The Annals of Probability
1983, Vol. 11, No. 1, 1–15

SUPERCRITICAL CONTACT PROCESSES ON Z

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In this paper we introduce a percolation construction which allows us to reduce problems about supercritical contact processes to problems about 1-dependent oriented percolation with density p close to 1. Using this method we obtain a number of results about the growth of supercritical contact processes and the wet region in oriented percolation. As a corollary to our results we find that the critical probability for oriented site percolation is greater than (>) that for bond percolation.

1. Introduction. In (1974) Harris introduced a class of Markov processes with state space $S = \{\text{all subsets of } Z\}$ which he called contact processes. The process $\xi_t^A(\lambda)$ with initial state $A \subset Z$ and parameter λ may be thought of as the evolution of an infection. In this interpretation $\xi_t^A(\lambda)$ is the set of infected site at time t and the evolution of the system may be described as follows: each infected site infects healthy neighbors at rate λ and recovers at rate 1.

As the reader can probably guess, the contact process changes from subcritical to supercritical as the parameter varies, i.e. for small λ (e.g. $\lambda \le 1$) $\xi_t^0(\lambda) \to \varnothing$ with probability 1; for large λ the infection persists for all t with positive probability; and there is a critical value λ_c where the change occurs. (Here $0 = \{0\}$.) Over the past several years many results have been proved about λ_c and about sub- and super-critical contact processes (see Griffeath (1981) for a survey). In this paper we will prove some new results about the supercritical contact process. To explain our results we need to recall briefly some of the known results. The reader should refer to Sections 1–6 of Griffeath (1981) for more details.

For each $\lambda > 0$ there is an invariant measure ν_λ for $\xi_t(\lambda)$ such that

$$\xi_t^Z(\lambda) \Rightarrow \nu_\lambda \quad \text{as} \quad t \to \infty$$

(here $\Rightarrow$ denotes weak convergence). The measure ν_λ has the property that

$$\nu_\lambda\{0 \text{ infected}\} = P(\xi_t^0(\lambda) \text{ is never } \varnothing)$$

so $\nu_\lambda = \delta_\varnothing$ (a point mass on $\varnothing$) if $\lambda < \lambda_c$ and is a nontrivial invariant measure if $\lambda > \lambda_c$. The reason for our interest in ν_λ is the following: the set of invariant measures for the contact process is $\{(1-\theta)\nu_\lambda + \theta\delta_\varnothing : \theta \in [0, 1]\}$ so ν_λ is the only nontrivial invariant measure (Liggett, 1978).

In this paper we will investigate properties of the contact process for $\lambda > \lambda_c$. The key to our investigation is a percolation construction inspired by recent work of Russo and Kesten which allows us to reduce results concerning supercritical contact processes to corresponding results about 1-dependent oriented percolation in which the density of wet sites is arbitrarily close to 1. In the latter situation the results are easy to prove using "contour methods" (i.e. "Peierls' argument").

Using this method we have proved a number of exponential estimates for supercritical

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Received August 1981; revised August 1982.

[1] This author was supported by NSF grant MCS80-02732 and by an NSF grant to Cornell University.

[2] This author was partially supported by NSF grants MCS78-01241 and MCS81-00256.

AMS 1970 *subject classifications.* Primary 60K35; secondary 60F15.

Key words and phrases. Contact processes, interacting particle systems, oriented percolation, large deviations, Ceminusgammatee.

contact processes. The most important of these is the following inequality. Let $\tau^A = \inf\{t: \xi_t^A = \varnothing\}$. If $\lambda > \lambda_c$, there are constants C, γ (which depend on λ but not on A) so that

$$P(t < \tau^A < \infty) \leq Ce^{-\gamma t}.$$

This rather curious result has many useful consequences. From this it follows that if $\lambda > \lambda_c$, ξ_t^Z converges to ν_λ exponentially rapidly, i.e. if $\rho_t(\lambda) = P(0 \in \xi_t^Z(\lambda))$ and $\rho(\lambda) = \nu\lambda\{0 \in \cdot\}$ then there are constants C, γ so that $|\rho_t(\lambda) - \rho(\lambda)| \leq Ce^{-\gamma t}$. (Here and below the constants depend upon λ and will change from line to line). Using the last result and the contact process duality equation, it follows that there are C, γ so that

$$0 \leq \nu_\lambda\{x, y \in \cdot\} - \rho(\lambda)^2 \leq Ce^{-\gamma|x-y|},$$

i.e. ν_λ has exponentially decreasing correlations. The last inequality implies that under ν_λ the number of infected sites in $[-n, n]$ obeys the classical laws of probability—the strong law, central limit theorem $\cdots$.

Another consequence of the exponential bound given above is a strong law for $|\xi_t^0|$ (the number of infected sites) when $\lambda > \lambda_c$. Durrett [1] has shown that if $r_t^0 = \sup \xi_t^0$ and $\ell_t^0 = \inf \xi_t^0$, then there is a constant $\alpha(\lambda) > 0$ so that as $t \to \infty$,

$$\text{(1)} \qquad \frac{r_t^0}{t} \to \alpha(\lambda), \quad \frac{\ell_t^0}{t} \to -\alpha(\lambda) \quad \text{a.s. on } \{\tau^0 = \infty\}.$$

From this and the coupling result

$$\xi_t^0 = \xi_t^Z \cap [\ell_t^0, r_t^0] \qquad \text{(see [1], Section 3)},$$

it is natural to conjecture that

$$\frac{|\xi_t^0|}{t} \to 2\alpha(\lambda)\rho(\lambda) \qquad \text{a.s. on } \{\tau^0 = \infty\}.$$

In [1] the convergence was shown to occur in L^1. In this paper we prove the a.s. convergence (settling a conjecture of Harris, 1978).

The results mentioned above are just a few of the exponential bounds we can obtain from our percolation construction. We can also show

$$P(\tau^A < \infty) \leq Ce^{-\gamma|A|}$$

and prove large deviations results for $P(r_t^0 < at, \tau^0 = \infty)$ when $a < \alpha(\lambda)$. (Griffeath (1981) gives results for $P(r_t^0 > bt)$ $b > \alpha(\lambda)$). These results are described in Section 4 and are useful in studying the contact processes in several dimensions (see Durrett and Griffeath, 1981).

Another type of application of the percolation construction (and historically the first) is to critical systems. In Section 3 we show that for the contact process $\alpha(\lambda_c) = 0$, suggesting that $\rho(\lambda_c) = 0$, i.e. the contact process dies out at the critical value, but we have not been able to prove this stronger result. The problem of determining whether $\rho(\lambda_c)$ is positive or zero is a very important one since it is a necessary first step in studying properties of the contact process near λ_c (which is the object of the study of phase transitions).

The result $\alpha(\lambda_c) = 0$, when generalized, is useful in proving strict inequalities between critical values. Consider the one parameter family of processes with jump rates as shown in Table (1).

TABLE 1

	at x	at rate	if $\lvert\xi_t \cap \{x-1, x+1\}\rvert =$
	$1 \to 0$	1	anything
(*)	$0 \to 1$	0	0
		λ	1
		$\theta\lambda$	2

For each θ there is a critical value $\lambda_c(\theta)$ defined in the obvious way. Our results will show that if $\alpha(\theta, \lambda)$ is the asymptotic speed of r_t^0 (in the sense of (1)) and $1 \le \theta \le 2$, then $\alpha(\theta, \lambda_c(\theta)) = 0$. A simple generalization of Lemma 4.2 of [1] shows that if $1 \le \theta < \bar{\theta} < \infty$ then $\alpha(\theta, \lambda) < \alpha(\bar{\theta}, \lambda)$, so it follows that if $1 \le \theta < \bar{\theta} \le 2$ then $\lambda_c(\theta) > \lambda_c(\bar{\theta})$, a result which is "obvious" but not easy to prove. In fact, the reader should note that we have not been able to prove the result for the full range of parameter values $\theta \in [0, \infty)$. The restriction to [1, 2] arises because in our construction we assume the process is *additive*, i.e. it can be constructed from a percolation structure of the type described in Harris (1978) (or Griffeath, 1979, or Durrett, 1981).

