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Oriented percolation in dimensions $d \geqslant 4$: bounds and asymptotic formulas

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Abstract. Let $p_c(d)$ be the critical probability for oriented percolation in $\mathbb{Z}^d$ and let $\mu(d)$ be the time constant for the first passage process based on the exponential distribution. In this paper we show that as $d \to \infty$, $dp_c(d) \to 1$ and $d\mu(d) \to \gamma$ where γ is a constant in $[e^{-1}, 2^{-1}]$ which we conjecture to be e^{-1}. In the case of $p_c(d)$ we have made some progress toward obtaining an asymptotic expansion in powers of d^{-1}. Our results show

$$d^{-1} + \tfrac{1}{2}d^{-3} + o(d^{-3}) \leqslant p_c(d) \leqslant d^{-1} + d^{-3} + O(d^{-4}).$$

The left hand side agrees, up to $o(d^{-3})$, with a (nonrigorous) series expansion of Blease (**1**, **2**):

$$p_c(d) = d^{-1} + \tfrac{1}{2}d^{-3} + d^{-4} + 3d^{-5} + \tfrac{21}{2}d^{-6} + \tfrac{479}{12}d^{-7} + O(d^{-8}).$$

1. *Introduction*

In this paper we will study oriented percolation and a companion process – first passage percolation. In these models we view $\mathbb{Z}^d$ as a graph with directed edges from each vertex $x \in \mathbb{Z}^d$ to $x+e_1, \ldots, x+e_d$ where $e_1, \ldots, e_d$ are the d unit vectors. In oriented percolation the edges b are independently open ($X(b) = 0$) or closed ($X(b) = \infty$) with probabilities p and $1-p$ respectively, the edge being open meaning that a fluid can move from one end to the other in the direction indicated. In this model, which was introduced by Hammersley (5), interest centres on whether or not there is an infinite open path in the resulting random network (the occurrence of an infinite open path indicates that the material is porous enough for the fluid to penetrate). A path r from x to y is a sequence $r = v_0, b_1, v_1, \ldots, b_n, v_n$ of vertices and edges, where $v_0 = x$, $v_n = y$, $v_i \in \mathbb{Z}^d$, and b_i is the (directed) edge joining v_{i-1} to v_i, $i = 1, 2, \ldots, n$. A path is open if and only if all its edges are open.

It is easy to show that if $p < 1/d$ there is no infinite open path (compare with a branching process) and, if one thinks about it for a while, one can show that if $d \geqslant 2$ and p is close enough to 1 there is an infinite open path (this is not as easy as it sounds). From the last two results and obvious monotonicity it follows that if we let $\Omega_\infty = \{$there is an infinite open path$\}$ then there is a critical probability $p_c = p_c(d) \in (0, 1)$ so that $P(\Omega_\infty) = 0$ if $p < p_c$ and $P(\Omega_\infty) > 0$ if $p > p_c$. The result mentioned above implies $p_c(d) \geqslant 1/d$. In this paper we will show that this trivial lower bound is asymptotically correct as $d \to \infty$. To state the result precisely we need a few definitions. Let $X_1, X_2, \ldots$ be iid with $P(X_n = e_i) = 1/d$ for $n \geqslant 1$, $1 \leqslant i \leqslant d$ and form the random walk $S_n = X_1 + \ldots + X_n$, with $S_0 = 0$. Let S'_n be an independent copy of S_n and let $\rho_d = \rho(d) = P$ (for some $m \geqslant 0$, $S_m = S'_m$ and $S_{m+1} = S'_{m+1}$) be the probability that the graphs of the random walks have an edge in common. Our upper bound is

$$p_c(d) \leqslant \rho(d). \tag{1·1}$$

This bound was first proved by Kesten (in response to a question asked by the authors). In Section 2 we give his simple proof and derive the asymptotic formula

$$\rho(d) = d^{-1} + d^{-3} + O(d^{-4}). \tag{1·2}$$

Combining (1·2) with the trivial lower bound $p_c(d) \geq d^{-1}$ we see that as $d \to \infty$ $dp_c(d) \to 1$. This gives a new proof of Mityugin's (6) result: $dp_c(d) \to \alpha$ and identifies the constant $\alpha = 1$.

In Section 3 we turn our attention to improving the lower bound on $p_c(d)$. By considering a kth order branching process approximation we show that

$$p_c(d) \geq d^{-1} + \frac{k-1}{2k} d^{-3} + O(d^{-4}),$$

and letting $k \to \infty$ conclude

$$p_c(d) \geq d^{-1} + \tfrac{1}{2} d^{-3} + o(d^{-3}). \tag{1·3}$$

This lower bound agrees up to $o(d^{-3})$ with a nonrigorous series expansion of Blease (1, 2). The reader should observe, however, that some new ideas will be needed to improve this since (a) the upper bound is $d^{-1} + d^{-3} + O(d^{-4})$ and (b) the occurrence of $k - \frac{1}{2}k < \frac{1}{2}$ in the kth approximation prevents us from getting more terms in the lower bound. Combining these results we obtain

THEOREM 1. *As* $d \to \infty$, $d^{-1} + \frac{1}{2}d^{-3} + o(d^{-3}) \leq p_c(d) \leq d^{-1} + d^{-3} + O(d^{-4})$.

