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# Extension of Domains with Finite Gauge

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Let $\{X_t\}$ be the Brownian motion in $R^d$, $d \geqq 1$; $E^x$ and $P^x$ denote respectively the expectation and probability for the process with $X_0 = x$. Let $D$ be a domain in $R^d$, $d \geqq 2$, with $m(D) < \infty$ where $m$ is the Lebesgue measure in $R^d$. All given sets and functions below are Borel measurable. For a bounded function $q$ in $R^d$, and positive ($\geqq 0$) $f$ on $\partial D$, we define

$$u(D, q, f; x) = E^x\{e_q(\tau_D) f(X(\tau_D))\} \tag{1}$$

where

$$e_q(t) = \exp\left(\int_0^t q(X_s)ds\right)$$

and

$$\tau_D = \inf\{t > 0 | X(t) \notin D\}.$$

It is proved in [4] that if $u(D, q, f; \cdot) \not\equiv \infty$ in $D$, then it is bounded in $\bar{D}$. We call $u(D, q, 1; \cdot)$ the *gauge* for $(D, q)$. Since $q$ is fixed in this paper but $D$ will vary, we will denote the gauge by $u_D$, and say it is finite when $u_D \not\equiv \infty$ in $D$. We write also $\|u_D\|$ for $\sup_{x \in \bar{D}} u_D(x)$. The importance of the gauge is evident from the results in [4]. In this paper we study the question: if $u_D$ is finite, can we enlarge $D$ to a domain $G$ so that $u_G$ is still finite? First, we prove that we can always add a finite number of balls centered on $\partial D$ to get such a $G$. But as we add more and more such balls, their radii may have to shrink to zero so that it is not always possible to cover the entire boundary of $D$. In fact, we shall give a simple example of a regular domain $D$ with $u_D < \infty$, such that if part of $D$ is added, the resulting domain $G$ may have $u_G = \infty$. However, if $D$ satisfies a uniform cone condition, in particular, if $D$ is Lipschitzian, then there exists a domain $G \supset \bar{D}$ such that $u_G < \infty$. A discussion of these results from the point of view of eigenvalues follows the theorems.

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Let $z\in\partial D$ and $B(z,\varepsilon)$ be the ball with center $z$ and radius $\varepsilon$. We write $B_\varepsilon$ for $B(z,\varepsilon)$ below, and put

$$\varphi_\varepsilon(x)=u(D,q,1_{B_\varepsilon};x). \tag{2}$$

If $u_D<\infty$, then $\lim_{\varepsilon\to 0}\varphi_\varepsilon(x)=0$ for all $x\in D$, because the singleton $\{z\}$ is a polar set. That is why we have supposed $d\geqq 2$. Moreover, $\varphi_\varepsilon\in C^{(1)}(D)$. Hence the convergence of $\varphi_\varepsilon$ as $\varepsilon\downarrow 0$ is uniform in each compact subset of $D$ by Dini's theorem. This is not sufficient for our later application, and we need the strengthening given below.

**Lemma.** *Suppose $u_D<\infty$. Let $A$ be a compact subset of $\bar{D}$ and $z\notin A$. Then $\lim_{\varepsilon\to 0}\varphi_\varepsilon(x)=0$ uniformly for $x\in A$.*

*Proof.* There is $\varepsilon>0$ such that $A$ is disjoint from $\overline{B(z,\varepsilon)}$. Let $x_0\in A$ and fix a number $r>0$ so that $r<\varrho(A,\overline{B(z,\varepsilon)})$, where $\varrho$ denotes the distance, and also such that

$$\sup_{x\in B(x_0,r)} E^x\{\exp(Q\tau_{B(x_0,r)})\}<\infty \tag{3}$$

where $Q=\sup_x|q(x)|$. Writing $\tau_r$ for $\tau_{B(x_0,r)}$, we have by the strong Markov property, for each $x\in B(x_0,r)$:

$$\varphi_\varepsilon(x)=E^x\{\tau_r<\tau_D;e_q(\tau_r)\varphi_\varepsilon(X(\tau_r))\}. \tag{4}$$

Put $\tilde{\varphi}_\varepsilon=1_D\varphi_\varepsilon$ and define for $x\in B(x_0,r)$:

$$\psi_\varepsilon(x)=E^x\{e_q(\tau_r)\tilde{\varphi}_\varepsilon(X(\tau_r))\}=u(B(x_0,r),q,\tilde{\varphi}_\varepsilon;x).$$

