

Contact Processes in Several Dimensions

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1. Introduction

This paper deals primarily with the basic contact processes introduced and studied by Harris [12–14]. These are random evolutions on the state space $S = \{\text{all subsets of } Z^d\}$ ($Z^d = \text{the } d\text{-dimensional integer lattice}$) with extremely simple local dynamics. Namely, if we think of the contact process (ξ_t) as representing the spread of an infection, $\xi_t \in S$ denoting the set of infected sites at time t , then $x \in \xi_t$ becomes healthy at exponential rate 1 while $x \in Z^d - \xi_t$ is infected at a rate proportional to the number of sites neighboring x where infection is present. The proportionality constant λ is called the infection parameter. Contact processes are perhaps the simplest S -valued Markov processes which exhibit a “critical phenomenon”: infection emanating from a single site dies out with probability one for small positive λ , but has positive probability of surviving for all time when λ is large. There is a critical value $\lambda_c^{(d)}$ where the “phase transition” occurs. Principal objectives of analysis are the precise formulation of the critical phenomenon, and detailed description of the ergodic behavior both below and above $\lambda_c^{(d)}$.

Our recent survey articles [10] and [6] provide introductions to contact processes, and to the general field of interacting S -valued systems, respectively. We will make constant use of notation and techniques from the surveys, assuming that the reader is familiar with them. The article [10] describes in some detail the current state of knowledge concerning one dimensional contact processes. The theory for $d=1$ is now fairly complete with two major exceptions: $\lambda_c^{(1)}$ is not even determined to one decimal place ($1.18 \leq \lambda_c^{(1)} \leq 2$ are rigorous bounds), and very little is known about the behavior of the critical contact process (the case $\lambda = \lambda_c^{(1)}$).

If $(\xi_t(\lambda))$ is the d -dimensional basic contact process with parameter λ , starting at time 0 with infection everywhere on Z^d , then for $\lambda > \lambda_c^{(d)}$ there is an

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invariant measure $\nu_\lambda \neq \delta_\emptyset$ (δ_\emptyset = “all healthy”) such that as $t \rightarrow \infty$ $\xi_t(\lambda) \Rightarrow \nu_\lambda$, i.e. $\xi_t(\lambda)$ converges weakly to ν_λ . One of the fundamental results which has been proved for $d=1$ is called the complete convergence theorem. It asserts that if $(\xi_t^A(\lambda))$ is the contact process with parameter λ and initial state $A \in \mathcal{S}$, and if τ^A is the time that (ξ_t^A) first hits the trap \emptyset , then

$$\xi_t^A \Rightarrow P(\tau^A < \infty) \delta_\emptyset + P(\tau^A = \infty) \nu_\lambda \tag{1}$$

for any $A \in \mathcal{S}$. This shows, in particular, that all the invariant measures for the Markov system $\{(\xi_t^A); A \in \mathcal{S}\}$ are convex combinations of δ_\emptyset and ν_λ . To prove (1), one studies the process (ξ_t^0) which starts with a single infected site at the origin. For $\lambda > \lambda_c$, this process lives forever with positive probability, and (1) with $A = \{0\}$ can be rewritten as

$$(\xi_t^0 | \tau^0 = \infty) \Rightarrow \nu_\lambda. \tag{1'}$$

It is not hard to deduce (1) from (1'), and in attempting to prove (1') it is natural to investigate the almost sure behavior of $(\bar{\xi}_t^0) = (\xi_t^0 | \tau^0 = \infty)$. Let

$$l_t^0 = \min \{x : x \in \xi_t^0\}, \quad r_t^0 = \max \{x : x \in \xi_t^0\}.$$

The edge processes (l_t^0) and (r_t^0) are defined for all t on $\{\tau^0 = \infty\}$. In [5] it is proved that for $\lambda > \lambda_c$ there is an asymptotic speed $\alpha(\lambda) > 0$ such that

$$\lim_{t \rightarrow \infty} -t^{-1} l_t^0 = \lim_{t \rightarrow \infty} t^{-1} r_t^0 = \alpha(\lambda) \quad \text{a.s. on } \{\tau = \infty\}. \tag{2}$$

Moreover, a copy of the contact process (ξ_t) can be defined on the same probability space with (ξ_t^0) in such a way that

$$\xi_t^0 = \xi_t \cap [l_t^0, r_t^0] \quad \text{for all } t \quad \text{a.s. on } \{\tau^0 = \infty\}. \tag{3}$$

Thus ξ_t^0 is “coupled” to ξ_t on a (random) interval of linearly increasing diameter; (1') follows rather easily from (2) and (3). See [5] for details.

In two or more dimensions, (1) has not been proved for any value of λ . The only known result along these lines, due to Harris [13], states that if μ is a translation invariant probability measure on \mathcal{S} , then for any $\lambda > \lambda_c$ the contact process $(\xi_t^\mu(\lambda))$ with initial distribution μ satisfies the analogue of (1). One of the main goals of this paper is to prove complete convergence for contact processes in several dimensions. The approach we adopt involves a multi-dimensional version of (2)–(3), i.e. an almost sure linear growth theorem for $(\bar{\xi}_t^0)$. As for $d=1$, (1) will follow as a corollary. Intuitively, it seems most plausible that the infection should expand linearly “in radius” with time, acquire an asymptotic “shape”, and converge to equilibrium within its “hull”. Our main result is a precise formulation of these heuristics.

Unfortunately, we are only able to carry out our program for sufficiently large λ . Presumably our results hold for all $\lambda > \lambda_c$, but the methodology has a serious shortcoming. Namely, we rely on imbedded one-dimensional contact processes, which must survive for the proofs to work. Thus we need to assume $\lambda > \lambda_c^{(1)}$.

The body of the paper is divided into two parts. Section 2 contains a rather general growth theorem for S -valued interacting systems with \emptyset as a trap. Systems of this sort have been studied extensively in the literature, especially since the appearance of Richardson’s paper [18]. Until now the models considered tended to expand as solid “blobs”, a property clearly not shared by contact processes. Nevertheless, techniques from previous work on growth models, notably from [18] and [3], can be modified to meet our needs. The main tool is Hammersley-Kingman subadditivity theory [11, 17].