In Section 4 we turn our attention to first passage percolation. In this model there is associated with each edge b in the graph an independent non-negative random variable $X(b)$ which gives the time it takes the fluid to pass through the edge in the direction indicated (movement in the other direction is impossible). For each path r we let $t(r) = \Sigma_{b \in r} X(b)$, and consider $t(x, y) = \min\{t(r) \mid r \text{ is a path from } x \text{ to } y\}$, the time it takes the fluid to get from x to y. Many results are known about the limiting behaviour of $t(x, y)$ as $|x - y| \to \infty$ (see Smythe and Wierman ((7); Cox and Durrett (3)). In this paper we will restrict our attention to the limiting behaviour of the 'point to line' process

$$\tau_n = \min\{t(0, x) : x \in H_n\},$$
$$H_n = \left\{x : \sum_{i=1}^{d} x_i = n\right\}.$$

It is a consequence of Kingman's subadditive ergodic theorem that if $E\tau_1 < \infty$ then as $n \to \infty$

$$n^{-1}\tau_n \to \inf_{n \geq 1} E(n^{-1}\tau_n) \text{ a.s.}$$

If we denote the limit by $\mu_F(d)$ to record the dependence upon the dimension d and the distribution F of the $X(b)$, then by generalizing Mityugin's (6) construction to first passage percolation we can obtain the following comparison result:

THEOREM 2. *If* $F_n(x) = 1 - (1 - F(x))^n$ *is the distribution of the minimum of* n *independent random variables with distribution* F, *then*

$$\mu_F(nd) \leq \mu_{F_n}(d). \tag{1·4}$$

If $E(x) = (1 - e^{-x})^+$ is the exponential distribution with mean 1 then

$E_n(x) \equiv 1-(1-E(x))^n = (1-e^{-nx})^+$ is the exponential distribution with mean $1/n$. So (1·4) becomes

$$\mu_E(nd) \leqslant \mu_{E_n}(d) = \frac{1}{n}\mu_E(d),$$

or

$$nd\mu_E(nd) \leqslant d\mu_E(d),$$

and it follows easily (see Section 4 for the details) that

$$\gamma \equiv \lim_{d\to\infty} d\mu_E(d)$$

exists.

To show that $\gamma \geqslant e^{-1}$ is simple. There are d^n paths of length n so

$$P\left(\tau_n(d) \leqslant \frac{na}{d}\right) \leqslant d^n P\left(U_n \leqslant \frac{na}{d}\right) = d^n \int_0^{(na/d)} e^{-s}\frac{s^{n-1}}{n-1!}\,ds \leqslant \frac{(na)^n}{n!},$$

where U_n is the sum of n iid random variables with distribution $E(x)$. For $a < e^{-1}$

$$\sum_{n=1}^{\infty} P\left(\tau_n(d) \leqslant \frac{na}{d}\right) < \infty,$$

i.e.

$$\liminf_{n\to\infty} n^{-1}\tau_n(d) \geqslant \frac{e^{-1}}{d} \text{ a.s.}$$

The lower bound calculated above is exactly analogous to the argument for oriented percolation which gave $p_c \geqslant 1/d$ so we conjecture the lower bound e^{-1}/d is asymptotically exact in this case as well. We have not been able to prove this result because we have not been able to get good upper bounds. It follows from subadditivity that, for each n,

$$\mu_E(d) \leqslant n^{-1}E\tau_n(d),$$

providing an infinite sequence of upper bounds for $\mu_E(d)$. We expect that $n^{-1}E\tau_n(d) = c_n d^{-1} + O(d^{-2})$ and letting $n \to \infty$ will produce the right asymptotics. The computations required are trivial for $n = 1$, easy for $n = 2$ but become horribly complicated when $n \geqslant 3$ and we see no hope of letting $n \to \infty$. Summarizing our results we have

THEOREM 3. *There is a constant γ such that $d\mu_d(E) \to \gamma$ as $d \to \infty$, and $e^{-1} \leqslant \gamma \leqslant 2^{-1}$.*

In addition to the problem of determining the constant γ there is the problem of investigating the limit of $d\mu_F(d)$ for a general distribution F. If $F'(0) = 1$ then n times the minimum of n such independent random variables converges weakly to an exponential with mean 1, so it seems reasonable to conjecture that for such a distribution

$$d\mu_F(d) \to \gamma_E,$$

where γ_E is the limit for the exponential distribution. Using Theorem 2 we have been able to show that if $\int x^\epsilon dF(x) < \infty$ for some $\epsilon > 0$ then

$$\limsup_{d\to\infty} d\mu_F(d) \leqslant \gamma_E, \tag{1·5}$$

but we have not been able to show

$$\liminf_{d\to\infty} d\mu_F(d) \geqslant \gamma_E.$$

We do not include the proof of (1·5) in this paper.