Since $\varphi_\varepsilon\leqq u_D$, $\varphi_\varepsilon$ is bounded in $\bar{D}$ and $\tilde{\varphi}_\varepsilon$ is bounded in $R^d$. Now (3) implies that $u_{B(x_0,r)}<\infty$, hence $\psi_\varepsilon\in C^{(1)}(B(x_0,r))$ by Theorem 2.1 of [4]. By Dini's theorem, $\lim_{\varepsilon\to 0}\psi_\varepsilon=0$ uniformly in $B(x_0,r/2)$. Since $\varphi_\varepsilon\leqq\psi_\varepsilon$, we have $\lim_{\varepsilon\to 0}\varphi_\varepsilon=0$ uniformly in $B(x_0,r/2)$. This being true for every $x_0\in A$, and $A$ being compact, the lemma follows.

**Theorem 1.** *Let $D$ be a domain in $R^d$, $d\geqq 2$, with $m(D)<\infty$ and $u_D<\infty$. For any $z\in\partial D$ there exists $\varepsilon>0$ such that if $G=D\cup B(z,\varepsilon)$, then $u_G<\infty$. Furthermore, for any $\delta>0$ there exists $\varepsilon(z,\delta)$ such that $\|u_G\|<\|u_D\|+\delta$ if $\varepsilon<\varepsilon(z,\delta)$.*

*Proof.* Given $0<\delta<1$, let $\eta$ be such that

$$\sup_{x\in B(z,\eta)} E^x\{\exp(Q\tau_{B(z,\eta)})\}<1+\delta. \tag{5}$$

Observe that $B(z,\eta)\cap D$ may not be connected (see the blackened area in Fig. 1). Let

$$A=(\partial B(z,\eta))\cap D.$$

We apply the lemma to find $\varepsilon$ so that $0<\varepsilon<\eta$ and

$$\sup_{x\in A}\varphi_\varepsilon(x)<\delta. \tag{6}$$

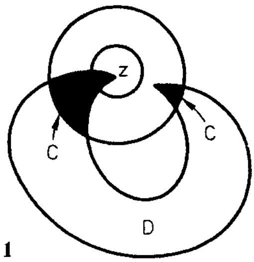

Fig. 1

Now put

$$C = G \cap B(z, \eta)$$
$$F = (\partial D) \cap B(z, \varepsilon)$$

where $G$ is given in the statement of the theorem. Let $x \in D \cap B(z, \varepsilon)$. We shall prove that $u(G, q, 1; x) < \infty$. Then $u_G$ will be finite by Theorem 1.2 of [4], as reviewed above.

The method of proof is similar to that of Theorem 1 in [1]*, which treats the case of a half line (instead of the $D$ here) in $R^1$. It is somewhat more complicated owing to the geometry of $R^d$. Define $T_0 = 0$, and for $n \geqq 1$:

$$T_{2n-1} = T_{2n-2} + \tau_C \circ \theta_{T_{2n-2}},$$
$$T_{2n} = T_{2n-1} + \tau_D \circ \theta_{T_{2n-1}},$$
$$R_n = T_n \wedge \tau_G .$$

On $\{T_{2n-1} < \tau_G\}$, we have $X(T_{2n-1}) \in A$; on $\{T_{2n} < \tau_G\}$, we have $X(T_{2n}) \in F$. On $\{T_n < \tau_G\}$, $T_n < T_{n+1}$. Let $T_\infty = \lim_n \uparrow T_n$. On $\{T_\infty < \infty\}$, the path of the Brownian motion undergoes infinitely many oscillations of distance exceeding $(\eta - \varepsilon)/2$ before the time $T_\infty$, since $\varrho(F, A) = \eta - \varepsilon$. The continuity of paths implies that $T_\infty = \infty$ a.s. (almost surely). Since $\tau_G < \infty$ a.s., it follows that there exists $n \geqq 1$ such that $T_{n-1} < \tau_G \leqq T_n$. But both sets $C$ and $D$ are subsets of $G$, and $T_n$ is either an exit time from $C$ or an exit time from $D$, the last inequalities entail that $\tau_G = T_n$, namely $\tau_G = R_n$. Hence if we define

$$N = \min\{n \geqq 0 | R_n = \tau_G\},$$

then $N < \infty$ a.s.