In Sect. 3 we check that the basic contact processes in any dimension with sufficiently large parameter λ satisfy the hypotheses, and hence the conclusions, of our growth theorem. Some new ergodic theorems for multi-dimensional contact processes, complete convergence for example, are established as corollaries.

2. A Limit Theorem for a Class of Growth Models

In this section we formulate and prove a theorem which describes the asymptotic behavior of certain Markov processes with state space $S = \{\text{all subsets of } Z^d\}$. Probably the simplest model of the type we have in mind is the following.

Example 1. Denote by $\{(\zeta_t^A(\lambda)); A \in S\}$ the family of Markov chains such that (ζ_t^A) starts with infection on the set A , such that $x \in \zeta_t^A$ remains infected at later times $u > t$ (i.e. no recovery takes place), and such that $x \in Z^d - \zeta_t^A$ is infected at rate

$$\lambda \cdot \text{the number of infected neighbors of } x.$$

Here $\lambda > 0$ is a parameter; note however that as λ varies the $(\zeta_t^A(\lambda))$ differ only by a change in the time scale. Thus there is essentially only one system with these dynamics, which we call *Richardson’s model* (cf. [18]).

The interacting systems we propose to call growth models may be viewed as generalizations of $\{(\zeta_t^A)\}$; for our purposes we will need four key properties.

Definition. A family $\{(\eta_t^A)\}$ of S -valued Markov processes is called a *growth model* if \emptyset is an absorbing state, and the family is

- (TI) translation invariant: the translated process $x + \eta_t^A$ is a copy of η_t^{x+A} ;
- (A) attractive: if $A \subset B \in S$ η_t^A and η_t^B can be defined on the same probability space in such a way that $\eta_t^A \subset \eta_t^B$ for all t ;
- (L) local: there is an $L < \infty$ so that if $A \cap \{x: \|x\| \leq L\} = \emptyset$ then $P(0 \in \eta_t^A) = o(t)$ as $t \rightarrow 0$.

Clearly Richardson’s model satisfies our axioms; here are three additional examples.

Example 2. Coalescing random walks with nearest neighbor births. Particles undergo continuous time rate-1 simple random walks. In addition, the particles give birth to new particles (i.e. they branch) at neighboring sites at rates

$\frac{\kappa - 1}{2d} > 0$. Whenever two particles attempt to occupy the same site they coalesce into one particle.

Example 3. Williams-Bjerknes growth models ([21, 2, 3]). This is an infection model such that $x \in Z^d - \eta_t^A$ is infected at rate

$$\kappa \cdot \text{the number of infected neighbors of } x,$$

while $x \in \eta_t^A$ becomes healthy at rate

$$1 \cdot \text{the number of healthy neighbors of } x.$$

Here $\kappa > 1$ is a parameter (called the ‘‘carcinogenic advantage’’ in [21]).

Example 4. The basic contact processes, described in Sect. 1.

A few remarks about the four examples are in order here. Richardson’s model is the simplest largely because of two features, both due to the absence of recovery:

(a) The growth of the system can be analyzed in terms of ‘‘chains of infection’’, i.e. imbedded Z^d -valued Markov chains which are controlled to move in a specified direction.

(b) The process starting from $\{0\}$ tends to evolve as a solid blob. (Z^d is a trap for $\{(\xi_t^A)\}$.)

Example 2 enjoys feature (a), since by exploiting the branching effect we can find imbedded random walks with drift. Due to the coalescence, however, (b) fails; intuition suggests that an equilibrium of occupied and vacant sites is approached within the cloud of particles. For Example 3, (a) fails. This obstacle was overcome in [2] with the aid of dual processes (which turn out to be those of Example 2). The Williams-Bjerknes models do exhibit feature (b), since recovery can only occur at the boundary of the infection. Lastly, the basic contact processes satisfy neither (a) nor (b). There is no tractable way to find an imbedded Markov ‘‘chain of infection’’, and as mentioned in the introduction, (ξ_t^0) should spread like a ‘‘blob in equilibrium’’. Thus the basic contact processes constitute the most complex of the four examples.

Let us now proceed to formulate our theorem. Given a growth model $\{(\eta_t^A)\}$, we introduce the random times

$$\begin{aligned} \tau^A &= \min\{t: \eta_t^A = \emptyset\} & A \in \mathcal{S}, \\ t^A(x) &= \min\{t: x \in \eta_t^A\} & A \in \mathcal{S}, x \in Z^d. \end{aligned}$$

Our attention will focus on the *one-site growth process* (η_t^0) starting from the singleton $\{0\}$, so we abbreviate

$$\tau = \tau^0, \quad t(x) = t^0(x).$$

Say that (η_t^0) is *permanent* if

$$P(\tau = \infty) > 0.$$

It will be convenient to identify each $x \in Z^d$ with the unit cube in R^d , having center x ; in this manner, the spread of the one-site process up to time t is represented by

$$H_t = \{y \in R^d : \exists x \in Z^d \text{ with } \|x - y\| \leq \frac{1}{2} \text{ and } t(x) \leq t\}.$$

(Here and throughout the paper we use the L^∞ norm on R^d .) Now for any attractive system, there is an invariant measure ν given by

$$\nu = \text{wk-lim}_{t \rightarrow 0} P(\eta_t^{Z^d} \in \cdot),$$

and the stationary process (η_t^ν) and the one-site process (η_t^0) can be defined on the same space in such a way that $\eta_t^0 \subset \eta_t^\nu \subset \eta_t^{Z^d}$ for all t . These matters are discussed in [6], for example. Thus we can introduce the *coupled region*

$$K_t = \{y \in R^d : \exists x \in Z^d \text{ with } \|x - y\| \leq \frac{1}{2} \text{ and } \eta_t^0(x) = \eta_t^\nu(x)\}.$$

Here and below we use the coordinate notation

$$\begin{aligned} \eta_t^0(x) &= 1 & x \in \eta_t^0 \\ &= 0 & x \notin \eta_t^0. \end{aligned}$$

Very loosely, η_t^0 is “in equilibrium” on K_t .

The main result of this section is a criterion for linear growth of $(\bar{\eta}_t^0) = (\eta_t^0 | \tau = \infty)$. By linear growth we mean that $t^{-1}H_t$ approaches a convex set U as $t \rightarrow \infty$, and that $t^{-1}(H_t \cap K_t)$ approaches the same U . Roughly, then, η_t is completely coupled on tU and completely absent from $Z^d - tU$. The criterion is that three sorts of probabilities should decay exponentially in time.