2. *Proof of the upper bound for* $p_c(d)$

In this section we will prove the following results which we stated in the introduction,

$$p_c(d) \leqslant \rho(d) \tag{2·1}$$

and

$$\rho(d) = d^{-1} + d^{-3} + O(d^{-4}) \quad \text{as} \quad d \to \infty. \tag{2·2}$$

Proof of (2·1). Let $\mathscr{R}_n$ be the collection of all possible (oriented) paths from 0 to $H_n = \{x: \Sigma_i x_i = n\}$ and let N_n be the number of open paths in $\mathscr{R}_n$. Let G_n be the collection of vertices and edges in the region $\Sigma_{i=1}^n x_i \leqslant n$ and let $\mathscr{F}_n$ be the σ-field generated by $\{X(b), b \in G_n\}$. Then

$$W_n = N_n/(d\rho)^n \quad \text{is martingale with respect to } \{\mathscr{F}_n\}. \tag{2·3}$$

To prove this suppose $b \in G_{n+1} - G_n$, and we let z_b be the end point of b which is in G_n, then

$$N_{n+1} = \sum_{b \in G_{n+1} - G_n} 1_{\{X(b)=0\}} N(z_b)$$

where $N(z_b)$ is the number of open paths from 0 to z_b. Since $N(z_b) \in \mathscr{F}_n$ and the $X(b)$, $b \in G_{n+1} - G_n$ are iid and independent of $\mathscr{F}_n$, it follows that

$$E(N_{n+1} \mid \mathscr{F}_n) = p \sum_{b \in G_{n+1} - G_n} N(z_b) = dpN_n,$$

i.e. $E(W_{n+1} \mid \mathscr{F}_n) = W_n$ a.s.

Remark. We use the notation $W_n = N_n/(dp)^n$ to suggest an analogy with the branching process result: Z_n/m^n is a martingale. Since $W_n \geqslant 0$ it follows immediately from (2·3) that, as $n \to \infty$,

$$N_n/(dp)^n \to W \text{ a.s.}, \tag{2·4}$$

$EW \leqslant 1$ and

$$P(N_n > 0 \text{ for all } n) \geqslant P(W > 0). \tag{2·5}$$

From (2·5) we see that to prove $p > p_c$ it suffices to show that $P(W > 0) > 0$. A common way to do this (and the one which gives the bound quoted above) is to show

$$EW_n^2 \to c < \infty \quad \text{as} \quad n \to \infty, \tag{2·6}$$

for then we have $EW = 1$ and an application of Cauchy–Schwarz

$$(EW)^2 = E(W 1_{(W<0)})^2 \leqslant EW^2 P(W > 0)$$

leads to the estimate

$$P(W > 0) \geqslant \frac{(EW)^2}{EW^2} \geqslant \frac{1}{c}. \tag{2·7}$$

To estimate EW_n^2 we observe that

$$EN_n^2 = \sum_{r, s \in \mathscr{R}_n} P(t(r) = 0, t(s) = 0) = \sum_{r, s \in \mathscr{R}_n} p^{2n - K(r,s)},$$

where $K(r, s)$ is the number of edges r and s have in common. Rewriting the last expression in terms of the random walks we have

$$EN_n^2 = (dp)^{2n} Ep^{-K(n)},$$

where $K(n)$ is the number of edges the two random walks paths S_m, $0 \leqslant m \leqslant n$, and

S'_m, $0 \leqslant m \leqslant n$, have in common. As $n \to \infty\, K(n) \uparrow K =$ the number of edges two random walks ever have in common so it follows from monotone convergence that

$$EW_n^2 \to E(p^{-K}).$$

By the strong Markov property

$$P(K = k) = \rho_d^k(1-\rho_d),$$

so $E(p^{-K}) < \infty$ if and only if $p > \rho_d$, in which case

$$E(p^{-K}) = \sum_{k=0}^{\infty} p^{-k}\rho_d^k(1-\rho_d) = (1-\rho_d)\left(1-\frac{\rho_d}{p}\right)^{-1}.$$

Combining this result with (2·7) we see that if $p > \rho_d$

$$P(W > 0) \geqslant \frac{1-\rho_d/p}{(1-\rho_d)} = \frac{p-\rho_d}{p(1-\rho_d)}, \tag{2·8}$$

proving (2·1). We turn now to the proof of

$$\rho(d) = d^{-1} + d^{-3} + O(d^{-4}). \tag{2·2}$$

Proof. Let $\tau = \inf\{k \geqslant 0 : S_k = S'_k, S_{k+1} = S'_{k+1}\}$ (where $\inf \varnothing = \infty$). In this notation $\rho(d) = P(\tau < \infty)$. Simple counting shows

$$P(\tau = 0) = d^{-1},$$
$$P(\tau = 1) = 0,$$
$$P(\tau = 2) = d^{-3} - d^{-4},$$

so the proof of (2·2) will be complete when we show

$$\lim_{d\to\infty} d^{l+1}P(\tau \geqslant l) = \lim_{d\to\infty} d^{l+1}P(\tau = l) \leqslant l!. \tag{2·9}$$

Remark. It is possible to compute the second limit explicitly when l is small but we have not been able to find a formula.

