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THE SHAPE OF THE LIMIT SET IN RICHARDSON'S GROWTH MODEL[1]

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Let C_p be the limiting shape of Richardson's growth model with parameter $p \in (0, 1]$. Our main result is that if p is sufficiently close to one, then C_p has a flat edge. This means that $\partial C_p \cap \{x \in R^2: x_1 + x_2 = 1\}$ is a nondegenerate interval. The value of p at which this first occurs is shown to be equal to the critical probability for a related contact process. For $p < 1$, we show that C_p is not the full diamond $\{x \in R^2: \|x\| = |x_1| + |x_2| \leq 1\}$. We also show that C_p is a continuous function of p, and that when properly rescaled, C_p converges as $p \to 0$ to the limiting shape for exponential site percolation.

1. Introduction and statement of results. In [10], Richardson introduced a discrete time growth model which may be described as follows:

(i) at time 0, all the points in Z^2 are colored white, except the origin, which is colored red;

(ii) if a site is red at time n, it remains red at all future times;

(iii) if a site $z \in Z^2$ is white at time n and all of its neighbors $z + (1, 0)$, $z + (0, 1)$, $z + (-1, 0)$ and $z + (0, -1)$ are white at that time, then z remains white at time $n + 1$;

(iv) if a site $z \in Z^2$ is white at time n and at least one of its neighbors is red at that time, then it becomes red at time $n + 1$ with probability p and remains white with probability $1 - p$.

Decisions in (iv) at different times and different sites are made independently. Richardson's main result in [10] is that the set of red sites has an asymptotic shape as $n \to \infty$. In order to state this result precisely, replace "red" and "white" by "1" and "0" respectively, and regard his model as a discrete time Markov process η_n with state space $\{0, 1\}^{Z^2}$. Thinking of η_n as a $\{0, 1\}$ valued function on Z^2, extend its definition to R^2 by letting $\eta_n(x)$ be the value of η_n at the $z \in Z^2$ which is closest to x, taking the maximum in case of ties. Let $A_n = \{x \in R^2: \eta_n(x) = 1\}$. The asymptotic shape result is then

THEOREM 1. (Richardson). *For any $p \in (0, 1)$, there is a norm φ_p on R^2 so that for any $\epsilon > 0$,*

$$\lim_{n\to\infty} P(\{\varphi_p \leq 1 - \epsilon\} \subset \frac{A_n}{n} \subset \{\varphi_p \leq 1 + \epsilon\}) = 1.$$

The norm in this result is defined by

$$\varphi_p(x) = \inf_{n\geq 1} \frac{Et_0(nx)}{n}$$

where $t_0(y) = \inf\{m: \eta_m(y) = 1\}$ is the first time the "infection" reaches y. It seems to be very difficult to use the above expression to compute $\varphi_p(x)$ for any $x \neq 0$ and $p \in (0, 1)$, and as far as we know, no one has succeeded in doing so, or even in saying very much about the norm in a qualitative way. In this paper we will prove some qualitative results about the unit ball $C_p = \{\varphi_p \leq 1\}$ by exploiting the relationship between Richardson's

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model and three other processes: percolation processes, contact processes, and branching random walks. Our results are described in Sections 1.1–1.3, and are organized primarily according to the auxiliary process which is used in the proofs. Several of the proofs are deferred to Section 2.

1.1 *Continuity results.* Richardson used computer simulations to get an idea of the shape of C_p for various p's. On the basis of these pictures, which are included in [10], he conjectured that "as p varies from 1 to 0 the unit ball of the associated norm varies from a diamond to a circle." As was observed by Cox in [2], one can use the connection between Richardson's model and certain percolation processes to prove the first part of Richardson's conjecture.