Theorem. *Let (η_t^0) be a permanent one-site growth process. Suppose that there are constants $\gamma, c, C \in (0, \infty)$ such that*

$$P(t < \tau < \infty) \leq C e^{-\gamma t} \quad t \geq 0, \tag{4}$$

and

$$P(t(x) > t, \tau = \infty) \leq C e^{-\gamma t} \quad t \geq 0, \|x\| < ct. \tag{5}$$

Then there is a convex subset U of R^d such that for any $\varepsilon > 0$,

$$(1 - \varepsilon)U \subset t^{-1}H_t \subset (1 + \varepsilon)U \text{ eventually} \tag{6}$$

a.s. on $\{\tau = \infty\}$. If, in addition,

$$P(x \notin K_t, \tau = \infty) \leq C e^{-\gamma t} \quad t \geq 0, \|x\| < ct, \tag{7}$$

then $(\bar{\eta}_t^0) = (\eta_t^0 | \tau = \infty)$ grows linearly, i.e.

$$(1 - \varepsilon)U \subset t^{-1}(H_t \cap K_t) \subset (1 + \varepsilon)U \text{ eventually} \tag{8}$$

a.s. on $\{\tau = \infty\}$.

Before we prove the theorem, some remarks concerning the hypotheses: In Sect. 3 we will verify (4), (5) and (7) for the basic contact processes with $\lambda > \lambda_c^{(1)}$.

The hypotheses can also be checked for Examples 1 through 3 and related growth models. Since we will only treat the contact processes in detail, let us discuss briefly the other applications. First, note that (4) is trivially satisfied if $P(\tau < \infty) = 0$ (as in Examples 1 and 2). Also, if Z^d is a trap (as in Examples 1 and 3), then $v(\{Z^d\}) = 1$ and

$$K_t = \{y \in \mathbb{R} : \|y - x\| \leq \frac{1}{2} \text{ for some } x \in \eta_t^0\}.$$

In this case $K_t \subset H_t$, so that (7) implies (5). If recovery is impossible (as in Example 1), then $K_t = H_t$, and (7) is equivalent to (5). Thus, to prove linear growth of Richardson's process using our theorem, it suffices to check (5). This can be done by considering the imbedded chain of infection in ζ_t^0 which travels from 0 to x along a path of minimal length. The exponential estimate then reduces to a simple large deviations result for sums of exponential random variables. This argument can be extended to any process η_t^A which dominates $\{\{\zeta_t^A(\lambda)\}\}$, i.e. which can be defined so that

$$\zeta_t^A(\lambda) \subset \eta_t^A \quad \text{for all } t \text{ a.s.}$$

In this case (5) holds by comparison with Richardson's process, and again (η_t^0) grows linearly; some examples of this sort are considered in [20].

For Example 2 one checks (5) and (7) by analyzing imbedded random walks which arise by following a tagged particle and switching to follow its offspring if this brings you closer to the desired location (see [2] for details). For Example 3 one needs to show (4) and (7). Hypothesis (4) can be verified using the fact that the cardinality of (η_t^0) is a (randomly) accelerated version of a simple random walk with positive drift on $\{1, 2, \dots\}$ and absorption at 0. Condition (7) follows from the methods of [2] and [3], although the proof given there of linear growth proceeds along somewhat different lines.

One final remark. For simplicity we will only prove our growth theorem for systems dominated by Richardson's process. By this we mean that for some $\lambda > 0$, (η_t^A) and $(\zeta_t^A(\lambda))$ can be defined on a common probability space so that $\eta_t^A \subset \zeta_t^A(\lambda)$ for all t a.s. Clearly Examples 1 through 4 are of this type. With minor changes our techniques handle the general local case.

Proof of the Theorem. Let $\{(\eta_t^A)\}$ be a growth model which satisfies the hypotheses. Our first goal is to establish (6) for the conditional process $(\bar{\eta}_t^0) = (\eta_t^0 | \tau = \infty)$. Henceforth we write $\bar{P}(\cdot) = P(\cdot | \tau = \infty)$, $\bar{E}[\cdot] = E[\cdot | \tau = \infty]$. After the argument for (6) is finished we will prove (8). Axioms which ensure linear growth have been developed by Richardson [18], Kesten [15] and Hamersley [11]. A fairly detailed description of their program is given in Sect. 2 of [3]; as explained in that paper, (6) follows from the following two properties:

I. (subadditivity). $t(x + y) \leq t(x) + s(y) + v(x, y)$ a.s., where $s(y)$ is an appropriately chosen copy of $t(y)$ which is independent of $t(x)$, and where

$$\bar{E}[v^2(x, y)] = O(\|x + y\|).$$

II. (regularity properties). There is an $r > 0$ such that

$$\bar{E}[t^2(kx)] \leq k^2 r^{-1} \|x\|^2 + O_x(k),$$

and such that for each $\delta > 0$,

$$\bar{P}(B_{x,r\delta} \cap \{y: |t(kx) - t(ky)| \leq \delta k\}) = O_\delta(k).$$

(Here and below, $B_{x,r} = \{y: \|x - y\| \leq r\}$ is the r -ball centered at x .) Our arguments for I, II and (8) will be modelled after ones in [3]; in particular we will prove

II'. For some $\gamma, c, C \in (0, \infty)$,

$$\bar{P}(B_{x,ct} \cap H_{t(x)+l^2+t} \quad \text{for some } t \geq 0) \leq C e^{-\gamma l}.$$

It is not hard to see that II follows from II'. To give structure to our proof, the remainder of this section is divided into three propositions, which assert respectively that II', I and (8) hold.

A word about notation in what follows: γ, c and C will denote positive finite constants whose values are unimportant and in general *will change* from line to line. This abuse of notation should help focus attention on the main ideas and alleviate clutter; when chaos threatens we will try to alert the reader.