It follows from (5) that

$$\sup_{x \in F} E^x\{e(\tau_C)\} < 1 + \delta . \tag{7}$$

Applying the strong Markov property repeatedly to $T_n$, $n \geqq 1$, and using the estimates (6) and (7), we obtain

$$\begin{aligned} &E^x\{e(\tau_G); N = 2n-1\} \leqq [(1+\delta)\delta]^{n-1}(1+\delta); \\ &E^x\{e(\tau_G); N = 2n\} \leqq [(1+\delta)\delta]^{n-1}(1+\delta)\|u_D\| . \end{aligned} \tag{8}$$

* There is a minor error on p. 351. Replace the definition of $S$ by $\tau_a \wedge \tau_c$, and put $N = \min\{n \geqq 0 | T_{2n+1} = \tau_c\}$

Let $(1+\delta)\delta<1$. Adding up (8) over $n \geqq 1$, we obtain

$$u_G(x) \leqq [1-(1+\delta)\delta]^{-1}(1+\delta)(1+\|u_D\|)<\infty .$$

In the above we have taken $x \in D \cap B(z, \varepsilon)$. A similar argument works for any $x \in D \backslash B(z, \varepsilon)$. Indeed, a slightly more refined argument shows that by taking $\varepsilon$ small enough, we can make $\|u_G\|$ as near to $\|u_D\|$ as we wish. To see this let $B=B(z, \eta)$ and replace (7) by the following, for $x \in F$:

$$E^x\{e(\tau_B) 1_A(X(\tau_B))\}<(1+\delta) \theta_x,$$

$$E^x\{e(\tau_B) 1_{\partial B-A}(X(\tau_B))\}<(1+\delta)(1-\theta_x);$$

where $\theta_x$ is the ratio of the spherical area of $A$ to the total area of $\partial B$. Although $\theta_x$ varies with $x$ on $F$, by taking $\varepsilon$ small enough in comparison with $\eta$, we can make $\theta'<\theta_x<\theta$ for all $x \in F$ and $\theta-\theta'<\delta$. The estimates on the right sides of (8) are then replaced by

$$[(1+\delta) \theta \delta]^{n-1}(1+\delta)(1-\theta') \quad \text{and} \quad [(1+\delta) \theta \delta]^{n-1}(1+\delta) \theta\|u_D\|,$$

respectively. The result is that

$$u_G(x) \leqq [1-(1+\delta) \theta \delta]^{-1}(1+\delta)(1-\theta'+\theta\|u_D\|), \tag{9}$$

for $x \in D \cap B(z, \delta)$; and similarly

$$u_G(x) \leqq [1-(1+\delta) \delta \theta]^{-1}[\|u\|_D+\delta(1+\delta)(1-\theta')], \tag{10}$$

for $x \in D \backslash B(z, \delta)$. Observe that $\|u_D\| \geqq 1$ because $u_D(z)=1$ if $z \in \partial D$ and $z$ is regular (for $D^c$). It follows that as $\delta \downarrow 0$, the right member of (9) approaches $\|u_D\|$ as well as that of (10). Thus $\|u_G\|$ approaches $\|u_D\|$ as claimed.

We come next to the example mentioned in the introduction.

*Example.* Let $q \equiv 1$ in $R^d$. It is well known that there exists a number $r_1$ such that

$$E^0\{e^{\tau B(0, r)}\}<\infty$$

if and only if $r<r_1$. Let $r_3<r_2<r_1$, $B_i=B(o, r_i)$, and $C=B_1-\bar{B}_3$. We may make $r_1-r_3$ so small that

$$\sup_{x \in C} u(C ; 1,1 ; x)<2 .$$

Since $u(B_2, 1,1 ; x)$ is continuous in $B_2$ and $u(B_2, 1,1_A ; x)$ decreases to zero as $A \downarrow \emptyset$, where $A$ is an arc on the circle $\partial B_2$, we may make $A$ small enough that

$$\sup_{x \in B_3} u(B_2, 1,1_A ; x)<\tfrac{1}{3} .$$

Let $L$ be a closed line segment connecting $\partial B_1$ and $\partial B_2-A$. Now define

$$D=B_1-[(\partial B_2-A) \cup L] .$$

Thus $D$ is a simply connected domain ($L$ is used only to ensure this). It is easy to see that $D$ is a regular domain. If we denote $(\partial B_2-A) \cup L$ by $E$, then $E \subset \partial D$ and $D \cup E=B_1$, and $u_{B_1}=\infty$ by the definition of $r_1$. Clearly, for any domain $G \supset \bar{D}$ we have $u_G=\infty$. It remains to show that $u_D<\infty$.

