuration $\xi_0^\lambda = 1_{(-\infty, 0]}$. If we let $r_t^\lambda = \sup\{y : \xi_t^\lambda(y) = 1\}$ and $\alpha_t(\lambda) = Er_t^\lambda$ then $\alpha_t(\lambda + \delta) - \alpha_t(\lambda) \geqslant \delta t$ for all $\delta \geqslant 0$.*

PROOF. Let t be a fixed time and let $\delta > 0$. Use the basic coupling to construct $\xi_s^{\lambda+\delta}$ and ξ_s^λ on the same space. Let $\tau = \inf\{s \geqslant 0 : r_s^{\lambda+\delta} > r_s^\lambda\}$. If $\tau > t$ then $r_t^{\lambda+\delta} = r_t^\lambda$. To compare $r_t^{\lambda+\delta}$ and r_t^λ on $\{\tau \leqslant t\}$ it is convenient to introduce a process $\hat{\xi}_t^{\lambda+\delta}$ which $= \xi_s^{\lambda+\delta}$ for $s \leqslant \tau$ and from τ until t evolves according to the rules of the contact process with parameter λ. Since τ is a stopping time we can use the basic coupling to construct $\hat{\xi}_s^{\lambda+\delta}$ on the same space with $\xi_s^{\lambda+\delta}$ and ξ_s^λ in such a way that $\xi_s^{\lambda+\delta} \geqslant \hat{\xi}_s^{\lambda+\delta} \geqslant \xi_s^\lambda$ for all $s \leqslant t$ and the first inequality is an equality for $s \leqslant \tau$. From this it follows that if we let $\hat{r}_t^{\lambda+\delta} = \sup\{y : \hat{\xi}_t^{\lambda+\delta}(y) = 1\}$ then

$$E\left(r_t^{\lambda+\delta} - r_t^\lambda\right) \geqslant E\left(\hat{r}_t^{\lambda+\delta} - r_t^\lambda; \tau \leqslant t\right).$$

Now at time τ, $\hat{r}_\tau^{\lambda+\delta} = r_\tau^{\lambda+\delta} \geqslant r_\tau^\lambda + 1$ and $\hat{\xi}_\tau^{\lambda+\delta} = \xi_\tau^{\lambda+\delta} \geqslant \xi_\tau^\lambda$ so using the Markov property and applying Lemma 4.1 gives that the last term above is $\geqslant P(\tau \leqslant t)$. Since the birth rate at $r_t^{\lambda+\delta} + 1$ is always $\lambda + \delta$ while birth rate at $r_t^\lambda + 1$ is always λ we have $P(\tau \leqslant t) \geqslant 1 - e^{-\delta t}$. Combining the inequalities above we get

$$\alpha_t(\lambda + \delta) - \alpha_t(\lambda) \geqslant 1 - e^{-\delta t}.$$

To strengthen this result to the desired conclusion (recall $x \geqslant 1 - e^{-x}$) we observe that

$$\begin{aligned}\alpha_t(\lambda + \delta) - \alpha_t(\lambda) &= \Sigma_{k=1}^n \alpha_t(\lambda + \delta k/n) - \alpha_t(\lambda + \delta(k-1)/n)\\ &\geqslant n(1 - e^{-\delta t/n})\end{aligned}$$

and the right-hand side converges to δt as $n \to \infty$.

This completes the proof of Lemma 4.2. Our next step will be to prove an inequality for the expected location of the leftmost particle. Let $\tilde{\xi}_t^\lambda$ be a one-sided or attractive two-sided contact process with parameter λ and initial configuration $\tilde{\xi}_0^\lambda = 1_{[0,\infty)}$. If $\delta > 0$ we can use the basic coupling to construct $\tilde{\xi}_t^{\lambda+\delta}$ and $\tilde{\xi}_t^\lambda$ on the same space in such a way that $\tilde{\xi}_t^{\lambda+\delta} \geqslant \tilde{\xi}_t^\lambda$ for all t. If we let $l_t^\lambda = \inf\{y : \tilde{\xi}_t^\lambda(y) = 1\}$ and $\beta_t(\lambda) = El_t^\lambda$ then we have $l_t^{\lambda+\delta} \leqslant l_t^\lambda$ and $\beta_t(\lambda + \delta) \leqslant \beta_t(\lambda)$. Combining this with the result of Lemma 4.1 gives

$$\alpha_t(\lambda + \delta) - \beta_t(\lambda + \delta) \geqslant \alpha_t(\lambda) - \beta_t(\lambda) + \delta t$$

