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Downcrossings and Local Time

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Let $\{W(t): t \geqq 0\}$ be the standard Brownian motion with all paths continuous. Let $M(t) = \max_{0 \leqq s \leqq t} W(s)$ be the maximum process and $Y(t) = M(t) - W(t)$ be reflecting Brownian motion. If $d_\varepsilon(t)$ is the number of times Y crosses down from ε to 0 before time t, then it was Paul Lévy's idea that

$$P\{\lim_{\varepsilon \to 0} \varepsilon d_\varepsilon(t) = M(t) \text{ for all } t \geqq 0\} = 1. \tag{1}$$

In [3] Itô and McKean demonstrated the almost sure convergence of $\varepsilon d_\varepsilon(t)$ using martingale methods. To identify the limit they used the hard fact, due to Lévy, that

$$P\{\lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \text{ measure } \{s: Y(s) < \varepsilon, s \leqq t\} = M(t) \text{ for all } t \geqq 0\} = 1 \tag{2}$$

and computed the second moment of the difference of the expressions in (1) and (2). In this paper, by examining the excursions in Brownian motion and using a new formula for the distribution of their maxima, we obtain a direct identification of the limit in (1) without using (2).

Let $T_x = \inf\{t: W(t) = x\}$, $T'_x = \inf\{t: Y(t) = x\}$. For $a > 0$ let $R_0^a = 0$, $R_1^a = T'_a + T'_0 \circ \theta_{T'_a}$, and for $n \geqq 2$ let $R_n^a = R_{n-1}^a + R_1^a \circ \theta_{R_{n-1}^a}$. Here $\{\theta_t, t \geqq 0\}$ is the usual collection of shift operators: $W(s, \theta_t \omega) = W(s+t, \omega)$ and if S is a random variable, $\theta_S = \theta_t$ on $\{S = t\}$. If S is a random variable, let $d_a(S) = \sup\{n: R_n^a \leqq S\}$. $d_a(S)$ is the number of downcrossings of $(0, a)$ by Y before time S.

Scaling shows that $d_{\varepsilon/m}(T_a)$ and $d_\varepsilon(T_{ma})$ have the same distribution. Using the strong Markov property $d_\varepsilon(T_{ma})$ is the sum of m independent random variables with the same distribution as $d_\varepsilon(T_a)$ so from the strong law of large numbers $\frac{\varepsilon}{m} d_{\varepsilon/m}(T_a)$ converges in probability to $E(\varepsilon d_\varepsilon(T_a))$ as $m \to \infty$.

To compute that $E(\varepsilon d_\varepsilon(T_a)) = a$ we examine the excursions in Brownian motion: (α, β) is an excursion interval of the path $Y(\cdot, \omega)$ if $\alpha < \beta$, $Y(\alpha, \omega) = 0 = Y(\beta, \omega)$ and $Y(s, \omega) > 0$ for $\alpha < s < \beta$; $\{Y(s, \omega), \alpha \leqq s \leqq \beta\}$ is called an excursion if

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(α, β) is an excursion interval – see [1] for a more complete discussion. Observe that we can count the number of down-crossings of $(0, \varepsilon)$ by Y before T_a by counting the number of excursions in $[0, T_a]$ with maxima $\geqq \varepsilon$. The advantage of this viewpoint is that the excursions in $[0, T_a]$ when scaled and suitably enumerated are independent and have the same law. To state this result precisely we need to introduce the enumeration of the excursions given in [3] on page 75. Let $Z(\omega)=\{t: Y(t, \omega)=0\}$. Since Y has continuous paths, Z is a closed subset of $[0, \infty)$. Let (γ_n, β_n) be the open interval of $[0, \infty)-Z$ containing the first number of the list $1, \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, 3, \ldots$ not included in Z or $\bigcup_{m<n} (\gamma_m, \beta_m)$. Let $e_n(t)=Y(\gamma_n+t\Delta_n)/\Delta_n^{\frac{1}{2}}$ where $\Delta_n=\beta_n-\gamma_n$. Now, if we modify j_1 of (4) on page 76 of [3] to be a function of $[(Y(s), M(s));\ s \leqq \gamma_1]$, then the proof on pages 75–78 gives that $\{e_n;\ n \geqq 1\}$ is independent of $\{(\gamma_n, \beta_n);\ n \geqq 1\}$ and M, so if we let $N_0=0$ and for $n \geqq 1$, $N_n=\inf\{k>N_{n-1};\ \beta_k<T_a\}$ and define $e'_n=e_{N_n}$, then $\{e'_n:\ n \geqq 1\}$ are independent, and each has the same law as e_1. Further, if $\Delta'_n=\beta_{N_n}-\gamma_{N_n}$, then $\{e'_n;\ n \geqq 1\}$ and $\{\Delta'_n;\ n \geqq 1\}$ are independent since $\{e_n;\ n \geqq 1\}$ and $\{\Delta_n;\ n \geqq 1\}$ are, and N_n is determined by $\{(\gamma_n, \beta_n);\ n \geqq 1\}$ and M.