As the last remarks may suggest, many of the results in this paper hold for interacting systems other than the basic contact process. Since one of our aims is to prove the comparison of critical values mentioned in the last paragraph, we have proved all our results for the systems described by (*). At the end of Section 2 we describe some of the possible generalizations and throughout the paper we mention some of the results which can be obtained. Perhaps the most noteworthy of these is that oriented percolation (site, bond, or mixed) can be treated by our methods and we can show that the critical probability for the site problem is larger than that for the bond problem. This result has been proved recently for unoriented percolation by Kesten (private communication). The referee informs us that related results have recently been obtained by Higuchi.

The paper is organized as follows. Section 2 describes the percolation construction. In Section 3 we obtain our results about $\alpha(\lambda_c)$ and the critical values $\lambda_c(\theta)$. Our exponential estimates are proved in Section 4 and, finally, Section 5 contains the proof of the strong law for $|\xi_t^0|$ mentioned above.

2. The percolation construction. In this section we will introduce a construction which allows us to reduce problems about supercritical contact processes to analogous questions about an oriented percolation process. To begin, we introduce the percolation process. Let $U = \{(m, n) \in Z^2 : n \ge 0, m + n \text{ even}\}$. By a *$d$-dependent random field η with density p on U*, we mean a collection of $\{0, 1\}$-valued random variables $\eta(z)$, $z \in U$, such that $P(\eta(z) = 1) \equiv p$, and

$$P(\cap_{i=1}^m \{\eta(z_i) = 0\}) = (1 - p)^m$$

for all $z_1, \cdots, z_m \in U$ with $\|z_i - z_j\| > d$ for all $i \ne j$ (here and throughout the paper we use the norm $\|(m, n)\| = (|m| + |n|)/2$ which is the most natural for percolation estimates on U). We say *there is a path from* x to y in η if there are sites $z_0 = x, z_1, \cdots, z_n = y$ with $\|z_i - z_{i-1}\| = 1$, increasing second coordinates, and $\eta(z_i) = 1$ for all $i \ge 1$. Put

$$W = \{y : \exists \text{ path from } (0, 0) \text{ to } y\} \text{ (the set of wet sites)},$$

$$H = \sup\{\eta : (m, n) \in W \text{ for some } m\} \text{ (the height of } W).$$

We say there is *percolation from* 0 in η if $H = \infty$.

Let ξ_t be a supercritical contact system with rates of the form (*). Let $\mathscr{P}$ be the percolation structure for ξ_t (see Griffeath, 1979, or Durrett, 1981, for a description). To relate the contact process to the percolation problem, we begin by covering $R \times R^+$ with rectangles $R_{jk} = (jJ, kL) + [-K, K] \times [0, 1.2\,L)$ (to be described below) and then use the a.s. properties of r_t^0 to define "good events" E_{jk} which are adapted to $\mathscr{F}_{jk}$ = the σ-algebra generated by $\mathscr{P}$ restricted to R_{jk}, in such a way that

(i) the field on U given by $\eta_{jk} = 1_{E_{jk}}$ has density p and is d-dependent, for some $d < \infty$,
(ii) {percolation for 0 in η} $\subset \{\xi_t^{[-K,K]}$ lives forever$\}$,
(iii) $p \to 1$ as J, K, L grow appropriately.

Now for the details of the percolation construction. As in Griffeath's (1981) survey we write

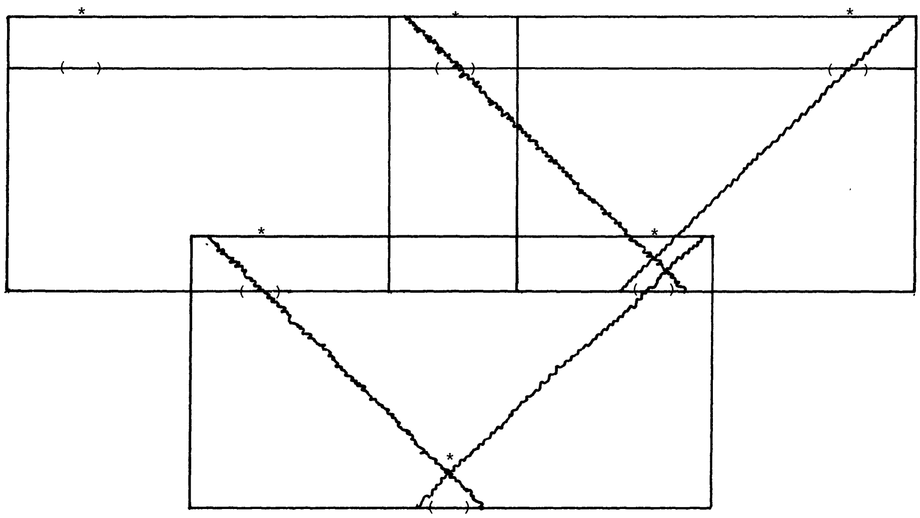

FIG. 1

$$\tau^A = \inf\{t : \xi_t^A = \varnothing\},$$
$$r_t^A = \sup \xi_t^A, \qquad \sup \varnothing = -\infty,$$
$$\ell_t^A = \inf \xi_t^A, \qquad \inf \varnothing = \infty.$$

A fundamental result from Durrett (1980) asserts that if $\lambda > \lambda_c(\theta)$ then there is a constant $\alpha = \alpha(\theta, \lambda) > 0$ so that for any finite $A \subset Z$ as $t \to \infty$,

$$\frac{r_t^A}{t} \to \alpha, \quad \frac{\ell_t^A}{t} \to -\alpha \quad \text{a.s. on } \{\tau^A = \infty\}. \tag{2}$$

The strong laws above have proved to be useful in studying the contact process (see Griffeath, 1981) and are indispensible in what follows. Let

$R \;= (-1.4\alpha L, 1.4\alpha L) \times [0, 1.2L),$ and for $(j, k) \in U$ let
$c_{jk} = ((1 - .1)j\alpha L, kL),$
$R_{jk} = c_{jk} + R,$
$I_{jk} = c_{jk} + ([-.1\alpha L, .1\alpha L] \times \{0\}),$
$E_{00} = \{$there is a path in R_{00} from $[-.11\alpha L, -.1\alpha L] \times \{0\}$
to $((1 - .1)\alpha L, \infty) \times \{1.2L\}$ which passes through
I_{11} and $(0), \infty) \times \{.2L\}\}$
$\cap$ {there is a path in R_{00} from $[.1\alpha L, .11\alpha L] \times \{0\}$
to $(-\infty, (-1 + .1)\alpha L) \times \{1.2L\}$ which passes through
I_{-11} and $(-\infty, 0) \times \{.2L\}\}$,
$E_{jk} = \{E_{00}$ occurs in the translated percolation structure $\mathscr{P} - c_{jk}\}$.

Figure 1 illustrates the notation introduced above and shows the paths which are required for E_{00} to happen. The constraints in the definition say that the paths must cross each other before the $*$, pass through the opening marked by ( ) and then end on one side of a $*$ at $1.2L$.