Proof of (2·9). We note first that

$$P(\tau = k) \leqslant d^{-1}P(S_k = S'_k),$$

and since S_k and S'_k are independent

$$P(S_k = S'_k) \leqslant \max_{x \in H_k} P(S_k = x).$$

When $1 \leqslant k \leqslant d$ the maximum occurs when all the steps have been taken in different directions so we have

$$P(S_k = x) \leqslant d^{-k}k!$$

and combining these estimates gives

$$P(l \leqslant \tau \leqslant d) \leqslant d^{-1}\sum_{k=l}^{d} d^{-k}k! \leqslant d^{-1}(d^{-l}l! + d^{-(l+1)}(l+1)! + (d-l)\,d^{-(l+2)}(l+2)!)$$
$$\leqslant d^{-(l+1)}l! + d^{-(l+2)}(l+3)!.$$

It remains then to estimate $P(\tau > d)$. A little thought shows that if $k \geqslant jd$

$$P(S_k = x) \leqslant d^{-jd}\frac{(jd)!}{(j!)^d}.$$

Using now the form of Stirling's formula given in Feller (4), p. 54

$$e^{-\frac{1}{12}} \leqslant \frac{n!}{n^n e^{-n}\sqrt{(2\pi n)}} \leqslant 1 \quad (n \geqslant 1), \tag{2·10}$$

we get

$$P(S_k = x) \leqslant \frac{j^{jd}(2\pi jd)^{\frac{1}{2}}}{(j^j(2\pi j)^{\frac{1}{2}}e^{-\frac{1}{12}})^d} = (2\pi d)^{\frac{1}{2}} j^{(1-d)/2}(e^{\frac{1}{12}}/\sqrt{(2\pi)})^d.$$

Combining the estimates above with the fact that $d \geqslant 4$

$$P(d < \tau < \infty) \leqslant d^{-1} \sum_{j=1}^{\infty} \sum_{k=jd+1}^{(j+1)d} P(S_k = S'_k) \leqslant (2\pi d)^{\frac{1}{2}} \left(\frac{e^{\frac{1}{12}}}{\sqrt{(2\pi)}}\right)^d \sum_{j=1}^{\infty} j^{-\frac{3}{2}},$$

which approaches zero exponentially rapidly as $d \to \infty$ ($e^{\frac{1}{12}} < \sqrt{(2\pi)}$).

3. *Lower bounds for the critical probability*

One of the first things that was noticed about oriented percolation is that if we let Z_n be a branching process with $Z_0 = 1$ and offspring distribution

$$P(Z_1 = j) = \binom{d}{j} p^j (1-p)^{d-j} \quad (0 \leqslant j \leqslant d),$$

then the branching process and the percolation process can be constructed (coupled) on the same probability space with the following property. If W_0 is the set of vertices we can reach from 0 along open paths, then

$$Z_n \geqslant |W_0 \cap H_n|,$$

where $|A|$ denotes the cardinality of the set A. Since the branching process becomes extinct almost surely when $EZ_1 = dp \leqslant 1$, it follows that

$$p_c(d) \geqslant d^{-1}. \tag{3·1}$$

It is easy to see how this bound can be improved and a sequence of lower bounds can be developed. For each positive integer m we construct a branching process $\{Z_{m,n}\}_{n=0,1,2,\dots}$ with $Z_{m,0} = 1$ and offspring distribution

$$P(Z_{m,1} = j) = P(|W_0 \cap H_m| = j).$$

Again, by coupling we have

$$P(Z_{m,n} > 0) \geqslant P(|W_0 \cap H_{mn}| > 0),$$

and it follows that

$$p_c(d) \geqslant \pi_m(d) \equiv \sup\{p \mid EZ_{m,1} \leqslant 1\}.$$

Let $f_m(p) = EZ_{m,1}$.

It is easy to compute $\pi_2(d)$. By considering the cases $x = 2e_i$ and $x = e_i + e_j$ we obtain

$$f_2(p) = dp^2 + \tfrac{1}{2}d(d-1)(2p^2 - p^4).$$

Setting $f_2(p) = 1$ and solving for p^2 gives

$$\pi_2(d) = \frac{d}{d-1}[1 - (1 - 2d^{-2} + 2d^{-3})^{\frac{1}{2}}]^{\frac{1}{2}},$$

and a little patience shows that

$$\pi_2(d) = d^{-1} + \tfrac{1}{4}d^{-3} + O(d^{-4}) \quad \text{as} \quad d \to \infty. \tag{3·2}$$

The computational procedure just employed breaks down when $m = 3$. This time if we consider the possible cases $3e_i$, $2e_i + e_j$, $e_i + e_j + e_h$ and use the inclusion–exclusion formula to compute the probability of at least one open path we find after several hours of computing and checking (see (3·5) below) that

$$f_3(p) = dp^3 + d(d-1)(3p^3 - 2p^5 - p^6 + p^7) \\ + \tfrac{1}{6}d(d-1)(d-2)(6p^3 - 6p^5 - 9p^6 + 6p^7 + 12p^8 - 4p^9 - 9p^{10} + 6p^{11} - p^{12}),$$

details left to the reader. It is clear that it is not possible to solve for $\pi_3(p)$.

Fortunately, to derive a bound for $\pi_3(d)$ of the form $a_1 d^{-1} + a_2 d^{-2} + a_3 d^{-3} + O(d^{-4})$ we do not have to solve the equation $f_3(p) = 1$ or worry about any terms in the expansion of $f_3(p)$ which are of order p^7 or larger. We will describe our method first in a casual fashion and at the end give the short supplementary argument needed to make our computation rigorous.

