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A NEW PROOF OF SPITZER'S RESULT ON THE WINDING OF TWO DIMENSIONAL BROWNIAN MOTION[1]

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Let $W(t)$ be a two dimensional Brownian motion with $W(0) = (1, 0)$ and let $\varphi(t)$ be the net number of times the path has wound around $(0, 0)$, counting clockwise loops as -1, counterclockwise as $+1$. Spitzer has shown that as $t \to \infty$, $4\pi\varphi(t)/\log t$ converges to a Cauchy distribution with parameter 1. In this paper we will use Levy's result on the conformal invariance of Brownian motion to give a simple proof of Spitzer's theorem.

Let B_t^1 and B_t^2 be two independent Brownian motions with $B_0^1 = 1$ and $B_0^2 = 0$, and let $C_t = B_t^1 + iB_t^2$ be a complex Brownian motion. Since C_t almost surely never hits 0, we can define the total angle swept out up to time t to be the unique process θ_t with continuous paths which has $\theta_0 = 0$ and $\sin(\theta_t) = B_t^2/|C_t|$ for all $t > 0$. In words, the process θ_t records the angle and keeps track of the number of times the path has wound around 0, counting clockwise loops as -2π and counterclockwise loops as $+2\pi$. Spitzer (1958) proved the following limit theorem for θ_t.

Theorem 1. *As* $t \to \infty$

$$P(2\theta_t/\log t \le y) \to \int_{-\infty}^{y} \frac{dx}{1 + x^2}.$$

Spitzer's proof is ingenious but requires a lot of computation; see Ito and McKean (1965) pages 270–271 for a succinct version. He introduces the double transform $f(\alpha, \beta, x) = \int_0^\infty e^{-\beta t} E_x(exp(i\alpha\theta_t))dt$ and observes that $f(\alpha, \beta, x) = g_{\alpha,\beta}(|x|)$ where $g_{\alpha,\beta}$ satisfies

$$\frac{1}{2}\left(g''(r) + \frac{1}{r}g'(r) - \frac{\alpha^2}{r^2}g(r)\right) - \beta g(r) = -1.$$

The last equation is solved to obtain an explicit formula for $g_{\alpha,\beta}$ which, with the help of Erdélyi (1954), is inverted in terms of Bessel functions. With an explicit formula for $E_x(\exp(i\alpha\theta_t))$, the last step is to let $t \to \infty$ and use a few facts about Bessel functions to complete the proof.

As the reader can imagine, filling in all the details in the outline above requires some work. The purpose of this note is to give a computationally simple proof of Spitzer's theorem based on a result of Lévy (1948) which we now pause to describe. (The reader might enjoy looking at Lévy's original "proof" (see page 270) for an intuitive but nonrigorous explanation. A complete proof (with a serious typo in the final formula) can be found in McKean (1969) on page 109.)

Let f be a nonconstant function which is analytic on the entire complex plane, and let D_t be a complex Brownian motion which starts at 0. Lévy's result is

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THEOREM 2. *$F_t = f(D_t)$ is a time change of a complex Brownian motion starting at $f(0)$. To be precise, if we let*

$$\sigma_t = \int_0^t |f'(D_s)|^2 \, ds$$

and

$$\gamma_t = \inf\{s : \sigma_s \geq t\},$$

then $F(\gamma_t)$ has the same distribution as a complex Brownian motion starting at $f(0)$.

We will use this result with $f(z) = e^{iz}$. *Since* $e^0 = 1$, so after the time change, $F_t = e^{iD_t}$ has the same distribution as C_t. The first advantage of constructing C_t as $F(\gamma_t)$ is that we can write down the angle process for $F(\gamma_t)$ — if $A_t = ReD_t$ and $B_t = ImD_t$, then $\theta_t = B(\gamma_t)$. The second and more crucial feature of this representation is that

$$\sigma_t = \int_0^t |\exp(iD_s)|^2 \, ds = \int_0^t \exp(2A_s) \, ds$$

so γ and B are independent.