(in the two-sided case we can replace δt by $2\delta t$). From the L^1 convergence in Theorem 2.1 we have that

$$\begin{aligned}\alpha(\lambda) &= \inf_t \alpha_t(\lambda)/t = \lim_{t\to\infty} \alpha_t(\lambda)/t\\ \beta(\lambda) &= \inf_t \beta_t(\lambda)/t = \lim_{t\to\infty} \beta_t(\lambda)/t.\end{aligned}$$

From the a.s. convergence in that result we get that

$$\alpha(\lambda) - \beta(\lambda) < 0 \Rightarrow P\left\{|\xi_t^0| \to 0\right\} = 1.$$

Combining the last two results with the inequality shoows that if $\lambda > \lambda_{cr} = \sup\{\lambda : \mathrm{P}\{|\xi_t^0| \to 0\} = 1\}$ then $\alpha(\lambda) - \beta(\lambda) \geqslant \lambda - \lambda_{cr} > 0$.

The reader should observe that up to this point all the computations in this section are valid for any additive contact process (or when properly reformulated for any additive growth model). To prove $\gamma > 0$ we need to show that if $\xi_t^{\lambda,Z}$ is a realization of the contact process with parameter $\lambda > \lambda_{cr}$ and initial configuration $\xi_0^{\lambda,Z} \equiv 1$ then

$$\rho(\lambda) = \lim_{t\to\infty} P\left(\xi_t^{\lambda,Z}(0) = 1\right) > 0.$$

We do not know how to prove this for a general contact process. In the case of a one-sided or basic two-sided contact process it follows from the duality theory associated with additive processes (combine (4.2) of Harris (1978) with (1.1) of Harris (1976)) we have in these cases that

$$P(\xi_t^{\lambda,Z}(0) = 1) = P(|\xi_t^{\lambda,0}| \neq 0)$$

so $\rho(\lambda) > 0$ if $\lambda > \lambda_{cr}$.

5. A convergence theorem for the basic two-sided process. In this section we will use the results of Section 4 to prove a convergence theorem for the basic contact process. The reader will see from the proof that the same conclusion holds for any additive two-sided ($N = \{-1, 1\}, \lambda_1 = \lambda, \lambda_2 = \theta\lambda, 1 \leqslant \theta \leqslant 2$) contact process. We do not know how to prove this result for $\theta > 2$.

THEOREM 5.1. *If $\lambda > \lambda_{cr}$ then as $t \to \infty$, ξ_t converges weakly to*

$$P(|\xi_t| \to 0)\,\nu_0 + P(|\xi_t| \nrightarrow 0)\,\nu_1$$

where ν_0 is the pointmass on $\xi \equiv 0$ and ν_1 is the limit distribution when the initial configuration is $\xi_0 \equiv 1$.

PROOF. We will first consider what happens when $\xi_0 = 1_{\{0\}}$. Let ξ_t^Z and ξ_t^0 be versions of the contact process starting from $\xi_0^Z \equiv 1$ and $\xi_0^0 = 1_{\{0\}}$ and use the basic coupling to construct these processes on the same probability space. From Lemma 3.4 it follows that if we let $l_t = \inf\{y : \xi_t^0(y) = 1\}$ and $r_t = \sup\{y : \xi_t^0(y) = 1\}$ then we have $\xi_t^Z(x) = \xi_t^0(x)$ for all $l_t \leqslant x \leqslant r_t$. If $\lambda > \lambda_{cr}$ then it follows from Theorem 2.1 and the results of Section 4 that there are constants α, β with $\alpha > \beta$ so that if $\Omega_t = \{|\xi_t^0| > 0\}$ and $\Omega_\infty = \cap_{t>0}\Omega_t$ then

$$(r_t/t)\,1_{\Omega_t} \to \alpha 1_{\Omega_\infty} \quad \text{in } L^1$$

and

$$(l_t/t)\,1_{\Omega_t} \to \beta 1_{\Omega_\infty} \quad \text{in } L^1.$$

Since the basic two-sided process is symmetric we have $\beta = -\alpha$. Combining this with the last result shows that $\alpha > 0$ so $r_t \to \infty$ in probability on Ω_∞ and by symmetry that $l_t \to -\infty$ in the same sense. When we combine the last conclusion with the result of Lemma 3.1 it becomes clear that ξ_t converges weakly to the indicated limit. To prove this we follow an argument given by Griffeath (look at the proof of Theorem 5.1 in (1979)). Let Λ be a finite subset of Z.