The motivating force behind the definitions given above is that if the sequence (j_n, k_n), $n = 0, 1, \cdots$, is a path in U and all the events $E_{j_n k_n}$ occur, then there is a "path to ∞" in $\mathscr{P}$ starting in $c_{j_0 k_0} + ([-.11\alpha L, .11\alpha L] \times \{0\})$. To see this observe that on $E_{00} \cap E_{11}$ we can find a path from $[-.11\alpha L, .11\alpha L] \times \{0\}$ through I_{11} and on to both I_{02} and I_{22} (see Figure 1 for a picture proof). And so on.

To prove (ii) we observe that the induced field η has constant density and $R_{0,0}$ intersects only $R_{-1,1}$, $R_{1,1}$, $R_{-2,0}$, $R_{2,0}$, $R_{-1,1}$, and $R_{1,-1}$, so η is 1-dependent.

To prove (iii) we observe that as $M \to \infty$, $P(\tau^{[-M,0]} < \infty) = P(\xi^Z_\infty \cap [-M, 0] = \phi) \to 0$ (see Griffeath, 1979, page 39) so we can choose M large enough that $P(\tau^{[-M,0]} < \infty) < \varepsilon$. Temporarily write $r_t = r_t^{(-\infty,0]}$ and $\ell_t = \ell_t^{[0,\infty)}$. According to Durrett (1980), $t^{-1}r_t \to \alpha$ a.s. as $t \to \infty$, so we can pick L large enough that each of the following probabilities is also less than ε:

(a) $P(r_{.2L} < .15\alpha L)$,
(b) $P(r_L < .95\alpha L)$,
(c) $P(r_L > 1.1\alpha L)$,
(d) $P(r_{1.2L} < 1.1\alpha L)$,
(e) $P(\sup_{t \le 1.2L} r_t > 1.25\alpha L)$,
(f) $P(\inf_{t \le 1.2L} r_t < -.1\alpha L)$.

Let $A(L) = [-.11\alpha L, -.1\alpha L]$. On $\{\tau^{A(L)} > t\}$, $r_t^{A(L)} = r_t^{(-\infty,-.1\alpha L)} := r_t - .1\alpha L$, so if L is large enough then with probability at least $1 - 7\varepsilon$ the right edge $r_t^{A(L)}$, $0 \le t \le 1.2L$ has the properties we want. Unfortunately the right edge is not a path, so we have to do a bit more work to finish the construction. Let

$$s_t^L = \sup\{x \in \xi_t^{A(L)} : \exists \text{ path from } (x, t) \text{ to } Z \times \{1.2L\}\}.$$

Then s_t^L, $0 \le t \le 1.2L$, is the rightmost path from $A(L) \times \{0\}$ to $Z \times \{1.2L\}$. By definition $s_t^L \le r_t^{A(L)}$, so the path does not wander too far to the right. To ensure that it doesn't fall too far left, we have saved an extra .05 in (a) and (b) above. Since ν is ergodic (cf. [1]) we can choose N large enough that

$$\nu(\cdot \cap [0, N] \text{ contains an interval of length } M) > 1 - \varepsilon.$$

By monotonicity, if $.05\alpha L > N$, then with probability at least $1 - \varepsilon$, $\xi^Z_{.2L} \cap [0, .05\alpha L]$ contains an interval of length M. Hence with probability at least $1 - 2\varepsilon$ there is a path from $Z \times \{0\}$ to $[0, .05\alpha L] \times \{.2L\}$ and on up to $Z \times \{1.2L\}$. On $\{r_{.2L}^{A(L)} > .05\alpha L\} \cap \{\ell_{.2L} < -.15\alpha L\}$ we can use the last path and paths from $[0, \infty) \times \{0\}$ to $(-\infty, -.15\alpha L) \times \{.2L\}$ and from $A(L) \times \{0\}$ to $(r_{.2L}^{A(L)}, .2L)$ to get a composite path from $A(L) \times \{0\}$ to $[0, .05\alpha L] \times \{.2L\}$ and on up to $Z \times \{1.2L\}$, so that $s_{.2L}^L > 0$.

A similar argument shows that when $r_L^{A(L)} > .85\alpha L$, then with probability $1 - 2\varepsilon$, $s_L^L > .8\alpha L$. Since $s_{1.2L}^L = r_{1.2L}^{A(L)}$, there is no need to worry about the position at time $1.2L$. Finally, at time t we clearly have $s_t^L \ge \ell_t^{[-.11\alpha L,\infty)}$, so from (d) it follows that $s_t^L \ge -1.4\alpha L$ for all $t \le 1.2L$ with probability at least $1 - \varepsilon$. Combining all our estimates shows that for large L there is a path of the first type needed for E_{00} with probability at least $1 - 13\varepsilon$. By symmetry $P(E_{00}) \ge 1 - 26\varepsilon$, which proves (iii).

At this point we have developed the correspondence between the contact process and 1-dependent percolation. This will be exploited in later sections to prove results about the contact process. We conclude this section by mentioning variations on the construction just completed which allow us to obtain results for other systems as corollaries.

(a) *Discrete time systems.* Let $\xi_0^A = A$ and for $n \ge 0$ if we are given ξ_n^A, then independently for each x let $x \in \xi_{n+1}^A$

with probability	if $\lvert \xi_n^A \cap \{x-1, x\} \rvert =$
0	0
ab	1
$a(2b - b^2)$	2

where $a, b \in [0, 1]$. These are the discrete time counterparts of the systems with jump rates (*). As explained in Griffeath (1981), the case $a = p$, $b = 1$ is equivalent to *oriented site percolation* on $\mathbb{Z}^2$ and the case $a = 1$ $b = p$ is equivalent to *oriented bond* percolation on $\mathbb{Z}^2$. Since our techniques apply in the discrete setting, the discrete versions of several theorems in this paper give new results for oriented percolation. Some of these will be mentioned later as corollaries.

(b) *Contact processes on the half line* $\{0, 1, 2, \cdots\}$. (See Durrett and Griffeath, 1981, for application of these systems). The dynamics are given by (*), except that now $S = \{$all subsets of $\mathbb{Z}^+\}$, so that site 1 is the only neighbor of site 0. Some thought reveals that our percolation scheme can be modified to work in this case. The key observation is that if p is close enough to 1, then with positive probability there will be an infinite path up in $\mathscr{P}$ which never enters a rectangle further left than the one it starts in. Letting λ_c^+ be the critical value for the contact process on $\{0, 1, 2, \cdots\}$, an immediate corollary is $\lambda_c^+ = \lambda_c$.

(c) *One sided contact processes.* Consider a contact process with jump rates

at x	at rate
$1 \to 0$	1
$0 \to 1$	$\lambda \mid \xi_t \cap \{x-1\} \mid$.

In Durrett (1980) it was shown that if $\lambda > \lambda_c$ there are constants $1 < \beta < \gamma < \infty$ so that as $t \to \infty$

$$t^{-1}\ell_t \to \beta,\ t^{-1}r_t \to \gamma \quad \text{a.s. on } \Omega_\infty.$$

Using this as a substitute for (2), it is possible to repeat the argument above: One lets T be the linear transformation which maps $(-.9\alpha, -1) \to (\beta + \delta, -1)$ and $(.9\alpha, 1) \to (\gamma - \delta, 1)$ ($\delta > 0$ sufficiently small) and then creates new "gates" from the old by applying T.