Combining the last observation with the scaling relationship $cB_t =_d B(c^2 t)$ reduces the proof of Theorem 1 to studying the limiting behavior for $4\gamma_t/(\log t)^2$ (more details on this point are given at the end of the proof). The first step in doing this is to realize

$$P(\gamma_t \leq y(\log t)^2/4) = P(\sigma(y(\log t)^2/4) \geq t)$$

and changing variables $t = e^u$ gives

$$P(\sigma(y(\log t)^2/4) \geq t) = P(\sigma(yu^2/4) \geq e^u) = P(\log \sigma(yu^2/4) \geq u).$$

To compute the limit of the last quantity, we use the following.

LEMMA. *Let $M_t = max_{0 \leq s \leq t} A_s$. As $t \to \infty$*

$$(\log \sigma(t))/2M(t) \to 1 \textit{ in probability.}$$

PROOF. To get an upper bound, observe that $\log \sigma(t) \leq \log(t \exp(2M_t))$ so

$$\frac{\log \sigma(t)}{2M(t)} \leq 1 + \frac{\log t}{2M(t)}.$$

From the scaling relationship $M(t) =_d t^{1/2} M(1)$, so we have shown that for all $\varepsilon > 0$

$$P\left(\frac{\log \sigma(t)}{2M(t)} \leq 1 + \varepsilon\right) \to 1 \quad \text{as} \quad t \to \infty.$$

This proves half of the lemma. To prove the other half, observe that if we let $L_\varepsilon(t)$ be the Lebesgue measure of $\{s \in [0, t] : A_s \geq (1 - \varepsilon)M_t\}$, then $\sigma(t) \geq \exp((2 - 2\varepsilon)M(t))L_\varepsilon(t)$ so

$$\frac{\log \sigma(t)}{2M(t)} \geq 1 - \varepsilon + \frac{\log L_\varepsilon(t)}{2M(t)}.$$

From the scaling relationship we see that $(L_\varepsilon(t), M(t)) =_d (tL_\varepsilon(1), t^{1/2}M(1))$ so

$$P\left(\frac{\log \sigma(t)}{2M(t)} \geq 1 - 2\varepsilon\right) \geq P\left(\frac{\log(tL_\varepsilon(1))}{t^{1/2}M(1)} \geq -\varepsilon\right) \to 1$$

as $t \to \infty$, completing the proof of the lemma.

With the lemma proved, it is routine to complete the proof. Write

$$P(\log \sigma(yu^2/4) \geq u) = P\left(\frac{\log\sigma(yu^2/4)}{2M(yu^2/4)} \cdot \frac{M(yu^2/4)}{(u/2)} \geq 1\right).$$

From the lemma and the scaling relationship, it follows that if $\delta > 0$, then for u sufficiently large the right hand side is

$$\geq P\left((1-\delta)\frac{M(yu^2/4)}{u/2} \geq 1\right) - \delta = P\left(M(y) \geq \frac{1}{1-\delta}\right) - \delta.$$

Since δ is arbitrary and $P(M(y) = 1) = 0$, it follows that

$$\liminf_{u\to\infty} P(\log \sigma(yu^2/4) \geq u) \geq P(M(y) \geq 1).$$

A similar argument using $1 + \delta$ in place of $1 - \delta$ shows that the lim sup $\leq P(M(y) \geq 1)$, so as $u \to \infty$

$$P(\log \sigma(yu^2/4) \geq u) \to P(M(y) \geq 1)$$

and using the computation above the lemma, we have

$$P(\gamma(t) \leq y(\log t)^2/4) \to P(M(y) \geq 1).$$

Introducing the random variable $T_1 = \inf\{t > 0 : A_t = 1\}$, we can write the last conclusion as

$$P(\gamma(t) \leq y(\log t)^2/4) \to P(T_1 \leq y).$$

The last result gives the limiting behavior of $\gamma(t)$. To obtain the conclusion of the theorem, we observe that

$$P(2\theta(t)/\log t \leq y) = P(2B(\gamma_t)/\log t \leq y)$$

and since B and γ are independent, it follows from the scaling relationship that the right hand side

$$= P(B(4\gamma_t/(\log t)^2) \leq y).$$

Since Brownian motion has continuous paths, it follows now that as $t \to \infty$

$$P(2\theta(t)/\log t \leq y) \to P(B(T_1) \leq y).$$

The last term, being the hitting distribution of the line $\{(x, y) : x = 1\}$ for a two dimensional Brownian motion, is known to be a Cauchy distribution with parameter 1, so the proof is complete.

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