Let $p = \Sigma_{n=1}^{4} a_n d^{-n}$. When we put this into $f_3(p)$ we get something of the form $\Sigma_{n=0}^{48} b_n d^{-n}$. When $p = \pi_3(d)$, $f_3(p) = 1$, and so setting $b_0 = 1$, $b_1 = b_2 = b_3 = 0$ gives four equations in the a_i, $1 \leqslant i \leqslant 4$. The solution is $a_1 = 1$, $a_2 = 0$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{6}$. (The reader who is interested in seeing the details without too much work should try this method to derive the result for $\pi_2(d)$.)

This computation suggests that

$$\pi_3(d) = d^{-1} + \tfrac{1}{3}d^{-3} + O(d^{-4}). \tag{3·3}$$

To prove this, we now set $p_0 = d^{-1} + \frac{1}{3}d^{-3} + \frac{1}{6}d^{-4}$ and for $\epsilon > 0$ substitute $p_0 \pm \epsilon d^{-4}$ in f_3 to obtain

$$f_3(p_0 + \epsilon d^{-4}) = 1 + 3\epsilon d^{-3} + O(d^{-4}),$$
$$f_3(p_0 - \epsilon d^{-4}) = 1 - 3\epsilon d^{-3} + O(d^{-4}).$$

Clearly $f_3(p_0 - \epsilon d^{-4}) < 1 < f_3(p_0 + \epsilon d^{-4})$ for sufficiently large d, which implies

$$p_0 - \epsilon d^{-4} < \pi_3(d) < p_0 + \epsilon d^{-4},$$

a somewhat stronger statement than (3·3).

Neither $\pi_2(d)$ or $\pi_3(d)$ is as good as the lower bound given in Theorem 1. To achieve this bound we must consider $\pi_k(d)$ and let $k \to \infty$. Let $p = \Sigma_{n=1}^{4} a_n d^{-n}$ and expand $f_k(p)$ in the form

$$f_k(p) = \sum_{n,\, m \geqslant 0} c_k(m, n)\, d^n p^m = \sum_{n \geqslant 0} b_n d^{-n}.$$

Setting $b_0 = 1$, $b_1 = b_2 = b_3 = 0$ and solving for the a_i, $1 \leqslant i \leqslant 4$, will involve only the terms $c_k(m, n)\, d^n p^m$ with $m - n \leqslant 3$, which means we can ignore the higher order terms. We will prove that

$$f_k(p) = d^k p^k - \frac{k-1}{2} d^k p^{k+2} + \frac{k-1}{2} d^{k-1} p^{k+2} - \tfrac{3}{2}(k-2)\, d^k p^{k+3} + \sum_{m-n>3} c_k(m, n)\, d^n p^m. \tag{3.4}$$

Consequently, if

$$p_0 = d^{-1} + \frac{k-1}{2k} d^{-3} + \frac{2k-5}{2k} d^{-4},$$

$$f_k(p_0 + \epsilon d^{-4}) = 1 + k\epsilon d^{-3} + O(d^{-4}),$$
$$f_k(p_0 - \epsilon d^{-4}) = 1 - k\epsilon d^{-3} + O(d^{-4}).$$

This shows that, for fixed k, as $d \to \infty$

$$\pi_k(d) = d^{-1} + \frac{k-1}{2k} d^{-3} + O(d^{-4}).$$

We turn now to the proof of (3·4), assuming always that $k < d$. Let $\Pi(k) = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_j): \alpha_i$ are integers, $1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \alpha_j, \Sigma_{i=1}^j \alpha_i = k\}$ be the set of partitions of the integer k. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_j) \in \Pi(k)$ let z^α be the element of H_k whose ith coordinate is α_i, $1 \leqslant i \leqslant j$, and let $H_k(\alpha) = \{z \in H_k$: a permutation of the coordinates of z yields $z^\alpha\}$. Then H_k is the disjoint union $H_k = \cup_{\alpha \in \Pi(k)} H_k(\alpha)$. Let $\mathscr{R}(z)$ be the set of paths from 0 to z. If $z \in H_k(\alpha)$, then $|\mathscr{R}(z)| = |\mathscr{R}(z^\alpha)| = k!/\alpha_1! \ldots \alpha_j!$ and $P(\cup_{r \in \mathscr{R}(z)} \{r \text{ is open}\}) = P(\cup_{r \in \mathscr{R}(z^\alpha)} \{r \text{ is open}\})$. It is also clear that $|H_k(\alpha)|$ is a polynomial in d of degree at most k.