Proposition 1. *Assuming (4) and (5), II' holds.*

Proof of Proposition 1. If $\|x\| < cl^2$ the verification is easy:

$$\begin{aligned} \bar{P}(B_{x,ct} \cap H_{t(x)+l^2+t} \quad & \text{for some } t \geq 0) \\ & \leq \bar{P}(B_{x,ct} \cap H_{l^2+t} \quad \text{for some } t \geq 0) \\ & \leq \bar{P}(B_{0,ct} \cap H_t \quad \text{for some } t \geq l^2). \end{aligned}$$

Using (5) now gives the above

$$\begin{aligned} & \leq \left(C |B_{0,cl^2}| e^{-\gamma l^2} + \sum_{x \notin B_{0,cl^2}} P\left(t(x) > \frac{\|x\|}{c}, \tau = \infty\right) \right) / P(\tau = \infty) \\ & \leq C e^{-\gamma l^2}. \end{aligned}$$

Thus we can assume $\|x\| > cl^2$ and let T_1, T_2 be the respective times at which $(\tilde{\eta}_t^0)$ first hits $B_{x,l}, B_{x,l/2}$. Suppose we can construct

(a) a random time σ so that

$$\bar{P}(\sigma \geq T_2) \leq C e^{-\gamma l}, \tag{9}$$

and

(b) a copy $(\tilde{\eta}_{\sigma+t})$ of $(\tilde{\eta}_t^{\tilde{x}})$ for some random $\tilde{x} \in B_{x,l}$ with

$$\tilde{\eta}_{\sigma+t} \subset \bar{\eta}_{\sigma+t}^0 \quad \bar{P}\text{-a.s.} \tag{10}$$

Then by (9), (10) and (5), with overwhelming probability (i.e. $\geq 1 - C e^{-\gamma l}$) $\tilde{\eta}_{\sigma+t}$ starts at $\tilde{x} \in B_{x,l}$ at time $\sigma \leq T_2 \leq t(x)$ and expands at a linear rate. Hence, after

the additional “lag time” l^2 , $(\tilde{\eta}_{\sigma+t})$ will cover $B_{x,ct}$ with overwhelming probability. To finish the proof of II', it therefore suffices to construct σ and $(\tilde{\eta}_{\sigma+t})$ and check (9) and (10). The construction is carried out as follows. Let $v_0=0$, and for $k \geq 1$ inductively define a *restart* procedure by:

$$\begin{aligned}
 u_k &= \inf\{t \geq v_{k-1} : \tilde{\eta}_t^0 \cap B_{x,t} \neq \emptyset\}; \\
 x_k &= \text{a randomly chosen site in } \tilde{\eta}_{u_k}^0 \cap B_{x,t}; \\
 (\tilde{\eta}_{u_k+t}^{(k)}) &= \text{a copy of } (\eta_t^{x_k}) \text{ such that} \\
 &\tilde{\eta}_{u_k+t}^{(k)} \subset \tilde{\eta}_{u_k+t}^0 \quad \text{for all } t \text{ } \bar{P}\text{-a.s.}
 \end{aligned}$$

(this can be accomplished because $\{(\eta_t^A)\}$ is monotone);

$$v_k = \sup\{t : \tilde{\eta}_t^{(k)} \neq \emptyset\}.$$

Stop at

$$k_0 = \inf\{k : v_k = \infty\}.$$

Now let

$$\sigma = u_{k_0}, \quad \tilde{\eta}_{\sigma+t} = \tilde{\eta}_{\sigma+t}^{(k_0)}.$$

The inclusion (10) holds by construction, so we need only check (9). To do so, we consider the occupation times

$$\psi(t) = \int_0^t 1_{\{\tilde{\eta}_s^0 \cap B_{x,t} \neq \emptyset\}} ds,$$

and show that for some C, γ

$$\bar{P}(\psi(\sigma) \geq m) \leq C e^{-\gamma m}, \tag{11}$$

and

$$\bar{P}(\psi(T_2) \leq \varepsilon l) \leq C e^{-\gamma l}. \tag{12}$$

Inequality (9) follows immediately from (11) and (12), since

$$\begin{aligned}
 \bar{P}(\sigma \geq T_2) &\leq \bar{P}(\psi(\sigma) \geq \psi(T_2)) \\
 &\leq \bar{P}(\psi(\sigma) \geq \varepsilon l) + \bar{P}(\psi(T_2) \leq \varepsilon l) \\
 &\leq C e^{-\gamma l}.
 \end{aligned}$$

To prove (11) write

$$\psi(\sigma) = \sum_{j=1}^{k_0-1} v_j - v_{j-1},$$

and observe that

(i) The \bar{P} -distribution of k_0 is geometric (each restart is the last with probability $\rho = P(\tau^x = \infty)$),

(ii) given $k_0 = k$, the $v_j - v_{j-1}$ ($1 \leq j \leq k-1$) are i.i.d. with a distribution which has exponential tail by (4), and

(iii) facts (i) and (ii) imply (11).

To get (12) we let

$$T(t) = \sup\{s : \psi(s) \leq t\},$$

choose λ so that Richardson's process $\{(\zeta_t^A(\lambda))\}$ dominates $\{(\eta_t^A)\}$, and note that we can construct a coupling with

$$\eta_{T(t)}^0 \subset \zeta_t^{B_{x,l}}(\lambda) \quad \text{for all } t \text{ } \bar{P}\text{-a.s.}$$

Thus it suffices to show that

$$\bar{P}(\zeta_t^{B_{x,l}}(\lambda) \cap B_{x,l/2} \neq \emptyset \text{ for some } t \leq \varepsilon l) \leq C e^{-\gamma l}. \tag{13}$$

For the event on the left side to occur, some imbedded chain starting from the boundary of $B_{x,l}$ must make at least $l/2$ rate- λ jumps in time εl . The probability is therefore bounded by

$$|\partial B_{x,l}|(2d)^{l/2} P\left(\frac{l}{2} \text{ mean-}\lambda^{-1} \text{ exponentials sum to } \leq \varepsilon l\right).$$

For small ε , a standard large deviations result yields (13), and hence (12). The proof of Proposition 1 is finished.