$$P(\xi_t^0 = 0 \text{ on } \Lambda) = P(\Omega_t^c) + P(\xi_t^0 = 0 \text{ on } \Lambda, \Omega_t).$$

As $t \to \infty$ the first term converges to

$$P(\Omega_\infty^c) = P(|\xi_t^0| \to 0)\,v_0\{\xi : \xi(x) = 0 \text{ on } \Lambda\}.$$

To compute the limit of the second we observe that $r_t \to \infty$ and $l_t \to -\infty$ a.s. on Ω_∞ so

$$P(\xi_t^0 = 0 \text{ on } \Lambda, \Omega_t) - P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_t) \to 0.$$

To compute the limit of $P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_t)$ we observe that if $s < t$ then it follows from considering the distribution of $(\xi_s^Z | \Omega_s)$ that

$$P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_s) \geqslant P(\Omega_s) P(\xi_{t-s}^Z = 0 \text{ on } \Lambda)$$

so

$$\liminf_{t\to\infty} P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_s) \geqslant P(\Omega_s) P(\xi_\infty^Z = 0 \text{ on } \Lambda).$$

Similarly

$$\liminf_{t\to\infty} P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_s^c) \geqslant P(\Omega_s^c) P(\xi_\infty^Z = 0 \text{ on } \Lambda).$$

Since $\lim_{t\to\infty} P(\xi_t^Z = 0 \text{ on } \Lambda) = P(\xi_\infty^Z = 0 \text{ on } \Lambda)$ it follows from the last two inequalities that

$$\lim_{t\to\infty} P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_s) = P(\Omega_s) P(\xi_\infty^Z = 0 \text{ on } \Lambda)$$

so we have shown

$$\lim_{s\to\infty} \lim_{t\to\infty} P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_s) = P(\Omega_\infty) P(\xi_\infty^Z = 0 \text{ on } \Lambda).$$

To compute the limit of $\lim_{t\to\infty} P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_t)$ from this we observe that

$$\begin{aligned} 0 &\leqslant \lim_{s\to\infty} \lim_{t\to\infty} \big(P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_s) - P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_t) \big) \\ &\leqslant \lim_{s\to\infty} P(\Omega_s - \Omega_\infty) = 0 \end{aligned}$$

so

$$\lim_{t\to\infty} P(\xi_t^Z = 0 \text{ on } \Lambda, \Omega_t) = P(\Omega_\infty) P(\xi_\infty^Z = 0 \text{ on } \Lambda).$$

Combining this with previous results shows that for all finite $\Lambda \subset Z$

$$\lim_{t\to\infty} P(\xi_t^0 = 0 \text{ on } \Lambda) = P(\Omega_\infty^c) + P(\Omega_\infty) P(\xi_\infty^Z = 0 \text{ on } \Lambda).$$

Now if f is a function on $\{0,1\}^Z$ whose value at any ξ only depends upon the values of $\xi(x)$ for $x \in [-L, L]$ then f can be written as a linear combination of the indicator functions of the sets $\{\xi_t^0 = 0 \text{ on } \Lambda\}$ for $\Lambda \subset [-L, L]$ so it follows from the last equation that

$$\lim_{t\to\infty} Ef(\xi_t^0) = P(\Omega_\infty^c) \int f \, d\nu_0 + P(\Omega_\infty) \int f \, d\nu_1.$$

This proves Theorem 5.1 in the case $\xi_0 = 1_{\{0\}}$.

To extend this result to an arbitrary initial distribution we will use what can be called a "restart" coupling. Let A be a nonempty subset of Z and let ξ_t^A be a version of the contact process starting from an initial configuration with $\xi_0^A(x) = 1$ for $x \in A$ and $= 0$ for $x \notin A$. Let y be the element of A which minimizes $|y - 1/3|$ and let ξ_0^y be a version of the contact process starting from $\xi_0^y = 1_{\{y\}}$. Using the basic coupling we can construct ξ_t^y, ξ_t^A, and ξ_t^Z on the same space in such a way that $\xi_t^y \leqslant \xi_t^A \leqslant \xi_t^Z$. If we do this it follows from Lemma 3.1 that $\xi_t^y = \xi_t^A = \xi_t^Z$ on $[l_t^y, r_t^y]$ where $l_t^y = \inf\{x : \xi_t^y(x) = 1\}$ and $r_t^y = \sup\{x : \xi_t^y(x) = 1\}$. From the argument we have given for ξ_t^0 it follows that if we let $\tau_1 = \inf\{t : |\xi_t^y| = 0\}$ we have

$$P(\xi_t^y = 0 \text{ on } \Lambda, \tau_1 > t) - P(\xi_t^Z = 0 \text{ on } \Lambda, \tau_1 > t) \to 0$$

and

$$P(\xi_t^Z = 0 \text{ on } \Lambda, \tau_1 > t) \to P(\xi_\infty^Z = 0 \text{ on } \Lambda)P(\tau_1 = \infty).$$