To describe this connection, we will now introduce planar site percolation processes. (In [2], Cox studies bond percolation, which is somewhat different). Let $\{\sigma(z), z \in Z^2\}$ be a collection of nonnegative i.i.d. random variables, where $\sigma(z)$ is interpreted as the time required to penetrate an obstruction at z. A path $r = (x_0, \cdots x_n)$ is a finite sequence of points in Z^2 such that $\| x_i - x_{i-1} \| = 1$ for $1 \le i \le n$. The travel time associated with the path r is defined by

$$t(r) = \sum_{i=1}^n \sigma(x_i).$$

The travel time from x to y for $x \neq y$ is

$$t(x, y) = \inf\{t(r): r \text{ is a path from } x \text{ to } y\},$$

while $t(x, x) = 0$ by convention. The connection with Richardson's model is that if $\sigma(z)$ is chosen to have a shifted geometric distribution with parameter p, that is the distribution with $P(\sigma(z)) = k) = p(1-p)^{k-1}$, $k \ge 1$, then

$$\{t(0, z), z \in Z^2\} =_{\mathscr{D}} \{t_0(z), z \in Z^2\}.$$

To prove this, let $t_b(z) = \min\{t_0(y): \| y - z \| = 1\}$, and observe that from the definition of Richardson's model, $\{t_0(z) - t_b(z), z \neq 0\}$ are i.i.d. random variables with the shifted geometric distribution. Therefore it suffices to show that $t(0, z) = t_0(z)$ as for $z \in Z^2$ where the percolation process is constructed using $\sigma(z) = t_0(z) - t_b(z)$ for $z \neq 0$ and $\sigma(0)$ some shifted geometric random variable which is independent of the others. If $r = (z_0, \cdots, z_n)$ is a path from 0 to $z \neq 0$, which can be taken without loss of generality to satisfy $z_i \neq 0$ for $i \ge 1$, then

$$t(r) = \sum_{i=1}^n \sigma(z_i) = \sum_{i=1}^n [t_0(z_i) - t_b(z_i)]$$
$$\ge \sum_{i=1}^n [t_0(z_i) - t_0(z_{i-1})] = t_0(z),$$

so that $t(0, z) \ge t_0(z)$. On the other hand, a path r from 0 to z can be chosen by letting z_{i-1} be determined from z_i by the requirement that $t_0(z_{i-1}) = t_b(z_i)$. This construction terminates at 0 after at most $t(0, z)$ steps since $\sigma(z) \ge 1$ a.s.

We will need the following two theorems from percolation theory. To simplify their statements, we have assumed that the distribution F has finite mean and satisfies $F(0) = 0$. Theorem 2 is a part of Theorem 3 in [3], and Theorem 3 is Theorem 1.14 in [2].

THEOREM 2. (Cox and Durrett). *Let $A_t = \{x: t(0, x) \le t\}$. Then there is a norm φ_F such that for each $\epsilon > 0$,*

$$P(\{\varphi_F \le 1 - \epsilon\} \subset t^{-1} A_t \subset \{\varphi_F \le 1 + \epsilon\} \text{ for sufficiently large } t) = 1.$$

THEOREM 3. (Cox). *Suppose that*
(i) *F_n converges weakly to F, and*
(ii) *there is a distribution U with finite mean so that*

$$1 - F_n(s) \le 1 - U(s) \text{ for all } n \text{ and } s.$$

Then $\varphi_{F_n}(x) \to \varphi_F(x)$ for all $x \in R^2$.

The proofs of these results in [2] and [3] are carried out for bond rather than site percolation, and in the case of Theorem 3, only for $x = (1, 0)$. The proof of Theorem 2 carries over immediately to site percolation. The proof of Theorem 3 for x with rational coordinates is essentially the same as that in [2]. To extend this to general x, one can use the relation

$$|\varphi_F(x) - \varphi_F(y)| \le \left[\int u\, dF(u)\right] \|x - y\|,$$

which is a consequence of the subadditivity of the travel times for the percolation process.

Using these results, we obtain immediately

THEOREM 4. *For all $x \in R^2$, $\varphi_p(x)$ is a continuous function of p in* $(0, 1]$.