Using the decomposition of H_k we obtain

$$\begin{aligned} f_k(p) &= \sum_{z \in H_k} P\Big(\bigcup_{r \in \mathscr{R}(z)} \{r \text{ is open}\}\Big) \\ &= \sum_{\alpha \in \Pi(k)} |H_k(\alpha)| \Big\{ \sum_{r \in \mathscr{R}(z^\alpha)} P(r \text{ is open}) - \tfrac{1}{2} \sum_{\substack{r, s \in \mathscr{R}(z^\alpha) \\ r \neq s}} P(r, s \text{ are open}) \\ &\quad + \tfrac{1}{6} \sum_{\substack{r, s, t \in \mathscr{R}(z^\alpha) \\ r, s, t \text{ distinct}}} P(r, s, t \text{ are open}) - \ldots \Big\}, \end{aligned} \tag{3·5}$$

where there are $|\mathscr{R}(z^\alpha)|$ terms in the inclusion–exclusion expansion. Since each of the probabilities above is a polynomial in p, and $|H_k(\alpha)|$ is a polynomial in d, this expresses $f_k(p)$ in the form $\Sigma_{m,n \geqslant 0} c_k(m, n) d^n p^m$, $c_k(m, n)$ independent of p and d.

The first term in the expansion is

$$\sum_{\alpha \in \Pi(k)} |H_k(\alpha)| \sum_{r \in \mathscr{R}(z^\alpha)} P(r \text{ is open}) = \sum_{z \in H_k} \sum_{r \in \mathscr{R}(z)} p^k = (dp)^k. \tag{3·6}$$

We will begin by showing that the third and higher order terms can be neglected. Note that if $r_1, r_2, \ldots, r_l$, $l \geqslant 3$, are distinct elements of $\mathscr{R}(z^\alpha)$, then $P(r_1, r_2, \ldots, r_l$ are open$) = p^{k+i}$ for some $i \geqslant 4$ since there must be at least $k+4$ distinct edges in the union of the paths $r_1, r_2, \ldots, r_l$. Since $|H_k(\alpha)|$ is a polynomial in d of degree at most k, this shows that the third and higher order terms in (3·5) contain only terms of the form $c_k(m, n) d^n p^m$ with $n \leqslant k$ and $m \geqslant k+4$, and hence can be neglected. It remains to evaluate the second term of the inclusion–exclusion expansion.

As in Section 2 let $K(r, s)$ be the number of edges r and s have in common. Then

$$\begin{aligned} &\sum_{\alpha \in \Pi(k)} |H_k(\alpha)| \sum_{\substack{r, s \in \mathscr{R}(z^\alpha) \\ r \neq s}} P(r, s \text{ are open}) \\ &\qquad = \sum_{\alpha \in \Pi(k)} |H_k(\alpha)| \sum_{r \in \mathscr{R}(z_\alpha)} \sum_{j=0}^{k-2} p^{2k-j} |\{s \in \mathscr{R}(z^\alpha): K(r, s) = j\}| \end{aligned} \tag{3·7}$$

($K(r, s) = k-1$ or k is impossible if $r \neq s$). Since $|H_k(\alpha)|$ is a polynomial in d of degree at most k, we can neglect the terms with $j \leqslant k-4$ in this expansion. For the remaining terms we have

$$\begin{aligned} &\sum_{\alpha \in \Pi(k)} |H_k(\alpha)| \sum_{r \in \mathscr{R}(z^\alpha)} \sum_{j=k-3}^{k-2} p^{2k-j} |\{s \in \mathscr{R}(z^\alpha): K(r, s) = j\}| \\ &\qquad = p^{k+2} \sum_{z \in H_k} \sum_{\substack{r, s \in \mathscr{R}(z) \\ K(r, s) = k-2}} 1 - p^{k+3} \sum_{z \in H_k} \sum_{\substack{r, s \in \mathscr{R}(z) \\ K(r, s) = k-3}} 1. \end{aligned} \tag{3·8}$$

Let $\mathscr{R}_k = \cup_{z\in H_k}\mathscr{R}(z)$, and for $r \in R_k$ let r_i be the ith point of $\mathbb{Z}^d$ of r. Then

$$\sum_{z\in H_k}\sum_{\substack{r,s\in\mathscr{R}(z)\\ K(r,s)=k-2}} 1 = \sum_{r\in\mathscr{R}_k}\left|\{s\in\mathscr{R}_k: K(r,s)=k-2\}\right|$$
$$= \sum_{j=1}^{k-1}\sum_{r\in\mathscr{R}_k}\left|\{s\in\mathscr{R}_k: s_j \neq r_j, s_i = r_i \text{ for } i\neq j\}\right|.$$

Since

$$\left|\{s\in\mathscr{R}_k: s_j \neq r_j, s_i = r_i \text{ for } i\neq j\}\right| = \begin{cases}1 & \text{if} \quad r_j - r_{j-1} \neq r_{j+1}-r_j,\\ 0 & \text{otherwise}\end{cases}$$

and

$$\left|\{r\in\mathscr{R}_k: r_j - r_{j-1} \neq r_{j+1}-r_j\}\right| = d^{k-1}(d-1),$$
$$\sum_{z\in H_k}\sum_{\substack{r,s\in\mathscr{R}(z)\\ K(r,s)=k-2}} 1 = (k-1)\,d^{k-1}(d-1). \tag{3·9}$$