Since $\xi_t^y \leqslant \xi_t^A \leqslant \xi_t^Z$ it follows from this that

$$P(\xi_t^A = 0 \text{ on } \Lambda, \tau_1 > t) \to P(\xi_\infty^Z = 0 \text{ on } \Lambda)P(\tau_1 = \infty)$$

and since $P(t < \tau_1 < \infty) \to 0$ that $P(\xi_t^A = 0 \text{ on } \Lambda, \tau_1 = \infty)$ has the same limit.

To finish our computation we have to consider what happens when $\tau_1 < \infty$. If $|\xi_{\tau_1}^A| = 0$ then $|\xi_t^A| = 0$ for all $t \geqslant \tau_1$. If $|\xi_{\tau_1}^A| > 0$ let y' be the location which has $\xi_{\tau_1}^A(y') = 1$ and minimizes $|y' - 1/3|$. Let $\xi_t^{y'}$ be a version of the contact process starting from $\xi_0^{y'} = 1_{\{y'\}}$. Using the basic coupling we can construct $\xi_t^{y'}$, ξ_t^A, and ξ_t^Z on the same space in such a way that $\xi_t^{y'} \leqslant \xi_{t+\tau_1}^A \leqslant \xi_t^Z$ and again if we do this it follows from Lemma 3.1 that these three processes are equal on $[l_t^{y'}, r_t^{y'}]$. If we let $\tau_2 = \inf\{t : |\xi_t^{y'}| = 0\}$ then by repeating the argument given above with some minor changes shows that

$$P(\xi_t^A = 0 \text{ on } \Lambda, \tau_1 < \infty, |\xi_{\tau_1}^A| > 0, \tau_2 = \infty)$$
$$\to P(\xi_\infty^Z = 0 \text{ on } \Lambda)P(\tau_1 < \infty, |\xi_{\tau_1}^A| > 0, \tau_2 = \infty).$$

By repeating the construction above we can define a sequence of times τ_n and a sequence of times $T_n = \tau_1 + \cdots + \tau_n$. Since $P(|\xi_t^y| \nrightarrow 0) > 0$ and is independent of y it follows from the strong Markov property that the sequence τ_n ends after a finite number of terms with either $|\xi_{T_n}^A| = 0$ or $|\xi_{T_n}^A| > 0$ and $\tau_{n+1} = \infty$. From this it follows that we can write

$$P(\xi_t^A = 0 \text{ on } \Lambda) = \Sigma_{n=1}^\infty P(\xi_t^A = 0 \text{ on } \Lambda, |\xi_{T_n}^A| = 0)$$
$$+ \Sigma_{n=0}^\infty P(\xi_t^A = 0 \text{ on } \Lambda, \tau_n < \infty, |\xi_{T_n}^A| > 0, \tau_{n+1} = \infty).$$

Now $\cup_{n=1}^\infty \{|\xi_{T_n}^A| = 0\} = \{|\xi_t^A| \to 0\}$ and we have that for each fixed n that the terms in the sum converge to $P(|\xi_{T_n}^A| = 0)$ and $P(\xi_\infty^Z = 0 \text{ on } \Lambda)P(\tau < \infty, |\xi_{T_n}^A| > 0, \tau_{n+1} = \infty)$ so it follows from Fatou's lemma that

$$\liminf_{t\to\infty} P(\xi_t^A = 0 \text{ on } \Lambda) \geqslant P(|\xi_t| \to 0)$$
$$+ P(|\xi_t| \nrightarrow 0)P(\xi_\infty^Z = 0 \text{ on } \Lambda).$$

By truncating at N and using $P(T_N < \infty)$ to bound the remainder it is easy to show that the $\limsup_{t\to\infty} P(\xi_t^A = 0 \text{ on } \Lambda) \leqslant$ the right-hand side. This completes the proof of Theorem 5.1.

6. Discrete time models. Since this paper was first written, discrete time contact processes have arisen in connection with Richardson's model (see [2]) and an interest has developed in having the results of Sections 2 and 3 available for discrete time processes. In this section we will describe the modifications of the proof which are necessary in discrete time.