Continuity at $p = 0$ in this form is not of very much interest, since as $p \to 0$, $\varphi_p(x) \to \infty$ for $x \neq 0$ and hence $C_p \to \{0\}$. With an appropriate rescaling, however, the geometric random variables converge to exponentials. Thus Theorems 2 and 3 imply also

THEOREM 5. *Let $H(x) = (1 - e^{-x})^+$. Then*

$$\lim_{p\to 0} p\varphi_p(x) \to \varphi_H(x)$$

for all $x \in R^2$.

For Richardson's full conjecture to be correct, it would have to be the case that $\varphi_H(x)$ is a constant multiple of $(x_1^2 + x_2^2)^{1/2}$. This would mean that the exponential distribution and the lattice structure of Z^2 would have to combine to produce a limit shape which is invariant under rotations. It appears that most people who have thought about this possibility now believe that it is unlikely to be the case.

1.2. *The flat edge result.* The results which were given in Section 1.1 were obtained by exploiting the relationship of Richardson's model to percolation processes. In this section, we will prove our main result by identifying an embedded contact process. For $x \in Z$, let $\xi_n(x) = \eta_n(x, n - x)$. Since a site on $x_1 + x_2 = n$ which is infected at time n must have just been infected by a site on $x_1 + x_2 = n - 1$, ξ_n is a Markov process which operates according to the following rules:

(i) if $\xi_{n-1}(x) = \xi_{n-1}(x - 1) = 0$, then $\xi_n(x) = 0$;

(ii) if $\xi_{n-1}(x) = 1$ or $\xi_{n-1}(x - 1) = 1$, then $\xi_n(x) = 1$ with probability p; and

(iii) in the transition from ξ_{n-1} to ξ_n, the decisions at all the sites are made independently.

The above process is, in the terminology of Harris [6], a discrete time contact process. It is known [7] that there is a $p_0 < 1$ so that if $p > p_0$, then $P(\Omega_\infty) > 0$, where $\Omega_\infty = \{\xi_n \not\equiv 0 \text{ for all } n\}$. Since

$$\{x \in Z^2 : \eta_n(x) = 1\} \supset \{(y, n - y) : \xi_n(y) = 1\}$$

$P(\Omega_\infty) > 0$ implies that $C_p \cap \{x : x_1 + x_2 = 1\} \neq \varnothing$, and by symmetry that $(\frac{1}{2}, \frac{1}{2}) \in C_p$. Since $\varphi_p(x) \ge \varphi_1(x) = |x_1| + |x_2|$, we have proved the following, in which $p_{cr} = \inf\{p : P(\Omega_\infty) > 0\} < 1$.

THEOREM 6. *If $p > p_{cr}$, then $(\frac{1}{2}, \frac{1}{2}) \in C_p$ and $\varphi_p(\frac{1}{2}, \frac{1}{2}) = 1$.*

More can be said if we use stronger results on contact processes. Theorem 7 below is a consequence of results in Section 6 of [4] (note that for this process, hypothesis (a) in [4] is satisfied). A continuous time analogue of Theorem 8 is proved in [4]. For the discrete time version, a new computation is required, so that a proof is given in Section 2. To state these theorems, we need the following notation:

$$r_n = \sup\{y:\xi_n(y) = 1\}$$

$$l_n = \inf\{y:\xi_n(y) = 1\}$$

$$\Omega_n = \{\xi_n \not\equiv 0\}.$$

THEOREM 7. (Durrett). *If $P(\Omega_\infty) > 0$, there are constants α and β so that*

$$\frac{r_n}{n} 1_{\Omega_n} \to \alpha 1_{\Omega_\infty}, \text{ and}$$

$$\frac{l_n}{n} 1_{\Omega_n} \to \beta 1_{\Omega_\infty}$$

almost surely and in L'.