The argument for the second term is similar but slightly more involved. The first step is

$$\sum_{z\in H_k}\sum_{\substack{r,s\in\mathscr{R}(z)\\ K(r,s)=k-3}} 1 = \sum_{r\in\mathscr{R}_k}\left|\{s\in\mathscr{R}_k: K(r,s)=k-3\}\right|$$
$$= \sum_{j=1}^{k-2}\sum_{r\in\mathscr{R}_k}\left|\{s\in\mathscr{R}_k: s_j\neq r_j, s_{j+1}\neq r_{j+1}, s_i = r_i \text{ for } i\neq j, j+1\}\right|.$$

Letting $\Delta_i = r_{j+i} - r_{j+i-1}$,

$$\left|\{s\in\mathscr{R}_k: s_j\neq r_j, s_{j+1}\neq r_{j+1}, s_i = r_i \text{ for } i\neq j, j+1\}\right|$$
$$= \begin{cases}3 & \text{if } \Delta_0, \Delta_1, \Delta_2 \text{ distinct},\\ 1 & \text{if} \quad \Delta_0 = \Delta_1 \neq \Delta_2 \quad\text{or}\quad \Delta_1 = \Delta_2 \neq \Delta_0,\\ 0 & \text{otherwise},\end{cases}$$

$$\left|\{r\in\mathscr{R}_k: \Delta_0, \Delta_1, \Delta_2 \text{ distinct}\}\right| = d^{k-2}(d-1)(d-2),$$

and

$$\left|\{r\in\mathscr{R}_k: \Delta_0 = \Delta_1 \neq \Delta_2, \text{ or } \Delta_1 = \Delta_2 \neq \Delta_0\}\right| = 2d^{k-2}(d-1).$$

This gives us

$$\sum_{z\in H_k}\sum_{\substack{r,s\in\mathscr{R}(z)\\ K(r,s)=k-3}} 1 = (k-2)\,d^{k-2}(d-1)(3d-4). \tag{3·10}$$

By combining (3·5)–(3·10) we obtain (3·4).

4. *First passage percolation proofs*

In this section we will prove the results about first passage percolation which we stated in the introduction. We will start with

THEOREM 2. *If $F_n(x) = 1-(1-F(x))^n$ is the distribution of the minimum of n independent random variables with distribution F, then*

$$\mu_F(nd) \leqslant \mu_{F_n}(d). \tag{4·1}$$

Proof. Suppose the nd-dimensional process is constructed from iid variables $X(b)$, $b\in\mathbb{Z}^{nd}$ with distribution F, defined on some probability space $(\Omega, \mathscr{F}, P)$, and let $t(z)$ be the associated passage time from 0 to z. To prove the lemma we will define a mapping $\phi: \Omega \times \mathbb{Z}^d_+ \to \mathbb{Z}^{nd}_+$ and iid variables $Y(b)$ $b \in \mathbb{Z}^d$ with distribution F_n in such a way that, if $t'(x)$ is the associated passage time from 0 to x for the $Y(b)$ variables, then

$$t(\phi x) \leqslant t'(x) \quad \text{and} \quad \|\phi x\|_1 = \|x\|_1, \tag{4·2}$$

where for $x = (x_1, x_2, \ldots, x_d)$, $\|x\|_1 = \Sigma_{i=1}^d |x_i|$.

Since (4·2) clearly implies (4·1) the proof will be done when we have described how to construct ϕ and the variables $Y(b)$.

Step 0. Let $\phi(\omega, 0) = 0$ for all ω.

Step 1. Let a_{ij} be the edge from 0 to $e_{(j-1)n+i}$ in $\mathbb{Z}^{nd}$ and let b_j be the edge from 0 to e_j in $\mathbb{Z}^d$. We let

$$Y(b_j) = \min\{X(a_{ij}), 1 \leqslant i \leqslant n\},$$
$$i(j) = \min\{i: X(a_{ij}) = Y(b_j)\},$$
$$\phi(\omega, e_j) = e_{(j-1)n+i(j)}.$$

Induction Step. Let $\pi: \mathbb{Z}^{nd}_+ \to \mathbb{Z}^d_+$ be the projection operator

$$(\pi x)_j = \sum_{i=(j-1)n+1}^{jn} x_i,$$

and assume that ϕ has been defined on $\Omega \times G_k$ in such a way that $\pi\phi(\omega, y) \equiv y$ (this is true when $k = 1$). Let $z \in H_{k+1}$ and for each j for which $z - e_j \in H_k$ let $z_j = z - e_j$ and let b_j be the edge from z_j to z. Our problem is to define $Y(b_j)$ and $\phi(z)$. The first definition is handled as before: let a_{ij} be the edge from $\phi(z_j)$ to $\phi(z_j) + e_{(j-1)n+i}$ and let

$$Y(b_j) = \min\{X(a_{ij}): 1 \leqslant i \leqslant n\}.$$

The definition of $\phi(z)$ is a little more subtle: using the passage times for the edges in G_k and the $Y(b_j)$'s just defined compute $t'(z)$, the travel time form 0 to z. Find all the routes from 0 to z which have this travel time and let $C(z)$ be the set of points in H_k which appear in some route. Clearly $C(z) \neq \varnothing$ so we can let z^* be the lexicographic minimum of $C(z)$, define h by $z_h = z^*$ and let

$$i(h) = \min\{i: X(a_{ih}) = Y(b_h)\},$$
$$\phi(z) = \phi(z^*) + e_{(h-1)n+i(h)}.$$

It is clear from the construction that the new Y's are iid and independent of the ones previously constructed, that $\pi\phi(\omega, z) \equiv z$ and consequently that $\|\phi z\|_1 = \|z\|_1$. It remains then to show $t'(z) \geqslant t(\phi z)$. To prove this observe that the travel time from 0 to z^* on an optimal route from 0 to z must equal $t'(z^*)$ so

$$t'(z) = t'(z^*) + Y(b_h) \geqslant t(\phi(z^*)) + X(a_{ih}) \geqslant t(\phi(z)).$$

This completes the proof of Theorem 2.