(d) *Nearest neighbor additive growth models.* These terms are explained in Durrett (1980), and it is clear from results in that paper and above that the percolation construction can be carried out for these systems. Since this is a rather limited class of models, and the already cumbersome details become more complicated, we will leave it to the reader to see that our techniques apply.

3. Applications to critical values. Our first applications of the percolation construction in the last section are to the critical contact processes and the critical values of contact processes. Our first result is

THEOREM 1. *If* $1 \le \theta \le 2$, *then* $\alpha(\lambda_c(\theta)) = 0$.

Combining this with an idea due to Liggett (see Section 4 in Durrett, 1980), we easily get

THEOREM 2. *If* $1 \le \theta < \bar{\theta} \le 2$, *then* $\lambda_c(\theta) > \lambda_c(\bar{\theta})$.

(This is obvious from the rates but not so easy to prove.) As a corollary to Theorem 2 we show that the critical probability of oriented site percolation is larger than that for oriented bond percolation.

The key to the proof of Theorem 1 is the following.

LEMMA. *If η is a d-dependent random field on U with density* $p > 1 - 3^{-24(d+1)^2}$ *then*

$$P(n \le H < \infty) \le 3^{-n} \tag{3}$$

and

$$P(H = \infty) > .888. \tag{4}$$

PROOF. We use the contour method. Since similar arguments have appeared many times in the literature ([4], [6], [7], [9], [21], [22]) we will not give all the details. Let

$$D(z) = \{y\colon \|z - y\| \le \tfrac{1}{2}\}, \quad \bar{W} = \cup_{z \in W} D(z),$$

$$\Gamma = \partial(\text{unbounded component of } \bar{W}^c).$$

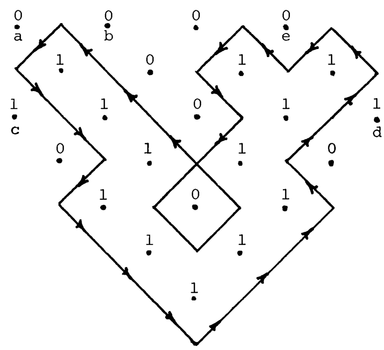

FIG. 2

Γ is the contour associated with W. In words, we inflate W to $\bar{W}$ by placing a diamond of radius ½ at each point of W and then Γ is the outer boundary of W (see Figure 2).

Orient Γ so that the segment $(0, -1) \to (1, 0)$ is oriented in the direction indicated. A little thought (and a peek at Figure 2) shows that a boundary segment for which the x coordinate is decreasing in the orientation must have an unoccupied site on its right and that at most two segments may share the same boundary site (see a, b, c, d, e on Figure 2), so if the length of Γ is $2k$ (in our metric $\| \|$), then there are at least $k/2$ sites z adjacent to Γ with $\eta(z) = 0$. These sites are not independent but if $a = 1 + 8(1 + 2 + \cdots + d) = 1 + 4d(d + 1) \le 4(d + 1)^2$, there is a subset B of these vacant sites so that $|B| \ge k/2a$ and every two distinct sites in B are separated by a distance more than d. Now the total number of contours with $2k$ edges is at most 3^{2k-1}($(0, -1) \to (1, 0)$ always belongs to Γ and after that at each stage there are at most 3 choices), $\{n \le H < \infty\} \subset \{|\Gamma| \ge 2n\}$ and $1 - p \le 3^{-6a}$, so

$$P(n \le H < \infty) \le \textstyle\sum_{k=n}^{\infty} 3^{2k-1}(1-p)^{k/2a} \le 3^{-1} \sum_{k=n}^{\infty} 3^{-k} = 3^{-1}3^{-n}(2/3)^{-1} \le 3^{-n},$$

proving (3). To prove (4) note that

$$P(H \ge 2) \ge 1 - 3(1-p) > 1 - 3^{-23},$$

so $P(H = \infty) = P(H \ge 2) - P(2 \le H < \infty) \ge (1 - 3^{-23}) - 1/9 > .888.$ □

PROOF OF THEOREM 1. The main step is to show that if $\alpha(\theta, \lambda) > 0$ then $\lambda > \lambda_c(\theta)$. Fix θ, λ_0, with $\alpha(\lambda_0) = \alpha(\theta, \lambda_0) > 0$, and note that the percolation construction of the previous section applies. Hence we can choose $L < \infty$ so that $P(E_{00}(L)) > 1 - 3^{-100}$. Now, with L fixed, and R and c_{jk} determined by L and $\alpha(\lambda_0)$, consider the function

$$f(\lambda) = P(E_{00} \text{ with } \alpha = \alpha(\lambda_0) \text{ occurs for } \mathscr{P}(\lambda)).$$

Since $E_{00} \in \mathscr{F}_{00}$, the event in question depends on only a finite number of Poisson processes run for a fixed amount of time so it follows from simple estimates on the Poisson process that f is continuous, and we can choose $\lambda_1 < \lambda_0$ so that $f(\lambda_1) > 1 - 3^{-100}$. Arguing as for (i), and using (4), we get

$$P(\xi_t^{[-.2\alpha(\lambda_0)L, .2\alpha(\lambda_0)L]}(\lambda_1) \text{ lives forever}) > .888.$$

Thus $\lambda_1 \ge \lambda_c(\theta)$ and $\lambda_0 > \lambda_c(\theta)$ as desired. We conclude that $\alpha(\lambda_c) \le 0$. The easy fact that $\alpha(\lambda_c) \ge 0$ is noted in Griffeath (1981). Thus $\alpha(\lambda_c) = 0$. □

PROOF OF THEOREM 2. Fix λ, $\theta < \bar{\theta}$, and let r_t, $\bar{r}_t$ be the right edge processes for the (λ, θ) and $(\lambda, \bar{\theta})$, contact processes respectively, each starting from $(-\infty, 0]$. According to Durrett (1980), the asymptotic edge velocities are given by

$$(5) \qquad \alpha = \lim_{t\to\infty} \frac{\alpha_t}{t}, \quad \bar{\alpha} = \lim_{t\to\infty} \frac{\bar{\alpha}_t}{t},$$

where $\alpha_t = Er_t$ and $\bar{\alpha}_t = E\bar{r}_t$.

Suppose we can find an $\varepsilon > 0$ and $t_0 < \infty$, depending only on λ, such that

$$\bar{\alpha}_t - \alpha_t \geq \varepsilon(\bar{\theta} - \theta)t, \; t \geq t_0. \tag{6}$$

Letting $\lambda = \lambda_c(\theta)$, and using (5) we get

$$\bar{\alpha}(\lambda_c(\theta)) = \lim_{t\to\infty} \frac{\bar{\alpha}_t(\lambda_c(\theta))}{t} \geq \lim_{t\to\infty} \frac{\alpha_t(\lambda_c(\theta))}{t} + \varepsilon(\bar{\theta} - \theta) = \varepsilon(\bar{\theta} - \theta) > 0,$$

so it follows from Theorem 1 that $\lambda_c(\theta) > \lambda_c(\bar{\theta})$. Thus we need only check (6). For $A \subset Z$ infinite and bounded above, call ξ_t^A *alternating* whenever $r_t^A - 1 \notin \xi_t^A$, $r_t^A - 2 \in \xi_t^A$. Consider a comparison process $\tilde{\xi}_t^-$ which agrees with $\xi_t^- (= \xi_t^{(-\infty,0]}(\theta, \lambda))$, except that whenever $\tilde{\xi}_t^-$ is alternating the site $\tilde{r}_t - 1$ is infected at rate $\bar{\theta}\lambda$. Then ξ_t^-, $\tilde{\xi}_t^-$, $\bar{\xi}_t^- (= \xi_t^{(-\infty,0]}(\bar{\theta}, \lambda))$ can be constructed so that

$$\xi_t^- \subset \tilde{\xi}_t^- \subset \bar{\xi}_t^- \quad \text{for all} \quad t \quad \text{a.s.},$$

and in particular,

$$r_t \leq \tilde{r}_t \leq \bar{r}_t \quad \text{for all} \quad t \quad \text{a.s.}$$

Note that $r_t = \tilde{r}_t$ until a chain of three effects occur:

(i) $\xi_t^- = \tilde{\xi}_t^-$ are alternating,

then

(ii) with $x = r_t = \tilde{r}_t$, $x - 1$ becomes infected in the $\tilde{r}.$ process but remains healthy in the $r.$ process,

then

(iii) recovery occurs at x in both processes before recovery occurs at $x - 1$ in the $\tilde{r}.$ process.