THEOREM 8. (Durrett). *If $p > p_{cr}$, then $\alpha - \beta \geq 2[p - p_{cr}]$.*

Our main result is now an immediate consequence of Theorems 7 and 8:

THEOREM 9. *If $p > p_{cr}$, then $\partial C_p \cap \{x:x_1 + x_2 = 1\}$ is an interval of length $\geq 2\sqrt{2}$ $[p - p_{cr}]$.*

We conclude this subsection with two observations. The slightly weaker version of this theorem which would assert that for p sufficiently close to one, $\partial C_p \cap \{x:x_1 + x_2 = 1\}$ is a nondegenerate interval could have been obtained with some additional work by using Theorem 12.2 in [7] in place of Theorems 7 and 8. Secondly, even though we have proved Theorem 9 only for Richardson's model, simple comparisons give similar results for other growth models. For example, one could let the probability of a white site turning red if it has k red neighbors be a function p_k of k. One natural choice is $p_k = 1 - (1 - p)^k$. We would again get a flat edge result for p sufficiently close to one.

1.3. *Upper bounds; equality of critical values.* In this section, we will discuss results which can be obtained by comparing Richardson's model with some branching random walks. For the first results, consider the branching random walk in which a particle at x at time n gives rise at time $n + 1$ to a particle at x with probability one and to a particle at each of the four neighbors of x independently with probability p per neighbor. Let $N_n(x)$ be the number of particles at x in this branching random walk when initially there is a single particle at the origin. Then N_n and η_n can be coupled so that $N_n(x) \geq \eta_n(x)$ for all n and x. Therefore $\{x:N_n(x) \geq 1\} \supset \{x:\eta_n(x) = 1\}$, so that the set of occupied sites in the branching random walk gives an upper bound for A_n in Richardson's model.

Biggins [1] has proved a result which describes the asymptotic shape of multidimensional branching random walks. In order to state it, we need some notation. Let m_p be the measure which assigns mass 1 to (0, 0), mass p to each of the points (1, 0), (0, 1), (−1, 0), (0, −1), and mass 0 to the rest of R^2. This is the mean measure of the branching random walk described above. Let

$$k(\theta) = \log\left(\int e^{-\langle\theta,x\rangle}\, dm_p(x)\right), \quad \text{and}$$

$$k^*(y) = \inf\{k(\theta) + \langle\theta, y\rangle:\theta \in R^2\}.$$

We have used the fact that our functions k and k^* are well behaved to simplify the following statement:

THEOREM 10. (Biggins). *Let H_n be the convex hull of $\{x:N_n(x) > 0\}$, and let $D = \{y:$*

$k^*(y) \geq 0\}$. *Then*

$$\liminf \frac{H_n}{n} = \limsup \frac{H_n}{n} = D.$$

While this gives an "explicit" upper bound for C_p, it is hard to use because of the two dimensional minimizations required in the computation of D. However, the information we want can be obtained by projecting D first on the line $x_2 = 0$ and then on the line $x_1 = x_2$. By doing this, we obtain problems concerning the one-dimensional branching random walks with respective mean measures

(a) $$m(0) = 1 + 2p, \qquad m(-1) = m(1) = p,$$

and

(b) $$m(0) = 1, \qquad m(-1) = m(1) = 2p.$$

The asymptotic shapes of these branching random walks can be computed in principle from the one dimensional version of Theorem 10. The computations are still difficult, so we will use the following result, which is proved in Section 2.

THEOREM 11. *If a one-dimensional branching random walk on the integers has a mean measure with* $m(A, \infty) = 0$, $m(\{A\}) < 1$, *and* $m(-\infty, A) < \infty$, *then the limit set D is an interval lying strictly to the left of A.*

Applications of Theorem 11 to the branching random walks with mean measures in (a) and (b) above give respectively the following results. The first of these was mentioned to us by H. Kesten.

THEOREM 12. *If* $p < 1$, *then* $\sup\{x_1 : x \in C_p\} < 1$, *so that* C_p *is not the full diamond* $\{|x_1| + |x_2| \leq 1\}$.

THEOREM 13. *If* $p < \frac{1}{2}$, *then* $\sup\{|x_1| + |x_2| : x \in C_p\} < 1$, *so that* C_p *lies strictly inside the diamond* $\{|x_1| + |x_2| \leq 1\}$.