The first thing we have to do is to introduce the discrete time model. It is described by giving the transition probabilities

$$p(x,\eta) = P(\eta_{n+1}(x) = 1 | \eta_n = \eta)$$

which we will assume to be of the form

$$p(x,\eta) = f(\eta(x-L),\cdots,\eta(x+L)).$$

Given the transition probability one defines the process inductively by computing $P(\eta_{n+1}(x) = 1)$ for the given η_n and then determining the outcomes at different sites by independent coins.

From the description of the process it is clear that if $p(0,\eta)$ is an increasing function of η the system is attractive in the sense that if $\eta_0^1 \leqslant \eta_0^2$ joint realizations can be constructed so that $\eta_n^1 \leqslant \eta_n^2$ for all n. If we assume in addition that $p(0,0) = 0$ then in the terminology of Section 1 we have a discrete time growth model and it is reasonable to expect that under these conditions we have

THEOREM 6.1. *There is a constant α so that $\bar{r}_n/n \to \alpha$ almost surely. If $\alpha > -\infty$ then the convergence also occurs in L^1.*

PROOF. A proof of this result can be obtained by following the proof of Theorem 2.1. (The resulting proof is somewhat simpler since estimates for intermediate t are no longer required.)

Differences between discrete and continuous time appear when we consider generalizing the results of Section 3. Lemma 3.1 is not valid for every discrete time growth model.

EXAMPLE. Suppose $L = 1$ and we have $f(1,1,1) > f(0,1,1)$ and $f(0,0,1) > 0$ then the following can happen with positive probability.

Time 0	1		1		1	│	1	0	0
	0		0		0	│	1	0	0
Time 1	1		1	│	1		1	0	0
	0		0	│	1		1	0	0
Time 2	1	│	1		1		1	0	0
	0	│	1		0		1	0	0

so the conclusion of Lemma 3.1 fails.

A similar argument shows that Lemma 3.1 is false if $f(0,1,0) < f(1,1,0)$ and $f(0,0,1) > 0$. If this and the preceding case are excluded, i.e., if we assume (i) $f(0,1,1) = f(1,1,1)$ and $f(0,1,0) = f(1,1,0)$ or (ii) $f(0,0,1) = 0$ then the conclusion of Lemma 3.1 is valid. To see this observe that in case (i) the values at l_n at time $n+1$ will always be the same and in case (ii) $l_{n+1} \geqslant l_n$. Once we have lemma 3.1 it is trivial (as it was in Section 3) to deduce.

THEOREM 6.2. *If* (i) *or* (ii) *is satisfied there is a constant* α *so that*

$$r_n/n \to \alpha \text{ a.s. } on\ \Omega_\infty.$$

If $\alpha > -\infty$ *then*

$$(r_n/n)\, 1_{\Omega_n} \to \alpha\, 1_{\Omega_\infty} \ in\ L^1.$$

Applying Theorem 6.2 to $\tilde{\eta}_n(x) = \eta_n(-x)$ shows that if we assume (iii) $f(1,1,0) = f(1,1,1)$ and $f(0,1,0) = f(0,1,1)$ or (iv) $f(1,0,0) = 0$ then we can obtain a limit theorem for l_n. Combining this with the previous result and using the methods of Section 3, it it is easy to show that if (i) or (ii), and (iii) or (iv) hold then the conclusion of Lemma 3.4 holds and we can in the same way prove the L^1 convergence of $|\eta_n|/n$. Since in a growth model either one of the conditions

(a) $$\rho(x,\eta) = f(\eta(x), \eta(x+1))$$

or

(b) $$\rho(x,\eta) = f(\eta(x-1), \eta(x), \eta(x+1)) \text{ and } f \text{ is constant on } \{\eta(x) = 1\}$$

are sufficient for (i) or (ii), and (iii) or (iv) ((a) implies (i) and (iv), (b) implies (i) and (iii)) we have the following result.

THEOREM 6.3. *If* (a) *or* (b) *is satisfied there is a constant* γ *so that*

$$|\eta_n|/n \to \gamma^1\Omega_\infty \ in\ L'.$$

At this point we have generalized the results of Sections 1-3 to discrete time. If the growth model is additive (see the definition in Section 4) then the results of Sections 4 and 5 also hold but some of the proofs (e.g., Lemma 4.2) need slight modifications. Details for a special case are given in Section 2 of Durrett and Liggett (1979).

Note added in proof. In the last year many of the problems mentioned above have been solved due to the work of David Griffeath alone (see [3]) and in conjunction with Maury Bramson and Larry Gray (see [16]). The reader should look at these papers to see the latest results on a condensed version of my paper (Section 4 of [3]).

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