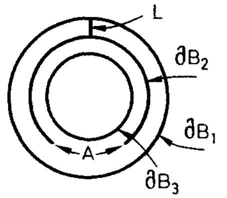


Fig. 2

The proof is similar to and simpler than that of Theorem 1.
Let $T_0=0$, and for $n \geqq 1$:

$$T_{2n-1}=T_{2n-2}+\tau_{B_2}\circ\theta_{T_{2n-2}},$$
$$T_{2n}=T_{2n-1}+\tau_C\circ\theta_{T_{2n-1}},$$
$$N=\min\{n\geqq 0 | T_n=\tau_D\}.$$

Then $N<\infty$ a.s. We have

$$E^0\{e(\tau_D); N=2n-1\}=(2/3)^{n-1}u_{B_2},$$
$$E^0\{e(\tau_D); N=2n\}=(2/3)^n.$$

Hence $E^0\{e(\tau_D)\}<\infty$.

The cone condition is well known in Dirichlet's boundary value problem. Let us denote by $C(z,\theta)$ the cone with vertex $z$ and relative angle $\theta$; namely the intersection of the cone with the sphere $\partial B(z,1)$ has an area in the ratio $\theta:1$ to the total area of the sphere. A domain $D$ is said to satisfy a cone condition at $z\in\partial D$ iff there exist $a>0$, $\theta>0$, so that $C(z,\theta)\cap B(z,a)\subset D^c$. If so, $z$ is regular for $D^c$ (see, e.g., [2] where a weaker cone condition is given). The condition is uniform iff the numbers $a$ and $\theta$ can be taken to be the same for all $z\in\partial D$. If $D$ satisfies a cone condition at every $z\in\partial D$ (not necessarily uniform), then it follows from Lebesgue's density theorem for measurable sets that $m(\partial D)=0$. It is easy to see that a bounded Lipschitz domain satisfies a uniform cone condition. We owe the last two remarks to N. Falkner.

**Theorem 2.** *Let $D$ be a bounded domain with $u_D<\infty$, and satisfying a uniform cone condition. Then there exists a domain $G$ containing $\bar{D}$ with $u_G<\infty$.*

*Proof.* Put

$$G=\{x\in R^d | \varrho(x,D)<\varepsilon\}, \tag{11}$$

for some $\varepsilon>0$ to be determined later. Given $\delta>0$, let $0<\varepsilon_0<a$ such that

$$E^x\{\exp(Q\tau_{B(x,\varepsilon_0)})\}<1+\delta; \tag{12}$$

the number in (12) being independent of $x$. We decrease $\varepsilon_0$ if necessary so that for any domain $E$ with $\bar{E}\subset D$ and $\varrho(\bar{E},\partial D)=\varepsilon_0$, and any $G$ defined in (11) with $\varepsilon<\varepsilon_0$, $m(G-\bar{E})$ is small enough to satisfy the following conditions, where $C=G-\bar{E}$:

$$\sup_{x\in C} E^x\{\exp(Q\tau_C)\}<1+\delta; \tag{13}$$

$$\sup_{x\in\partial E} u_D(x)<1+\delta. \tag{14}$$

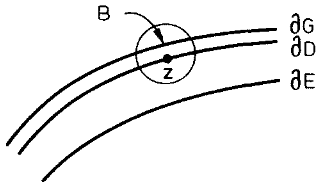


Fig. 3

Since $m(\partial D)=0$, $m(C)\to 0$ as $\varepsilon<\varepsilon_0\downarrow 0$; hence (13) is satisfied for small enough $\varepsilon_0$ by Lemma A of [4]. Since $D$ is regular, $u_D$ is continuous on $\bar{D}$ with boundary value one by Theorem 1.3 of [4]. Hence (14) is also satisfied for small enough $\varepsilon_0$.

Now let $\varepsilon_1<\varepsilon_0$. Then $\partial B(z,\varepsilon_1)\cap D^c$ has relative area $>\theta$ by the uniform cone condition, since $\varepsilon_1<$a. For any $0<\theta'<\theta$, by shrinking the angle of the cone, we obtain a subset $S(z,a)$ of $\partial B(z,a)\cap D^c$ which has relative area $>\theta'$, but with the additional property that

$$0<\varrho(S(z,a),D)<\varepsilon_0\,. \tag{15}$$

This number $\varrho(S(z,a),D)$ may be taken to be the same for all $z\in\partial D$, and we use it as the $\varepsilon$ in the definition (11) of $G$. This choice of $\varepsilon$ makes $G$ disjoint from $S(z,\varepsilon_1)$, so that

$$\partial B(z,\varepsilon_1)\cap G^c \text{ has relative area } >\theta'\,. \tag{16}$$

At the same time, since $\varrho(\bar{E},\partial D)=\varepsilon_0>\varepsilon_1$, we have

$$B(z,\varepsilon_1)\cap\bar{E}=\emptyset\,. \tag{17}$$

The geometrical preparation is now complete, and we are ready for the key estimate below.