Let $p_I = \sup\{p : C_p \cap \{x : |x_1| + |x_2| = 1\} = \emptyset\}$. By combining the results of Theorems 6 and 13, it follows that

$$\tfrac{1}{2} \leq P_I \leq p_{cr}.$$

The lower bound can be improved if we follow η for two units of time before making the branching random walk approximation. To see the estimate that gives, note that $P[\eta_2(0, 2) = 1] = p[\eta_2(2, 0) = 1] = p^2$ and $p[\eta_2(1, 1) = 1] = 2p^2 - p^4$, so that the mean measure of the projection on the line $x_1 = x_2$ has $m(2) = 4p^2 - p^4$. This is less than 1 if $p < (2 - \sqrt{3})^{1/2}$, so that $P_I \geq .518$. This procedure can be continued, of course, but the lower bounds for p_I increase very slowly and require large amounts of computation. Let p_J be the supremum of the lower bounds obtained by observing η_n for increasingly long periods before making the branching walk approximation. Then clearly $p_J \leq p_I \leq p_{cr}$. We will show that equality holds throughout by using the following result which is due to Griffeath [5]. We will give a proof of his result in Section 2 because it as yet unpublished.

THEOREM 14. (Griffeath). *If* $p < p_{cr}$, *then there are numbers* $A < \infty$ *and* $b > 0$ *depending on p so that*

$$P[\xi_n \not\equiv 0] \leq Ae^{-bn}.$$

THEOREM 15. $p_J = p_I = p_{cr}$.

PROOF. Let $|\xi_n| = |\{x : \xi_n(x) = 1\}|$. Since $|\xi_n| \leq n$, Theorem 14 implies that for $p < p_{cr}$, $\lim_{n\to\infty} E|\xi_n| = 0$. Choose an N so that $E|\xi_N| < 1$, and apply the branching walk approximation after η_n has evolved for N steps. This gives $p \le p_J$. Since $p < p_{cr}$ was arbitrary, $p_J \ge p_{cr}$, which proves the result.

Note that we now have the following situation:

$$C_p \cap \{x : x_1 + x_2 = 1\} = \emptyset \qquad \text{if } p < p_{cr}$$

$$C_p \cap \{x : x_1 + x_2 = 1\} \text{ is a nondegenerate interval} \qquad \text{if } p > p_{cr}.$$

2. Proofs. This section is devoted to the proofs which were omitted from Section 1.

THEOREM 8. *If $p > p_{cr}$, then $\alpha - \beta \ge 2[p - p_{cr}]$.*

PROOF. The proof follows the outline of that of the analogous result for continuous time contact processes in Section 4 of [4]. The key to that proof was an additivity equation which was a consequence of a graphical representation of the process [7]. To introduce the corresponding construction in discrete time, let V be the graph $Z \times \{0, 1, 2, \cdots\}$ with directed edges from each (m, n) to $(m, n + 1)$ and to $(m + 1, n + 1)$. Each vertex (m, n) with $n \ge 1$ is independently marked open with probability p and closed with probability $1 - p$. The contact process ξ_n^A with initial configuration 1_A can then be defined on this structure by letting $\xi_n^A(x) = 1$ exactly if there is a path along directed edges passing through open vertices from some point $(y, 0)$ with $y \in A$ to (x, n). This family of processes has the additivity property

$$\xi_n^{A \cup B}(x) = \xi_n^A(x) \vee \xi_n^B(x),$$

where $a \vee b = \max\{a, b\}$. Let $r_n^A = \sup\{x : \xi_n^A(x) = 1\}$. The proof of Lemma 4.1 in [4] can now be used to show that if $B \subset (-\infty, 1]$ is infinite, then

$$E(r_n^{B \cup \{0\}} - r_n^B) \ge 1,$$

so that adding a one to the right of all the ones in the initial configuration has the effect of increasing the expected location of the rightmost one at time n by at least one.