Proposition 2. *Assuming (4) and (5), I holds.*

Proof. Fix $x, y \in \mathbb{Z}^d$, and write $r = \|x\|^{1/2}$. First, using the restart construction of Proposition 1, we find a time σ , and a one-site process $(\tilde{\eta}_{\sigma+t})$ which starts in $B_{x,r}$ and lives forever. From (9),

$$\bar{P}(\sigma \leq t(x)) \geq 1 - C e^{-\gamma r}. \tag{14}$$

Now there are two cases: $\sigma \leq t(x)$ and $\sigma > t(x)$. When $\sigma \leq t(x)$ we begin a restart construction at time $t(x)$ to find a time $\check{\sigma} > t(x)$ and a one-site process $(\check{\eta}_{\check{\sigma}+t})$ which lives forever and is imbedded in $(\eta_{\check{\sigma}+t})$. When $\sigma > t(x)$ we simply begin a restart construction at time $t(x)$ to find a process $(\check{\eta}_{\check{\sigma}+t})$ which lives forever and is imbedded in $(\tilde{\eta}_{\check{\sigma}+t})$. In either case let $\{z\} = \check{\eta}_{\check{\sigma}}$, and define

$$s(y) = \inf\{t: \check{\eta}_{\check{\sigma}+t} \ni z + y\}.$$

Clearly $s(y)$ has the same distribution as $t(y)$, and is independent of $t(x)$. Moreover, if we define

$$\begin{aligned} v(x, y) &= [\check{\sigma} - t(x)] + [t(x+y) - \check{\sigma} - s(y)]^+ \\ &= v_0 + v_1, \end{aligned}$$

then

$$t(x+y) \leq t(x) + s(y) + v(x, y).$$

We now proceed to estimate v_1 and v_2 in order to show that $\bar{E}[v^2(x, y)] = O(\|x\|)$. Note that v_0 is a sum of a geometric number of i.i.d. variables with the distribution of $\tau | \tau < \infty$. From (4) we conclude that $E[v_0^2] < \infty$ is independent of x . Thus it suffices to handle v_1 , which we break in two: $v'_1 = v_1 I_{\{\sigma \leq t\}}$, $v''_1 = v_1 I_{\{\sigma > t\}}$. Temporarily write $P' = \bar{P}(\cdot \cap \{\sigma \leq t\})$, $P'' = \bar{P}(\cdot \cap \{\sigma > t\})$. To bound $\bar{E}[(v'_1)^2]$, we first show that z is close to x with overwhelming

probability. Let T be the time at which $\tilde{\eta}_t^0$ first hits $B_{x,r}$. Since $T \leq \sigma$ we can find c, C and γ such that

$$\begin{aligned} P' \left(t(x) - \sigma > \frac{r}{c} + l^2 \right) &\leq P' \left(t(x) - T > \frac{r}{c} + l^2 \right) \\ &\leq \sum_{y \in \partial B_{x,r}} P \left(t(x) - t(y) > \frac{r}{c} + l^2 \right) \\ &\leq C r e^{-\gamma l}. \end{aligned}$$

Moreover, the restart construction yields

$$P'(\check{\sigma} - t(x) > l^2) \leq C e^{-\gamma l^2}, \tag{15}$$

so for a new γ and C we get

$$P' \left(\check{\sigma} - \sigma > \frac{r}{c} + l^2 \right) \leq C r e^{-\gamma l}.$$

Since $(\tilde{\eta}_{\sigma+t})$ spreads at most linearly,

$$\begin{aligned} P' \left(\check{\sigma} - \sigma \leq \frac{r}{c} + l^2, \|z - x\| > r + K \left(\frac{r}{c} + l^2 \right) \right) \\ \leq C e^{-\gamma \left(\frac{r}{c} + l^2 \right)} \end{aligned}$$

for some K and some new choice of c, C, γ . Combining the last two inequalities,

$$P'(\|z - x\| > K(r + l^2)) \leq C e^{-\gamma l}. \tag{16}$$

Now by monotonicity, $t(z + y) \leq \check{\sigma} + s(y)$, and hence

$$P'(v_1 > K(r + l^2)) \leq P'(t(x + y) - t(z + y) > K(r + l^2)). \tag{17}$$

Moreover, Proposition 1 asserts that

$$P \left(t(x + y) - t(z + y) > \frac{\|z - x\|}{c} + l^2 \right) \leq C e^{-\gamma l}. \tag{18}$$

From (15)-(18) we conclude that

$$P'(v_1 > K(r + l^2)) \leq C e^{-\gamma l}, \tag{19}$$

and so $\bar{E}[(v_1')^2] = O(\|x\|)$. Finally, to estimate $\bar{E}[(v_1'')^2]$ we note that (5) and (15) imply

$$P''(\check{\sigma} > K(\|x\| + l)) \leq C e^{-\gamma(\|x\| + l)}$$

and, since (η_t^0) spreads at most linearly,

$$P''(\max\{\|x\|, \|z\|\} > K(\|x\| + l)) \leq C e^{-\gamma(\|x\| + l)}$$

for suitably altered constants. Arguing as for (19) we get

$$P''(v_2 > K(\|x\| + l^2)) \leq C e^{-\gamma l}.$$

Together with (15), this last inequality yields

$$E''[(v_1')^2] \leq C \|x\| e^{-\gamma r} = o(\|x\|).$$

The proof of Proposition 2 is finished.

Proposition 3. *Assuming (4), (5), and (7), (8) holds.*

Proof. An easy Borel-Cantelli argument using (7) yields

$$\bar{P}(B_{0,ct} \subset K_t \text{ eventually in } t) = 1.$$

Next we mimic the construction of Proposition 1 to get an appropriate imbedded one-site process $(\tilde{\eta}_{\sigma+t})$ which starts near x and lives forever. It follows that

$$\bar{P}(B_{x,ct} \not\subset K_{t(x)+l^2+t} \text{ for some } t \geq 0) \leq C e^{-\gamma t}. \tag{20}$$

(Monotonicity ensures that the coupled set $\tilde{K}_{\sigma+t}$ for $(\tilde{\eta}_{\sigma+t})$ is contained in $K_{\sigma+t}$.) Now the proof of (8) proceeds as in [3]. Recall from [18] that the limit set U is defined by

$$U = \{x: \varphi(x) \leq 1\}, \quad \text{where} \quad \varphi(x) = \lim_{n \rightarrow \infty} \frac{t(nx)}{n}.$$

Fixing $\varepsilon > 0$, and with c as in (20), we can cover $(1-\varepsilon)U$ with finitely many balls $B_i = B_{x_i, (1-\varphi(x_i))c/2}$, $1 \leq i \leq N(\varepsilon)$. Using (20), one can make sure that $B_i \subset t^{-1}(H_t \cap K_t)$ for each t with overwhelming probability and hence by Borel-Cantelli that $(1-\varepsilon)U \subset t^{-1}(H_t \cap K_t)$ eventually \bar{P} -a.s. The opposite inclusion follows from (6). See [3] for more details in a special case.