Now consider the exponential distribution $E(x) = (1 - e^{-x})^+$. The next result is

THEOREM 3. *There is a constant* γ *such that*

$$d\mu_E(d) \to \gamma \quad \text{as} \quad d \to \infty, \tag{4·3}$$

and

$$e^{-1} \leqslant \gamma \leqslant 2^{-1}. \tag{4·4}$$

Proof. For $x > 0$, $E_n(x) = 1 - (1 - E(x))^n = 1 - e^{-nx}$, so $\mu_{E_n}(d) = n^{-1}\mu_E(d)$, and (4·1) becomes

$$\mu_E(nd) \leqslant n^{-1}\mu_E(d).$$

To show $\lim_{d\to\infty} d\mu_E(d)$ exists, suppose $nd \leqslant j < n(d+1)$, then

$$j\mu_E(j) \leqslant j\mu_E(nd) \leqslant jn^{-1}\mu_E(d) \leqslant (d+1)\,\mu_E(d).$$

Since $\mu_E(d) \leqslant Et_1 = 1/d$, it follows that

$$\limsup_{j\to\infty} j\mu_E(j) \leqslant \liminf_{d\to\infty} d\mu_E(d) \leqslant 1,$$

and

$$\gamma \equiv \lim_j j\mu_E(j) \quad \text{exists.}$$

At this point we have proved (4·3) so we turn our attention to the inequality (4·4). The lower bound was proved in the Introduction. There are several ways of obtaining the upper bound $\gamma \leqslant 2^{-1}$. The simplest way is to use 'ordinary' oriented percolation to construct paths with small travel times. Let $t_d > 0$ such that $1-e^{-t_d} > p_c(d)$. Call an edge b open if $X(b) \leqslant t_d$ and closed otherwise. Then for all ω in some $\Omega_\infty \subset \Omega$, with $P(\Omega_\infty) > 0$, there is an infinite connected path of open edges starting at 0. Let R_∞ be the minimal (in the lexicographical order) infinite path of open edges and let R_k be the part of R_∞ from 0 to H_k. Then for $\omega \in \Omega_\infty$

$$\tau_k \leqslant \sum_{b\in R_k} X(b),$$

where R_k is random, and the variables $X(b)$, $b \in R(k)$ are independent of R_k and they are conditioned to be no larger than t_d. By the strong law, a.s. on Ω_∞,

$$\lim_{k\to\infty} k^{-1}\tau_k \leqslant \lim_{k\to\infty} k^{-1} \sum_{b\in R_k} X(b) = E(X(b) \mid X(b) \leqslant t_d) = \frac{1-e^{-t_d}-t_d e^{-t_d}}{1-e^{-t_d}}.$$

The right hand side above is therefore an upper bound for $\mu_E(d)$ for all t_d such that $1-e^{-t_d} > 1-p_c(d)$. Consequently

$$\mu_E(d) \leqslant 1+p_c(d)^{-1}(1-p_c(d))\ln(1-p_c(d)) = \sum_{n=1}^{\infty} p_c(d)^n/n(n+1).$$

Since $dp_c(d) \to 1$ as $d\to\infty$, $\gamma \leqslant 2^{-1}$.

An approach suggested by Kesten shows that

$$\mu_E(d) \leqslant \rho(d)/2, \tag{4·5}$$

which gives the same bound $\gamma \leqslant 2^{-1}$. The inequality (4·6) is proved using the method of Section 2. Fix $a > \rho(d)/2$ and let N_n be the number of paths r from 0 to H_n with $t(r) \leqslant an$. Somewhat detailed estimates of EN_n and EN_n^2 show

$$\liminf_{n\to\infty} \frac{(EN_n)^2}{EN_n^2} > 0,$$

which implies $\mu_E(d) \leqslant a$. Since this method does not improve the upper bound for γ we will not present the details.

Finally, as mentioned in the introduction, subadditivity gives us the sequence of inequalities

$$\mu_E(d) \leqslant n^{-1}E\tau_n(d).$$

The case $n=1$ was considered in the proof of (4·2); it yields $\mu_E(d) \leqslant d^{-1}$. Explicit calculations can be made for $n=2$; it can be shown that

$$P(\tau_2(d) > s) = e^{-s}\left[1+\frac{1}{d-1}(1-e^{-(d-1)s})\right],$$

and that

$$E\tau_2(d) \sim d^{-1}(e-1).$$

This yields $\gamma \leqslant 2^{-1}(e-1)$, which is not as good as (4·3). Presumably the cases $n \geqslant 3$ give better bounds, but the details become horrendously complicated.

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