Using the above fact, we can complete the proof by following the proof of Lemma 4.2 in [4]. Let ξ_n^p denote the contact process with parameter p and initial value $1_{(-\infty;0]}$, and let $r_n^p = \sup\{y : \xi_n^p(y) = 1\}$. If $\delta > 0$, $\xi_n^{p+\delta}$ and ξ_n^p can be constructed on the same state space in such a way that $\xi_n^{p+\delta}(x) \ge \xi_n^p(x)$ for all n and x, so that $r_n^{p+\delta} \ge r_n^p$. Let $\tau = \inf\{m \ge 1 : r_m^{p+\delta} > r_m^p\}$. To compare $r_n^{p+\delta}$ and r_n^p on $\{\tau \le n\}$, it is convenient to introduce a process $\hat{\xi}_m$ which agrees with $\xi_m^{p+\delta}$ for $m \le \tau$ and evolves according to the rules of ξ_n^p for $m > \tau$. Since τ is a stopping time, $\hat{\xi}_m$ can be constructed on the same space with $\xi_m^{p+\delta}$ and ξ_m^p in such a way that $\xi_m^{p+\delta} \ge \hat{\xi}_m \ge \xi_m^p$ for all $m \le n$. Letting $\hat{r}_n = \sup\{y : \hat{\xi}_n(y) = 1\}$, it then follows that

$$E(r_n^{p+\delta} - r_n^p) \ge E(\hat{r}_n - r_n^p; \tau \le n).$$

On the other hand, as shown above,

$$E(\hat{r}_N - r_N^p; \tau \le n) \ge P(\tau \le n).$$

Since the probability of a birth at $r_n^{p+\delta} + 1$ in $\xi_n^{p+\delta}$ is always δ greater than the probability of a birth at $r_n^p + 1$ in ξ_n^p, we have $P(\tau \le n) \ge 1 - (1 - \delta)^n$, so that

$$E(r_n^{p+\delta} - r_n^p) \ge 1 - (1 - \delta)^n.$$

Replacing δ by δ/k and summing gives

$$E(r_n^{p+\delta} - r_n^p) \ge k\left[1 - \left(1 - \frac{\delta}{k}\right)^n\right].$$

Letting $k \to \infty$ gives

$$\frac{1}{n} E[r_n^{p+\delta} - r_n^p] \ge \delta,$$

so that $\alpha(\Delta + \delta) - \alpha(p) \geq \delta$. Here we have used the fact that just as in the continuous time case, the α which occurs in the statement of Theorem 8 is equal to $\lim_{n\to\infty}(1/n)Er_n$ when $p > p_{cr}$. Since $\alpha + \beta = 1$ by a symmetry argument, the result follows.

Theorem 11. *If a one dimensional branching random walk on the integers has a mean measure with $m(A, \infty) = 0$, $m(\{A\}) < 1$, and $m(-\infty, A) < \infty$, then the limit set D is an interval lying strictly to the left of A.*

Proof. By monoticity, it suffices to consider the case $m(\{A\}) = p < 1$ and $m(\{A - 1\}) = M < \infty$, with m putting no mass on the complement of $\{A - 1, A\}$. In this case,

$$\int e^{-\theta x}\,dm = pe^{-\theta A} + Me^{-\theta(A-1)},$$

so

$$\begin{aligned} k(\theta) + \theta y &= \log[pe^{-\theta A} + Me^{-\theta(A-1)}] + \theta y \\ &= \log[pe^{\theta(y-A)} + Me^{\theta(y+1-A)}]. \end{aligned}$$

Pick $\theta_0 < 0$ so that $Me^{(1/2)\theta_0} < 1 - p$ and pick $y_0 \in (A - \tfrac{1}{2}, A)$ so that

$$p(e^{\theta_0(y_0-A)} - 1) < 1 - p - Me^{(1/2)\theta_0}.$$

Then if $y > y_0$,

$$pe^{\theta_0(y-A)} + Me^{\theta_0(y-A+1)} < 1,$$

which gives $k^*(y) < 0$. Therefore $D \subset (-\infty, y_0]$.