Let τ be the first time such a chain of effects occurs; at time τ, $\xi_\tau^- \subset \tilde{\xi}_\tau^-$ and $r_\tau \leq \tilde{r}_\tau - 1$. By modifying $\tilde{\xi}_\cdot^-$ to have the same dynamics as $\xi_\cdot^-$ after time τ, and arguing as in Lemma 4.2 of Durrett (1980), we get

$$E[\bar{r}_t - r_t] \geq P(\tau \leq t). \tag{7}$$

(Note that additivity of $\{(\xi_t^A(\theta, \lambda))\}$ is used at this point.) Next, observe that

(a) $\inf_A P(\xi_1^A \text{ is alternating}) = \gamma_\lambda > 0$ ($|A| = \infty$, $\sup A < \infty$),

(b) effect (ii) occurs at rate $\lambda(\bar{\theta} - \theta)$ whenever $\xi_t^- = \tilde{\xi}_t^-$ is alternating,

(c) effect (iii) occurs with probability ½ whenever effect (ii) has taken place.

By virtue of (a) – (c) and the Markov property, we can find constants $\varepsilon > 0$, $t_0 < \infty$ depending only on λ, so that

$$P(\tau > t) \leq e^{-\varepsilon(\bar{\theta}-\theta)t}, \quad t \geq t_0. \tag{8}$$

Combining (7) and (8) we get

$$\bar{\alpha}_t - \alpha_t \geq 1 - e^{-\varepsilon(\bar{\theta}-\theta)t}, \quad t \geq t_0. \tag{9}$$

To improve this to (6) we write $\alpha_t^{(k)} = E[r_t^{(k)}]$, where $r_t^{(k)}$ is the right edge of the $(\theta + (k/n)\,(\bar{\theta} - \theta), \lambda)$ process starting from $(-\infty, 0]$ $(0 \leq k \leq n)$. From (9),

$$\bar{\alpha}_t - \alpha_t = \sum_{k=1}^n \alpha_t^{(k)} - \alpha_t^{(k-1)} \geq n(1 - e^{-\frac{\varepsilon}{n}(\bar{\theta}-\theta)t}).$$

Let $n \to \infty$ to complete the proof. □

In light of the remarks at the end of Section 2, Theorems 1 and 2 have many variants. For example, if $p_c(\theta)$ is the critical probability for the family of discrete time systems (a) with $a = \theta + (1 - \theta)p$ and $b = p/a$, then an analogous argument shows that $p_c(\theta)$ is strictly decreasing. In particular, if p_s denotes the oriented site percolation ($\theta = 0$) critical probability, p_b the oriented bond percolation ($\theta = 1$) critical probability, then

$$p_s > p_b.$$

As another variant, consider the coalescing branching processes $\{\hat{\xi}_t^A(\theta, \lambda)\}$ $\lambda > 0$, $\theta \in [1, 2]$ (see Griffeath, 1979). The evolution of these processes may be described as follows:

$\{x\}$ dies	at rate 1,
$\{x\}$ branches to $\{x, x-1\}$	at rate $(\theta - 1)\lambda$,
$\{x\}$ branches to $\{x, x+1\}$	at rate $(\theta - 1)\lambda$,
$\{x\}$ branches to $\{x-1, x, x+1\}$	at rate $(2-\theta)\lambda$,

and two particles which try to occupy the same site coalesce to one. Let $\hat{\lambda}_c(\theta) = \sup\{\lambda: \lim_{t\to\infty} P(\hat{\xi}_t^Z(\lambda, \theta) \ni 0) = 0\}$. Since one of the rates decreases as θ increases, it is not obvious (to us at least) that $\hat{\lambda}_c$ is a decreasing function of θ. This is true, however, because

(10) $$\hat{\lambda}_c(\theta) = \lambda_c(\theta) \quad \text{for each} \quad \theta.$$

To prove (10), write $\alpha_t(\theta, \lambda) = E[r_t^-(\theta, \lambda)]$, $\hat{\alpha}_t(\theta, \lambda) = E[\hat{r}_t - (\theta, \lambda)]$ (as above "$-$" denotes $(-\infty, 0]$) and recall that

(11) $$\lambda_c(\theta) = \inf\left\{\lambda: \lim_{t\to\infty} \frac{\alpha_t(\theta, \lambda)}{t} > 0\right\},$$
$$\hat{\lambda}_c(\theta) = \inf\left\{\lambda: \lim_{t\to\infty} \frac{\hat{\alpha}_t(\theta, \lambda)}{t} > 0\right\},$$

so it suffices to show

(12) $$\hat{\alpha}_t(\theta, \lambda) = \alpha_t(\theta, \lambda) \quad \text{for all} \quad t, \theta, \lambda.$$

Identity (12) seems to have been overlooked until now. The proof is easy: using the duality equation

$$P(\xi_t^A \cap B \neq \varnothing) = P(\hat{\xi}_t^B \cap A \neq \varnothing)$$

and translation invariance we see that

$$\begin{aligned}
\hat{\alpha}_t &= \textstyle\sum_{k=1}^{\infty} P(\hat{r}_t^- \geq k) - P(\hat{r}_t^- \leq -k) \\
&= \textstyle\sum_{k=1}^{\infty} P(\hat{\xi}_t^- \cap [k, \infty) \neq \varnothing) - P(\hat{\xi}_t^- \cap (-k, \infty) = \varnothing) \\
&= \textstyle\sum_{k=1}^{\infty} P(\xi_t^- \cap [k, \infty) \neq \varnothing) - P(\xi_t^- \cap (-k, \infty) = \varnothing) \\
&= \textstyle\sum_{k=1}^{\infty} P(r_t^- \geq k) - P(r_t^- \leq -k) = \alpha_t.
\end{aligned}$$

In passing we note that this implies for nearest neighbor additive systems, using the notation of [1],

$$\alpha - \beta > 0 \Rightarrow \hat{\alpha} - \hat{\beta} > 0 \Rightarrow P(\hat{\xi}_t^0 \quad \text{never} \quad \varnothing) > 0 \Rightarrow \nu(\cdot \ni 0) = \lim_{t\to\infty} P(\xi_t^Z \ni 0) > 0$$

so combining this with the trivial observation that

$$\alpha - \beta < 0 \Rightarrow \hat{\alpha} - \hat{\beta} < 0 \Rightarrow P(\hat{\xi}_t^0 \text{ never } \varnothing) = 0 \Leftrightarrow \nu(\cdot \ni 0) = 0,$$

we see that the two notions of "supercriticality"

(13) $$P(\xi_t^0 \text{ never } \varnothing) > 0$$

(14) $$\nu(\cdot \ni 0) > 0$$

coincide, except possibly when $\alpha = \beta$.