With the preliminaries on independence established, we are ready to compute the desired expectation. If we let $M'_j=\sup_{0 \leqq s \leqq 1} e'_j(s)$, then from [1] (4.5, p. 23) or [2] (5.1, p. 21) we have

$$F(x)=P(M'_j \leqq x)=1-2\sum_{n=1}^{\infty}(4n^2x^2-1)\exp(-2n^2x^2)$$

and since e'_n and Δ'_n are independent,

$$E(\varepsilon d_\varepsilon(T_a))=\varepsilon\sum_{j=1}^{\infty}\int_0^{\infty} P(M'_j>\varepsilon u^{-\frac{1}{2}})\,P(\Delta'_j \in du).$$

Now the excursion intervals $(\gamma_{N_n}, \beta_{N_n})$ correspond to jumps of the passage time process $\{T_x: x \leqq a\}$, so from the Lévy decomposition ((12), p. 27 in [3]) we know that $a(2\pi u^3)^{-\frac{1}{2}}du$ is the expected number of Δ'_n with length in $(u, u+du)$, and using Fubini's theorem converts the above formula to

$$E(\varepsilon d_\varepsilon(T_a))=2a\varepsilon\int_0^{\infty}\sum_{n=1}^{\infty}\left(\frac{4n^2\varepsilon^2}{u}-1\right)\exp\left(-\frac{2n^2\varepsilon^2}{u}\right)(2\pi u^3)^{-\frac{1}{2}}du.$$

Computing the above integral requires some care, because a haphazard integration term by term gives the absurdity $E(\varepsilon d_\varepsilon(T_a))=0$. However, if we integrate only on $[0, K\varepsilon^2]$, then for $n \geqq K^{\frac{1}{2}}/2$ the summand in the integral is nonnegative on $[0, K\varepsilon^2]$ so we can invoke monotone convergence. To integrate the n-th term of the sum let $\alpha=4n^2\varepsilon^2$, change variables $x=\alpha/2u$ and integrate the second integral of the result by parts to get

$$\begin{aligned}2a\varepsilon\int_0^{K\varepsilon^2}\left(\frac{\alpha}{u}-1\right)e^{-\alpha/2u}(2\pi u^3)^{-\frac{1}{2}}du&=2\alpha\varepsilon(\pi\alpha)^{-\frac{1}{2}}\int_{\alpha/2K\varepsilon^2}^{\infty}(2x^{\frac{1}{2}}-x^{-\frac{1}{2}})e^{-x}dx\\&=a(8/\pi K)^{\frac{1}{2}}e^{-2n^2/K}.\end{aligned}$$

The remaining term

$$\int_{K\varepsilon^2}^{\infty}[1-F(\varepsilon n^{-\frac{1}{2}})](2\pi u^3)^{-\frac{1}{2}}du \leqq (2/\pi K)^{\frac{1}{2}}$$

so

$$E(\varepsilon d_\varepsilon(T_a)) = \lim_{K\to\infty} a(2/\pi K)^{\frac{1}{2}} \left[1 + 2\sum_{n=1}^{\infty} e^{-2n^2/K}\right].$$

Recognizing the term in brackets as Jacobi's theta function evaluated at $2/\pi K$ and using the identity $\theta(t) = t^{-\frac{1}{2}}\theta(t^{-1})$ we get

$$E(\varepsilon d_\varepsilon(T_a)) = \lim_{K\to\infty} a\theta\left(\frac{\pi K}{2}\right) = a.$$

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