Theorem 14. *If $p < p_{cr}$, then there are numbers $A < \infty$ and $b > 0$ depending on p so that*

$$P[\xi_n \not\equiv 0] \leq Ae^{-bn}.$$

Proof. Let ξ_n^R and ξ_n^L be contact processes with initial values $\xi_0^R = 1_{(-\infty,0]}$ and $\xi_0^L = 1_{[0,\infty)}$, and put $\bar{r}_n = \sup\{y: \xi_n^R(y) = 1\}$ and $\bar{l}_n = \inf\{y: \xi_n^L(y) = 1\}$. By results in Section 6 of [4], there are constants α and β with $-\infty \leq \alpha \leq 1$ and $0 \leq \beta \leq \infty$ so that $\bar{r}_n/n \to \alpha$ and $\bar{l}_n/n \to \beta$ a.s. These are the same constants which appear in Theorem 7 when $P(\Omega_\infty) > 0$. Note that still $\alpha + \beta = 1$ since $r_n = n - l_n$ in distribution. By using the construction in the proof of Theorem 8, ξ_n^L, ξ_n^R and ξ_n can be constructed on the same space in such a way that on $\Omega_n = \{\xi_n \not\equiv 0\}$,

$$(*)\qquad \xi_n(x) = \xi_n^L(x) = \xi_n^R(x) \quad \text{for } \bar{l}_n \leq x \leq \bar{r}_n$$

and

$$\xi_n(x) = 0 \quad \text{for all other } x\text{'s}.$$

Since $P(\Omega_\infty) = P(\xi_n \not\equiv 0 \text{ for all } n) = P(\bar{l}_n \leq \bar{r}_n \text{ for all } n)$, it follows that $\alpha < \beta$ implies $P(\Omega_\infty) = 0$ and $\alpha > \beta$ implies $P(\Omega_\infty) > 0$. We are now prepared to obtain the following useful characterization of p_{cr} which was discovered by Griffeath:

Lemma. *If $\alpha_n(p) = (1/n)E\bar{r}_n$ and $p_n = \sup\{p: \alpha_n(p) < \tfrac{1}{2}\}$, then $p_{cr} = \sup_n p_n$.*

Proof. Let $q = \sup_n p_n$. $\alpha(p) = \inf_n \alpha_n(p)/n$, so $p < p_n$ implies $\alpha(p) < \tfrac{1}{2}$ and $\beta(p) = 1 - \alpha(p) > \tfrac{1}{2}$, and hence $P(\Omega_\infty) = 0$. This shows that $p_{cr} \geq q$. Now if $p > p' > q$, then $\alpha(p') \geq \tfrac{1}{2}$. By the proof of Theorem 8, this gives $\alpha(p) - \alpha(p') \geq p - p'$, so that $\alpha(p) > \tfrac{1}{2}$, and hence $p > p_{cr}$. Hence $p_{cr} = q$.

Returning to the proof of Theorem 14, assume $p < p_{cr}$. By the lemma, there is an N so that $p < p_N$. Let $\bar{\xi}_n^N$ be a process with initial configuration $1_{(-\infty,0]}$ which evolves like a contact process except at times kN, $k = 1, 2, \ldots$. At those times, after the transition from time $kN - 1$ has been made, the process is reset to 1 at all points to the left of the rightmost one. (This is one of the reset processes used in [4]). Let $\bar{r}_n^N = \sup\{y: \bar{\xi}_n^N(y) = 1\}$. As a result of the choice of N and the resetting, $\bar{r}_n^N \geq \bar{r}_n$, and the increments $\{\bar{r}_{kN}^N - \bar{r}_{(k-1)N}^N, k \geq 1\}$ are independent and identically distributed random variables with mean $\mu_n \leq N/2$. Since the distribution of $\bar{r}_N^N$ is bounded above by N, a standard large deviations estimate ([9]) shows that if $\rho > \mu_N$, there is a $C < \infty$ and a $d > 0$ which depends on ρ so that $P(\bar{r}_{kN}^N \geq \rho k) \leq Ce^{-dk}$. A similar estimate holds for $\bar{l}_{kN}^N$. Combining these two with (*) completes the proof of the theorem.

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