4. Exponential estimates in the supercritical case. For the remainder of the paper we will deal only with supercritical contact systems (of type (*)). Our objective here is to derive exponential estimates of five basic types. In the statements and proofs of these results, C and γ will denote positive finite constants which may depend on θ, λ, but do not depend upon any other variables (unless this is explicitly indicated as in Theorem 4). The values of C and γ are unimportant. In fact, the values *will often change* from line to line in our proofs, an abuse of notation which helps suppress trivial computations. The theorems proved in the section are as follows:

THEOREM 3. $P(\tau^A < \infty) \le Ce^{-\gamma|A|}$, $|A| < \infty$.

THEOREM 4. *For any* $a < \alpha = \alpha(\theta, \lambda)$,

$$P(r_t^{(-\infty,0]} < at) \le Ce^{-\gamma t}, \quad t \ge 0,$$

where C *and* γ *depend on* a.

THEOREM 5. $P(t < \tau^A < \infty) \le Ce^{-\gamma t}$, $t \ge 0, |A| < \infty$.

THEOREM 6. $P(\sup_t r_t^0 > n, \tau^0 < \infty) \le Ce^{-\gamma n}$, $n \ge 0$.

THEOREM 7. *For* $x, y \in Z$, $t \ge 0$,

$$0 \le P(\{x, y\} \subset \xi_t^Z) - P(x \in \xi_t^Z)P(y \in \xi_t^Z) \le Ce^{-\gamma|x-y|}.$$

We will prove the results in the order they are stated. As the reader will see, our basic methodology is to derive analogous results for discrete percolation processes with parameter p close to 1, and then to exploit the construction of Section 2 to apply those results to contact processes with arbitrary $\lambda > \lambda_c$.

PROOF OF THEOREM 3. First suppose $A = [-2n, 2n]$. Consider oriented 1-dependent percolation on U, let $W_n = \{y$: there is a path from $[-2n, 2n] \times \{0\}$ to y in $U\}$, and say there is *percolation from* $[-2n, 2n]$ if $|W_n| = \infty$. If $|W_n| < \infty$ then the contour Γ for W_n has at least $4n + 2$ edges in $R \times R^+$ so by an argument in Section 3,

$$(15) \qquad P(|W_n| < \infty) \le P(\Gamma \text{ has} \ge 4n + 2 \text{ edges in } R \times R^+) \le \textstyle\sum_{m=4n+2}^{\infty} 3^m (1-p)^{m/72} \le 3^{-2n}$$

provided $p > 1 - 3^{-100}$. Now choose the grid of $2K \times 2L$ rectangles R_z, $z \in U$, for the construction of Section 2 so that the induced 1-dependent field η on U has density $p > 1 - 3^{-100}$. Then by the construction and (15),

$$P(\tau^{[-2nK, 2nK]} < \infty) \le P(\text{no percolation from } [-n, n] \text{ in } \eta) \le 3^{-2n}.$$

By monotonicity, we get $P(\tau^{[1,n]} < \infty) \le Ce^{-\gamma n}$ for suitable C, γ. To complete the proof we will show

$$(16) \qquad P(\tau^A < \infty) \le P(\tau^{[1,n]} < \infty) \quad \text{whenever} \quad |A| = n.$$

This inequality is due to T. Liggett (private communication), and is true for any nearest neighbor attractive system with constant death rates. For the applications in this paper, we only need Theorem 3 for blocks, so we will merely sketch Liggett's proof of (16). The idea is to couple $(\xi_t^{[1,n]})$ and (ξ_t^A) $(|A| = n)$ so that the latter process is always "more spread out," i.e. we can find a (random) function $\varphi_t: \xi_t^{[1,n]} \to \xi_t^A$ so that

$$(17) \qquad |\varphi_t(y) - \varphi_t(x)| \ge |y - x| \quad \text{for all} \quad x, y \in \xi_t^{[1,n]}.$$

(16) follows from (17).

To prove (17) we let a_i, $1 \le i \le n$, be the elements of A in inceasing order and let $\varphi_0(i) = a_i$. To define ξ_t^A, $\xi_t^{[1,n]}$ and φ_t we use the following coupling:

(i) recovery at $x \in \xi_t^{[1,n]}$ and $\varphi_t(x) \in \xi_t^A$ at the same (mean 1) exponential time,

(ii) whenever infection from x to y occurs in $\xi_t^{[1,n]}$ it also occurs from $\varphi_t(x)$ to $\varphi_t(x) +$

$(y - x)$ in ξ_t^A, and we define $\varphi_{t+}(y) = \varphi_t(x) + (y - x)$.
Further details are left to the reader. □

PROOF OF THEOREM 4. Again, we begin with oriented 1-dependent percolation on U. Introduce:

$$W^- = \{y\text{: for some } m \le 0 \text{ there is a path from } (m, 0) \text{ to } y \text{ in } U\},$$

$$r_n = \sup\{m\text{: } (m, n) \in W^-\}.$$

The first step is to prove:

(18) $$\text{If } q < 1 \text{ and } p > 1 - 3^{-200/(1-q)}, \text{ then } P(r_n < nq) \le 3^{-n+1} \quad \text{for all} \quad n \ge 1.$$

This involves another contour argument. Namely, fix n and let Γ be the contour for W^- up to time n; that is, let $W^* = \cup_{z \in W^-} D(z)$, $\Gamma = \partial$ (unbounded component of $(W^*)^c \cap R \times [0, n]$) · ($\Gamma$ runs from $(1, 0)$ to $(r_n + 1, n)$.) If Γ is given the orientation of Section 3, W^- is always to the left. Let N_1, N_2, N_3, N_4 be the number of edges (of length $\sqrt{2}$) in Γ oriented $\nwarrow$, $\swarrow$, $\nearrow$, $\searrow$ respectively. Then $N_3 + N_4 - N_1 - N_2 = r_n + 1$ so if Γ has length $n + 2k$ and $r_n < qn$ we have

$$2(N_1 + N_2) = N_1 + N_2 + N_3 + N_4 - r_n - 1 \ge (1 - q)n + 2k,$$

which implies

$$N_1 + N_2 \ge (1 - q)\left(\frac{n + k}{2}\right).$$

As before there are at least $(N_1 + N_2)/2$ points z adjacent to Γ which have $\eta(z) = 0$ and there is a subset of these sites of size $(N_1 + N_2)/36$ which are independent. Since there are at most 3^{n+2k} contours with $n + 2k$ edges, we arrive at the estimate

$$P(r_n < qn) \le \textstyle\sum_{k=0}^{\infty} 3^{n+2k}(1 - p)^{(1-q)(n+k)/72} \le \sum_{m=n}^{\infty} \{9(1 - p)^{(1-q)/72}\}^m.$$

Since $(1 - p) < 3^{-200/(1-q)}$, it follows that

$$P(r_n < qn) \le \textstyle\sum_n^{\infty} 3^{-m} = \frac{3}{2} 3^{-n} \le 3^{-n+1},$$

so (18) is proved. Now consider events E_{jk}^{δ} defined as in Section 2, except with .1 replaced by δ, .01 by δ^2 throughout. Given $a < \alpha$, choose $\delta < \alpha - a$ and pick $q < 1$ so that $q(\alpha - \delta) > a$. We can take L sufficiently large that the field η on U induced by the E_{jk}^{δ} is 1-dependent with density $p > 1 - 3^{-200(1-q)^{-1}}$. If r_n is defined as above and $r_t^- = r_t^{(-\infty,0]}$ is the right edge of the contact process, then from the percolation construction we have

$$r_t^- \ge r_n(\alpha - \delta)L - \frac{3}{2}\alpha L \quad \text{for all} \quad t \in [nL, (n + 1)L].$$