In combination, Propositions 1, 2, and 3 established our linear growth theorem.

3. Limit Theorems for Contact Processes on Z^d

Our main objective in this section is to check the hypotheses of the growth theorem for contact systems $\{(\xi_t^A)\}$ in several dimensions. At the end of the paper we discuss applications to the ergodic theory. As noted in the introduction, we are only able to verify (4), (5) and (7) for the case

$$\lambda > \lambda_c^{(1)} = \text{the critical value in one dimension.}$$

(It is known [9, 16] that $1.18 \leq \lambda_c^{(1)} \leq 2$.) When $\lambda > \lambda_c^{(1)}$ one can exploit imbedded one-dimensional processes, and thereby appeal to the simpler theory on the line. Instead of the basic $d=1$ contact system, we will need certain *truncated* one dimensional processes $(\xi_t^0(\lambda))$ which differ from the $(\xi_t^0(\lambda))$ only in that infection cannot arise at $x < 0$. For (ξ_t^0) , let τ^+ be the hitting time of \emptyset ,

$t^+(n)$ the first time n is infected. We will appeal to the following exponential estimates; proofs may be found in a companion paper [7] which deals with supercritical one-dimensional basic contact processes and related models.

Lemma. *Let (\cdot, ξ_t^0) be the one-dimensional truncated contact process with infection parameter $\lambda > \lambda_c^{(1)}$. Then $P(\tau^+ = \infty) > 0$, and there are constants $C, \gamma, a \in (0, \infty)$ such that*

$$P(t < \tau^+ < \infty) \leq C e^{-\gamma t}, \tag{21}$$

$$P(t^+(x) < \infty, \tau^+ < \infty) \leq C e^{-\gamma|x|}, \tag{22}$$

$$P(t < t^+(x), \tau^+ = \infty) \leq C e^{-\gamma t} \quad |x| \leq at. \tag{23}$$

Proof. See [7].

We now proceed to demonstrate (4), (5) and (7) for the basic d -dimensional contact processes $(\xi_t^0(\lambda))$ with $\lambda > \lambda_c^{(1)}$.

Proposition 4. *Exponential estimate (4) holds.*

Proof. Again we use a restart scheme. Fix $t < \infty$. Let $v_0 = 0$, and for $k \geq 0$ proceed inductively as follows: if $\xi_{v_k}^0 = \emptyset$, put $v_{k+1} = v_k$; otherwise let

$$x_k = (x_k^{(1)}, \dots, x_k^{(d)}) = a \text{ randomly chosen site in } \xi_{v_k}^0,$$

${}_{+} \xi_{v_k+t}^{(k)}$ a copy of the $d=1$ truncated contact process with parameter λ , starting from x_k , and living on

$$\{A \times \{x_k^{(2)}\} \times \dots \times \{x_k^{(d)}\}, A \subset Z\},$$

such that

$${}_{+} \xi_{v_k+t}^{(k)} \subset \xi_{v_k+t}^0 \quad \forall t \text{ a.s.},$$

$$v_{k+1} = \sup \{s: {}_{+} \xi_s^{(k)} \neq \emptyset\}.$$

Stop at

$$k_0 = \min \{k: v_k = v_{k+1} \text{ or } v_{k+1} = \infty\}.$$

On each trial there is a positive probability that $v_k = \infty$ so k_0 is majorized by a geometrically distributed random variable and

$$\{t < \tau < \infty\} = \{t < v_{k_0} < \infty\}.$$

Now given $k_0 = k$, the $v_j - v_{j-1}$ ($1 \leq j \leq k$) are i.i.d., with exponential tail by (21), so the claim follows from the argument given for (11) in Sect. 2.

The arguments for (5) and (7) are more involved and we need to make explicit use of percolation substructures. We will not review the basic properties here. The reader is referred to [9] or [10] for background.

Proposition 5. *Exponential estimate (5) holds.*

Proof. To keep matters as simple as possible, we present only the case $d=2$. After digesting what follows, the reader should be convinced that the technique applies in any dimension. Fix $x = (x_1, x_2) \in Z^2$. Without loss of generality as-

sume $x_1 \geq 0, x_2 \geq 0$. The basic idea is to find imbedded one-dimensional processes which live forever and move in desired directions. By linking such processes appropriately, a path of infection from the origin to x is constructed by time $c^{-1} \|x\|$ with overwhelming probability. In two dimensions this is typically a four step procedure involving random times T_1, T_2, T_3 and finally T_4 such that $t(x) \leq T_4$ \bar{P} -a.s., and for some C, γ, c

$$P(T_4 > t) \leq C e^{-\gamma t} \quad \|x\| \leq ct. \tag{24}$$

To check (24) we need estimates for the four differences $T_i - T_{i-1}, i = 1, 2, 3, 4$ ($T_0 = 0$). Thus the proof is divided into six steps: first the construction, then the four estimates, and finally we put all the pieces together.