Fix $z\in\partial D$, and write $B=B(z,\varepsilon_1)$. We shall prove that $u_G(z)<\infty$. Under $P^z$, $\partial C=(\partial G)\cup(\partial E)$, and $\{\tau_C<\tau_G\}=\{X(\tau_C)\in\partial E\}\subset\{\tau_B<\tau_C\}\cap\{X(\tau_B)\in G\}$. Hence the first inequality below follows from the strong Markov property:

$$\begin{aligned}E^z\{\exp(Q\tau_C);X(\tau_C)\in\partial E\}&\leqq E^z\{\tau_B<\tau_C;\exp(Q\tau_B)1_G(X(\tau_B))E^{X(\tau_B)}[\exp(Q\tau_C)]\}\\&\leqq E^z\{\exp(Q\tau_B)1_G(X(\tau_B))\}(1+\delta)\\&=E^z\{\exp(Q\tau_B)\}P^z\{X(\tau_B)\in G\}(1+\delta)\\&\leqq(1+\delta)^2(1-\theta')\,.\end{aligned}$$

The second inequality above follows from (13); the third from (12), (16), and spherical symmetry; the equality follows from the stochastic independence of $\tau_B$ and $X(\tau_B)$ under $P^z$. Since $\delta$ is arbitrarily small, the resulting bound may be made strictly less than one, which will suffice.

Define $T_0=0$, and for $n\geqq 1$:

$$\begin{aligned}T_{2n-1}&=T_{2n-2}+\tau_C\circ\theta_{T_{2n-2}},\\T_{2n}&=T_{2n-1}+\tau_D\circ\theta_{T_{2n-1}},\\N&=\min\{n\geqq 1\,|\,T_{2n-1}=\tau_G\}\,.\end{aligned}$$

Then $N<\infty$ a.s. We have

$$E^z\{e(\tau_G); N=2n-1\} \leqq [(1+\delta)^3(1-\theta')]^{n-1}(1+\delta),$$

where the third $1+\delta$ factor comes from (14), when the path moves from $\partial E$ back to $\partial D$. Choose $\delta$ so that $(1+\delta)^3(1-\theta')<1$. It follows by summing over $n$ that $u_G(z)<\infty$, indeed, $u_G(z)$ is arbitrarily near $\theta^{-1}$ for sufficiently small $\delta$, since $\theta'$ may be arbitrarily near $\theta$. For any $x\in G$, the same argument yields a bound for $u_G(x)$ arbitrarily near $\|u_D\|\theta^{-1}$. Thus, there exists $G$ containing $\bar{D}$ such that $\|u_G\|$ is arbitrarily near $\|u_D\|\theta^{-1}$.

We do not know whether the last inequality can be improved as in the case of Theorem 1.

The results above about enlarging the domain while keeping the gauge finite are intimately connected with the variation of eigenvalues with the domain. Consider the following eigen equation:

$$\begin{aligned}\left(\frac{\Delta}{2}+q\right)\varphi=\lambda\varphi \text{ in } D;\\ \varphi=0 \text{ on } \partial D.\end{aligned} \tag{18}$$

It is known that there exists a maximum eigenvalue $\lambda_1(D)$ for which (18) is solvable with $\varphi\in C^0(\bar{D})\cap C^{(2)}(D)$, provided that $q$ is Hölder continuous in $D$ (as well as bounded). If $D$ is regular, then it is shown in [3] and [6] by different methods that $u_D<\infty$ is equivalent to $\lambda_1(D)<0$. Now there is a "principle" in classical analysis which asserts that $\lambda_1(D)$ varies continuously with $D$ (at least when $q\equiv 0$). It should follow from this that if $\lambda_1(D)<0$, then for a domain $G$ "slightly larger" than $D$ we should have $\lambda_1(G)<0$. However, it is not clear under what precise conditions the said principle is valid. Conditions given in [5] are very strong in comparison with that used in Theorem 2 above, whereas Theorem 1 requires no condition on $D$ except $m(D)<\infty$. These results are proved without any reference to eigenvalues.

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**Added in proof.** Theorems 1 and 3 can be extended to the class of unbounded functions $q$, considered by Aizenman and Simon in Comm. Pure Appl. Math. **35**, 209–271 (1982)