Since $q(\alpha - \delta) > a$, it follows that for n sufficiently large and $t \in [nL, (n + 1)L]$,

$$P(r_t^- \le at) \le P(r_n \le qn) \le 3^{-n+1} \le Ce^{-\gamma t/L},$$

proving Theorem 4. □

Theorem 4 can be improved to assert that for $a < \alpha$

$$\lim_{t \to \infty} \frac{1}{t} \log P(r_t^- < at) = \gamma(a) < 0.$$

To do this we observe that

$$P(r_{t+s}^- < a(t+s)) \geq P(r_t^- < at)P(r_{t+s}^- - r_t^- < as \mid r_t^- < at)$$
$$\geq P(r_t^- < at)P(r_s^- < as),$$

and so it follows from a simple computation that

$$\lim_{t\to\infty} \frac{1}{t} \log P(r_t^- < at) = \gamma(a),$$

where $\gamma(a) = \sup_{t\geq 1} t^{-1} \log P(r_t^- < at) < 0$.

PROOF OF THEOREM 5. Again we *could* prove the corresponding result for 1-dependent percolation on U and then exploit the percolation construction. This approach is somewhat involved, so fortunately there is another alternative: we can obtain the result from Theorem 4. Namely, for $\lambda > \lambda_c$ and $a \in (0, \alpha)$, Theorem 4 yields

$$P(r_m^- < am \quad \text{for some integer} \quad m \geq N) \leq Ce^{-\gamma N}.$$

Also, r_t^- is majorized by a rate λ simple Poisson process, so

$$P(r_t^- < 0 \quad \text{for some} \quad t \in [m, m+1], r_{m+1}^- > a(m+1)) \leq Ce^{-\gamma m}.$$

Combining these two inequalities we see that

$$P(r_t^- < 0 \quad \text{for some} \quad t \geq N) \leq Ce^{-\gamma N},$$

and hence

$$P(r_t^- < 0 \quad \text{for some} \quad t \geq T) \leq Ce^{-\gamma T}, \quad (T \text{ real}).$$

By symmetry

$$P(\ell_t^+ \leq 0 \leq r_t^- \quad \text{for all} \quad t \geq T) \geq 1 - 2Ce^{-\gamma T}.$$

The desired result for $A = 0$ now follows from the fact that on $\{\xi_T^0 \neq \varnothing\}$, $\tau^0 = \inf\{t: \ell_t^+ > r_t^-\}$. The result for general A is obtained by an easy restart argument similar to ones in [1] and [3]. Only the $A = 0$ case is needed below, so we omit the extension.

PROOF OF THEOREM 6. This is quite easy in light of Theorem 5. On $\{\tau^0 > t\}$, $r_t^0 = r_t^-$ (Lemma 3.1 of Durrett, 1980, again). If there were no deaths, r_t^- would move right one unit at rate λ so a simple estimate on the rate λ Poisson process shows $P(r_t^- > 2\lambda t) \leq Ce^{-\gamma t}$. Combining this estimate with Theorem 5, we have

$$P(\sup_t r_t^0 > n, \tau^0 < \infty) \leq P\left(\frac{n}{2\lambda} < \tau^0 < \infty\right) + P\left(\sup_t r_t^0 > n, \tau^0 \leq \frac{n}{2\lambda}\right) \leq Ce^{-\gamma n}. \square$$

In the next section we will need an extension of Theorem 7 which establishes exponential decay of higher order correlation functions. Theorem 7 is the special case $k = 2$ of the following result.

THEOREM 8. *Let* $\rho_t = P(x \in \xi_t^z)$, $\bar{\xi}_t(x) = \xi_t^Z(x) - \rho_t$. *If* $|x_1 - x_i| \geq 2m$ *for* $i = 2, \cdots, k$, *then there are constants* $C < \infty$ *and* $\gamma > 0$ *depending only on* k *such that*

$$0 \leq E[\textstyle\prod_{i=1}^k \bar{\xi}_t(x)] \leq Ce^{-\gamma m}.$$

PROOF. The left inequality is due to Harris; an easy proof can be fashioned after Theorem (2.17) of [5]. It suffices to prove the right inequality for the variables $\zeta_t(x) = 1_{\{\xi_t^x \neq \phi\}} - \rho_t$, which have the same joint distribution as the $\bar{\xi}_t(x)$. The argument splits into two cases: $t \leq m/2\lambda$ and $t > m/2\lambda$. To handle the first case, let

$$G(x) = \{\xi_s^x \subset [-m+x, m+x] \ \forall s \leq m/2\lambda\}, \quad G = \cap_{k=1}^k G(x_i).$$

A by now familiar comparison with a rate λ simple Poisson process yields

$$P(G^c) \leq kCe^{-\gamma m/2\lambda}.$$

Conditional on G, $\bar{\zeta}_t(x_1)$ and $(\bar{\zeta}_t(x_2), \cdots, \bar{\zeta}_t(x_k))$ are independent if $t \le m/2\lambda$, so

$$E[\prod_{i=1}^m \zeta_t(x_i)] \le P(G^c) + | E[\prod_{i=1}^m \zeta_t(x_i), G]|.$$

The second term on the right is majorized by

$$\begin{aligned} | E[\prod_{i=1}^m \zeta_t(x_i)| G]| &= | E[\zeta_t(x_1)| G]E[\prod_{i=2}^m \zeta_t(x_i)| G]| \\ &\le | E[\zeta_t(x_1)| G]| = | E[\zeta_t(x_1]| G(x_1)]| \\ &= | E[\zeta_t(x_1), G(x_1)]|/P(G(x_1)) \\ &= | E[\zeta_t(x_1), G(x_1)^c]|/P(G(x_1)) \\ &\le P(G^c)/(1 - P(G^c)) \le 2P(G^c), \end{aligned}$$

this last provided m is large enough. Hence the result holds for $t \le m/2\lambda$. To establish the result for $t > m/2\lambda$, let $\zeta_t'(x) = 1_{\{\xi^{x}_{m/2\lambda} \ne \varnothing\}} - \rho_t$, $\zeta_t''(x) = 1_{\{\xi^{x}_{m/2\lambda} \ne \varnothing\}} - \rho_{m/2\lambda}$. By Theorem 5,

$$P(\zeta_t'(x) \ne \zeta_t(x)) \le P\left(\frac{m}{2\lambda} < \tau^x < \infty\right) \le Ce^{-\gamma m/2\lambda},$$

and hence

$$E[| \prod_{i=1}^k \zeta_t(x_i) - \prod_{i=1}^k \zeta_t'(x_i)|] \le 2kCe^{-\gamma m/2\lambda}.$$

Also, if we have numbers $-1 \le a_i \le b_i \le 1$,

$$| \prod_{i=1}^k b_i - \prod_{i=1}^k a_i | \le \sum_{i=1}^k (b_i - a_i),$$

and so

$$E[| \prod_{i=1}^k \zeta_t'(x_i) - \prod_{i=1}^k \zeta_t''(x_i)|] \le kP(m/2\lambda < \tau^x < \infty) \le kCe^{-\gamma m/2\lambda}.$$

Finally, the argument for $t \le m/2\lambda$ yields

$$E[| \prod_{i=1}^k \zeta_t''(x_i)|] \le Ce^{-\gamma m},$$

so combining the last three expectation estimates completes the proof for $t > m/2\lambda$. □

To close this section we simply remark that Theorems 3 through 8 hold for all the systems described at the end of Section 2.