Step 1. The construction. For $i = 1, 2, z \in Z^d, s \geq 0$, let $(+_s \xi_i^{z,s}(t))$ and $(-_s \xi_i^{z,s}(t))$ be truncated (one-dimensional) processes imbedded in the percolation substructure for $\{(\xi_i^A)\}$, starting at $z = (z_1, z_2)$ at time s . If $i = 1$, then infection only occurs in the first coordinate (\rightarrow or \leftarrow); if $i = 2$ infection occurs only in the second coordinate (\uparrow or \downarrow). The $+$ processes only infect sites $y = (y_1, y_2)$ with $y_i \geq z_i$, whereas the $-$ processes only infect sites with $y_i \leq z_i$. To begin, let $z_0 = 0, R_0 = 0$, and for $n \geq 0$ inductively define:

$$R_{n+1} = \inf\{t > R_n : \xi_1^{z_n, R_n}(t) = \emptyset\},$$

$$z_{n+1} = a \text{ random site in } \xi_{R_{n+1}}^0.$$

Put

$$M = \min\{n \geq 0 : R_{n+1} = \infty\}, \quad y = z_M, \quad T_1 = R_M.$$

Since $\lambda > \lambda_c^{(1)}, \bar{P}(T_1 < \infty) = 1$. $(+_s \xi_1^{y, T_1}(t))$ is a truncated process which lives forever, so each time this process hits a new site we can launch a new process of type $i = 2$ to try to reach the line (\cdot, x_2) . Thus, we introduce

$$S_k = \inf\{t \geq 0 : (y_1 + k, y_2) \in +_t \xi_1^{y, T_1}(t)\} \quad k \geq 0 \quad (S_0 = T_1),$$

and set

$$\zeta_k(t) = \pm \xi_2^{(y_1 + k, y_2), S_k}(t) \quad k \geq 0,$$

where the sign is chosen to direct infection toward x_2 from y_2 . The processes ζ_k are independent, each with a positive probability of living forever. Those which do will bring us to (\cdot, x_2) , where we start launching more truncated processes heading towards x . Let $J_0 = -1$, and for $n \geq 1$ inductively define

$$I_n = \min\{k > J_{n-1} : \zeta_k \text{ lives forever}\},$$

$$S'_n = \inf\{t : (y_1 + I_n, x_2) \in \zeta_{I_n}(t)\},$$

$$\zeta'_n(t) = +_t \xi^{(y_1 + I_n, x_2), S'_n}(t),$$

$$J_n = \max\{z : (z, x_2) \in \bigcup_t \zeta'_n(t)\}.$$

Note that the S'_n are not necessarily increasing in n . Nevertheless, since $I_n > J_{n-1}$ and the $+_s \xi_1$ processes are truncated, the fate of ζ'_n is not influenced by

that of $\zeta'_1, \dots, \zeta'_{n-1}$. The construction is completed by putting $N = \min\{n: J_n = \infty\}$, and defining

$$\begin{aligned} T_2 &= S_{I_N}, \\ T_3 &= S'_N, \\ T_4 &= \inf\{t: x \in \pm \zeta_1^{(y_1 + I_N, x_2)}(t)\}. \end{aligned}$$

Note that $t(x) \leq T_4$ by construction. We need to check (24).

Step 2. Estimating T_1 . Arguing just as for Proposition 4, we have

$$P(T_1 > t/10, \tau = \infty) = P(t/10 < T_1 < \infty) \leq C e^{-\gamma t}. \tag{25}$$

Step 3. Estimating $T_2 - T_1$. The \bar{P} -distribution of N is geometric with success probability $\rho^+ = P(\tau^+ = \infty) > 0$, and given $N = n$, the $I_k - J_{k-1}$ ($1 \leq k \leq n$) are i.i.d. geometric, also with success probability ρ^+ . Moreover, the $J_k - I_k$ ($1 \leq k \leq n$) are i.i.d., with exponential tail by (22), and are independent of the $I_k - J_{k-1}$. It follows that

$$\bar{P}(I_N > t) \leq C e^{-\gamma t}. \tag{26}$$

Also, by virtue of (23), we can choose $a > 0$ (and modify C, γ) so that if $k < at$

$$\bar{P}(S_k - T_1 > t) \leq C e^{-\gamma t}.$$

The last two inequalities imply that

$$\bar{P}(T_2 - T_1 > t/10) \leq C e^{-\gamma t} \tag{27}$$

for suitable C, γ .

Step 4. Estimating $T_3 - T_2$. Conditioned on $I_N = k$, $\zeta_k(t)$ has the same distribution as $\pm \zeta_2^{(y_1 + k, y_2), S_k}(t)$ conditioned on nonextinction, so by (23),

$$\bar{P}(T_3 - T_2 > t/10, |x_2 - y_2| < at/10) \leq C e^{-\gamma t} \tag{28}$$

for some C, γ . To estimate the random coordinate y_2 , note that

$$\bar{P}(\|y\| > at/20) \leq \bar{P}(T_1 > \varepsilon t) + \bar{P}(t(z) \leq \varepsilon t \text{ for some } z: \|z\| \geq at/20).$$

The first probability on the right decays exponentially in t for any fixed ε by (25). The second decays exponentially for ε sufficiently small, by comparison with Richardson's process. Hence

$$\bar{P}(\|y\| > at/20) \leq C e^{-\gamma t}, \tag{29}$$

and we have if $x_2 < at/20$,

$$\bar{P}(T_3 - T_2 > t/10) \leq C e^{-\gamma t} \tag{30}$$

for suitably chosen C, γ .

Step 5. *Estimating $T_4 - T_3$.* Arguing as in Step 4, we get that if $x_1 < at/20$,

$$\bar{P}(T_4 - T_3 > t/10, y_1 + I_N \leq x_1) \leq C e^{-\gamma t} \tag{31}$$

for some C, γ .

Step 6. *Denouement.* There are two cases.

Case 1. $\|x\| \geq at/100$. Then (29) and (26) show that $\bar{P}(y_1 + I_N > x_1) \leq C e^{-\gamma t}$, and (31) can be replaced by

$$\bar{P}(T_4 - T_3 > t/10) \leq C e^{-\gamma t}. \tag{32}$$

Now combine (25), (27), (30), (32) to get that if $\|x\| < at/20$ then

$$\bar{P}(T_4 > 4t/10) \leq C e^{-\gamma t},$$

proving (24) in this case.

Case 2. $\|x\| < at/100$. If x is too close to 0 we will not have $\bar{P}(y_1 + I_N > x_1) \leq C e^{-\gamma t}$. To circumvent this, we use the argument just completed in Case 1 to find a path of infection from the origin to $w = ([\delta t], [\delta t])$ where $\delta = at/50$. After hitting w , the idea is to “turn around” and head back toward x . This is accomplished by restarting in the $+\xi_1$ process which hits w_1 (and lives forever) until we find a $k \geq w_1$ and a time $T_5 \geq T_4$ such that $-\xi_1^{(k, w_2), T_5}$ lives forever. Then we can mimic the construction of Case 1 with x as our goal. If all the $\zeta'_i, 1 \leq i < N$, have died out by time T_4 , then the second construction is unaffected by the first; using the estimates from Case 1 it is not hard to check that the probability of a $\zeta'_i, 1 \leq i < N$, surviving to time T_4 is at most $C e^{-\gamma t}$. Finally, since the second construction fails with exponentially small probability, we conclude that if $\|x\| < at/100$,

$$\bar{P}(t(x) > t) \leq C e^{-\gamma t}$$

for suitable c, γ , proving the desired result in this case. Further details are left to the reader.

Proposition 6. *Exponential estimate (7) holds.*

Proof. There are two steps; the first is to show

$$P(\xi_t^{Z^d}(x) \neq \xi_t^v(x)) \leq C e^{-\gamma t} \quad x \in Z^d, \tag{33}$$

the second to show that (for some $c > 0$)

$$\bar{P}(\xi_t^0(x) \neq \xi_t^{Z^d}(x)) \leq C e^{-\gamma t} \quad \|x\| \leq ct. \tag{34}$$

The desired result (7) follows immediately from (33) and (34). To demonstrate the last two estimates we exploit the *self-duality* of basic contact processes. In particular, we assume that the reader is familiar with graphical representations whereby dual processes can be defined on a common probability space by means of “time/arrow reversal”. See e.g. [9] for background.

Argument for (33). If ξ_t^y and $\xi_t^{Z^d}$ are constructed on the same percolation substructure,

$$P(\xi_t^{Z^d}(x) \neq \xi_t^y(x)) = P(\xi_t^{Z^d}(x) = 1) - P(\xi_t^y(x) = 1) = P(t < \tau < \infty)$$

by self duality. Thus (33) is equivalent to (4), which was proved in Proposition 4.

Argument for (34). Now fix x, t , and for $s \geq 0$ define

$$\xi_s^0 = \{y: \exists \text{ path up from } (0, 0) \text{ to } (y, s)\},$$

$$\hat{\xi}_s^x = \{y: \exists \text{ path down from } (x, t) \text{ to } (y, t-s)\}.$$

Note that $(\hat{\xi}_s^x)$ is a basic contact process starting from x . Moreover,

$$\{\xi_t^0(x) \neq \xi_t^{Z^d}(x), \tau = \infty\} = \{\exists \text{ path from } Z^d \times \{0\} \text{ to } (x, t), \exists \text{ path from } (0, 0) \text{ to time } \infty, \exists \text{ path from } (0, 0) \text{ to } (x, t)\}.$$

Since a path from $(0, 0)$ to (y, s) and a path from (y, s) to (x, t) yield a path from $(0, 0)$ to (x, t) whenever $s \in [0, t]$, the event on the right implies

$$\{(\hat{\xi}_s^x) \text{ dies after time } t\} \cup \{(\xi_s^0) \text{ and } (\hat{\xi}_s^x) \text{ live forever but } \xi_s^0 \cap \hat{\xi}_{t-s}^x = \emptyset \forall s \in [0, t]\}.$$

By Proposition 4, the first event has probability at most $Ce^{-\gamma t}$. Call the second event E . Without loss of generality assume $x_1 > 0$, and for $\varepsilon \in (0, 1/2)$, introduce

$$F_1 = \{\tau = \infty, \exists \text{ path from } (0, 0) \text{ at time } 0 \text{ to } \{(y_1, 0): y_1 < 0\} \text{ at time } t - \varepsilon t \text{ such that at time } \varepsilon t \text{ the path is in } \{(y_1, 0): y_1 \geq 0\} \text{ and stays in the plane } \{(y_1, 0): y_1 \in Z\} \text{ after time } \varepsilon t\},$$

$$F_2 = \{(\hat{\xi}_t^x) \text{ lives forever, } \exists \text{ path from } \{(y_1, 0): y_1 < 0\} \text{ at time } \varepsilon t \text{ to } x \text{ at time } t \text{ such that at time } t - \varepsilon t \text{ the path is in } \{(y_1, 0): y_1 \geq 0\} \text{ and stays in the plane } \{(y_1, 0): y_1 \in Z\} \text{ at times before } t - \varepsilon t\}.$$

Using the construction in the proof of Proposition 5 to guide the paths, we can find $\varepsilon, c > 0$ so that $P(F_1) \geq 1 - Ce^{-\gamma t}$ and $P(F_2) \geq 1 - Ce^{-\gamma t}$ whenever $\|x\| < ct$. Since $F_1 \cap F_2$ implies that $\xi_s^0 \cap \hat{\xi}_{t-s}^x \neq \emptyset$ for some $s \in [ct, (1-c)t]$, we have $E \subset (F_1 \cap F_2)^c$. Hence $P(E) \leq Ce^{-\gamma t}$, and (34) holds.

Having completed the proof of our growth theorem for contact processes, let us now discuss briefly its applications to the ergodic theory. If A is any finite set, then an immediate consequence of (8) is

$$\bar{P}(\xi_t^0 \cap A = \xi_t^y \cap A \text{ eventually in } t) = 1. \tag{35}$$

In particular,

$$\bar{P}(\xi_t^0 \in \cdot) \Rightarrow \nu \text{ as } t \rightarrow \infty,$$

i.e. (1') holds. As mentioned in the introduction, complete convergence (1) follows without difficulty. See [5] or [9], for example. Another consequence of (35) is the complete pointwise ergodic theorem:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\xi_s^A) ds = \int f d\nu \quad \bar{P}\text{-a.s.},$$

for any continuous function f on S . Again, see [9]. Finally, a more subtle application of (8) is the law of large numbers for the number of particles:

$$\frac{|\xi_t^0|}{t^d} \rightarrow \rho |U| \quad \bar{P}\text{-a.s.},$$

where ρ is the density of ν and $|U|$ is the volume of the limit set U . This is proved in [7] for $d=1$; the proof when $d>1$ proceeds along the same lines.

In closing, it should be noted that the hypotheses (4), (5) and (7) can be checked for other additive growth models by a similar analysis. However the foremost open problem connected with our results is to prove linear growth of the d -dimensional basic contact processes for all $\lambda > \lambda_c^{(d)}$. This will require new techniques which rely less heavily on imbedded one dimensional systems.

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