5. A strong law for the size process. This section deals with the growth rate of supercritical contact processes on Z. Roughly speaking, since (ξ_t^0) spreads at rate 2α, and has limiting density ρ, if it survives we expect that $|\xi_t^0|$ should be about $2\alpha\rho t$. As an application of the exponential estimates in the last section, we prove the following strong law.

THEOREM 9. *Let $\{(\xi_t^A)\}$ be the contact system of type $(*)$ with parameters θ and $\lambda > \lambda_c(\theta)$. Let α be its edge velocity, ρ its equilibrium density. Then*

$$\lim_{t\to\infty} \frac{|\xi_t^0|}{t} = 2\alpha\rho \quad \text{a.s. } on \quad \{\tau^0 = \infty\}.$$

PROOF OF THEOREM 9. Let $\xi_t = \xi_t^Z$, $\xi_t(x) = 1$ if $x \in \xi_t$ (=0 otherwise), $\rho_t = P(x \in \xi_t)$, $\bar{\xi}_t(x) = \xi_t(x) - \rho_t$. The main step is to show that for any $a \in (0, \infty)$,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{x=-at}^{at} \xi_t(x) = 2a\rho \quad \text{a.s.} \tag{19}$$

To get (21), we first show that

$$\int_0^\infty P\left(\left| \frac{1}{t} \sum_{x=-at}^{at} \bar{\xi}_t(x)\right| > \varepsilon\right) dt < \infty. \tag{20}$$

Following Borel, we check (20) by means of a 4th moment computation. Let $\bar{Z}_t = \sum_{x=-at}^{at} \bar{\xi}_t(x)$; one then has

$$(21) \qquad E[\bar{Z}_t^4] \le \sum_{w,x,y,z\in[-at,at]^4} | E[\bar{\xi}_t(w)\bar{\xi}_t(x)\bar{\xi}_t(y)\bar{\xi}_t(z)]| \le 24 \sum_{-at\le w\le x\le y\le z\le at} | E[\bar{\xi}_t(w)\bar{\xi}_t(x)\bar{\xi}_t(y)\bar{\xi}_t(z)]|.$$

Now the number of terms with $\max\{|w-x|, |y-z|\} = 2m$ is at most $(2at+1)^2 2(m+1)$: the number of ways to pick w and z is at most $(2at+1)^2$, and having chosen w and z there are at most $2(2m+1)$ ways to pick x and y. If the max above is attained by the first term, we apply Theorem 8 with $x_1 = w$; otherwise we take $x_1 = z$. It follows that

$$E[\bar{Z}_t^4] \le \left(24 \sum_{m=0}^{\infty} (4m+2)Ce^{-\gamma m}\right)(2at+1)^2 \le Ct^2.$$

The claim (20) now follows from Chebyshev's inequality. To get from (20) to (19) we next introduce the stopping times

$$T_0 = 0, \quad T_{n+1} = \inf\left\{t > T_n + 1 : \frac{|\bar{Z}_t|}{t} > 10\,\varepsilon\right\}, \quad n \ge 0,$$

and show that

$$(22) \qquad \int_0^{\infty} 1_{\{t^{-1}|\bar{Z}_t|>\varepsilon\}}\, dt = \infty \quad \text{a.s. on} \quad \{T_n < \infty\ \forall n\}.$$

Since (20) shows the expectation of the last integral is finite, and ε is arbitrary, this will prove (19). Write $E = \{T_n < \infty\ \forall n\}$, $F_1 = E \cap \{T_n^{-1}\bar{Z}_{T_n} > 10\varepsilon \text{ i.o.}\}$, $F_2 = E \cap \{T_n^{-1}\bar{Z}_{T_n} < -10\varepsilon$ i.o.$\}$, and note that $E = F_1 \cup F_2$. We check (22) by arguing separately on F_1 and F_2. For the first argument, observe that it is enough to show

$$(23) \qquad \inf_{t\ge1} P(\bar{Z}_s > (2e)^{-1}\bar{Z}_t\ \forall s \in [t, t+1] | (\bar{Z}_u)_{0\le u\le t}) \ge \delta_1 > 0 \quad \text{a.s.}$$

This is established by (a) comparing (Z_t) with the process in which all infection is suppressed from time t to time $t+1$, so that each of the Z_t infected sites at time t independently remains infected until time $t+1$ with probability e^{-1}, and (b) noticing that ρ_t is decreasing so we can ignore the change in the normalization. For the second argument we choose $0 < h \le 1$ so that $2(e^{2\lambda h} - 1)(2a) < \varepsilon$, and observe that it is enough to show that for T sufficiently large,

$$(24) \qquad \inf_{t\ge T} P(\bar{Z}_s < \bar{Z}_t + 2\varepsilon t\ \forall s \in [t, t+h] | (Z_u)_{0\le u\le t}) \ge \delta_2 > 0 \quad \text{a.s.}$$

To get (24) one dominates (Z_s) $s \in [t, t+h]$ by a continuous time branching process with binary branching at rate 2λ, starting from $Z_t \le 2at$. If we pick T large enough that $2a(\rho_T - \rho) < \varepsilon$, (24) follows. Having demonstrated (19), the theorem follows quite easily. Recall that from [1] we have

$$\xi_t^0 = \xi_t^Z \cap [\ell_t^0, r_t^0] \quad \text{for all} \quad t \quad \text{on} \quad \{\tau^0 = \infty\},$$

and

$$\frac{\ell_t^0}{t} \to -\alpha, \quad \frac{r_t^0}{t} \to \alpha \quad \text{a.s. on} \quad \{\tau^0 = \infty\}.$$

Thus, for any $a \in (0, \alpha)$ we have

$$\liminf_{t\to\infty} \frac{|\xi_t^0|}{t} = \liminf_{t\to\infty} \frac{|\xi_t^Z \cap [\ell_t^0, r_t^0]|}{t} \ge \liminf_{t\to\infty} \frac{|\xi_t^Z \cap [-at, at]|}{t} = 2a\rho$$

a.s. on $\{\tau^0 = \infty\}$. Similarly, for any $a > \alpha$,

$$\limsup_{t\to\infty} \frac{|\xi_t^0|}{t} \le \limsup_{t\to\infty} \frac{|\xi_t^Z \cap [-at, at]|}{t} = 2a\rho$$

a.s. on $\{\tau^0 = \infty\}$. This forces

$$\lim_{t\to\infty} t^{-1}|\,\xi_t^0\,| = 2\alpha\rho \quad \text{a.s. on} \quad \{\tau^0 = \infty\},$$

as was to be proved. □

Versions of Theorem 9, with suitable modifications, can be proved for the systems mentioned in Section 2, and, as we prove in Durrett and Griffeath (1981), there is also a strong law for the size of the basic contact process on Z^d, $d > 1$. Our result states that if λ is sufficiently large (e.g. $\lambda \geq 2$) then $t^{-d}|\,\xi_t^0\,| \to$ const. a.s. on $\{\tau^0 = \infty\}$. This result is clearly not the best possible, but it will require new methods to prove the results for all $\lambda > \lambda_c^{(d)}$.

Added in revision. The authors would like to thank the referee for his careful reading of the paper. His conscientious effort eliminated many typos and bozos in the original version.

Since this work was completed, Larry Gray has developed a graphical representation for attractive processes which allows him to prove the results in this paper for nearest neighbor attractive systems. His results show in particular that Theorems 1 and 2 are valid for $1 \leq \theta < \infty$. Whether strict monotonicity holds for $\theta < 1$ is not known and would seem to require very different methods.

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