SOME PECULIAR PROPERTIES OF A PARTICLE SYSTEM WITH SEXUAL REPRODUCTION

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The purpose of this paper is to describe a two dimensional growth model with sexual reproduction (i.e. two particles are needed to produce a new one) and contrast its properties with those of a similar model with asexual reproduction (i.e. an additive process in the sense of Harris (1978) and Griffeath (1979)). The results reported here were obtained in collaboration with Larry Gray. Detailed proofs are given in Durrett and Gray (1985), a source which will be referred to as DG(1985) below.

In both models the state of the system at time t is ξ_t a subset of Z^2, and particles die at rate one, that is, if $x \in \xi_t$ then $P(x \notin \xi_{t+\delta} | \mathfrak{F}_t) = \delta + O(\delta)$ as $\delta \to 0$, where $\mathfrak{F}_t = \sigma(\xi_\delta : s \leq t)$ = the σ-field generated by the process up to time t. The two models then are distinguished by their birth rates which can be described as follows:

Example 1. (asexual reproduction).
If $x \notin \xi_t$ and $x + (1,0)$ OR $x + (0,1) \in \xi_t$ then

$$P(x \in \xi_{t+\delta} | \mathfrak{F}_t) = \lambda\delta + O(\delta).$$

Example 2. (sexual reproduction)
If $x \notin \xi_t$ and $x + (1,0)$ AND $x + (0,1) \in \xi_t$ then

$$P(x \in \xi_{t+\delta} | \mathfrak{F}_t) = \lambda\delta + O(\delta).$$

The reasons for the names and the difference between the models can be seen in the two capitalized words. The rest of this paper is devoted to explaining how this simple change results in drastic differences in the behavior of the two processes. We will look at four aspects of their behavior below.

I. If we consider what happens when we start from $\xi_0^A = A$, A a finite set, then immediately we see differences between the two models.

Ex. 1. If $\lambda \geq 4$ then $P(\xi_t^A \neq \emptyset$ for all $t) > 0$.

Proof. If we restrict the process to $\{(x,0): x \in Z\}$ then it is a one-sided contact process so the conclusion follows from a result of Holley and Liggett (1978).

Ex. 2. If A is finite then $P(\xi_t^A \neq \emptyset$ for all $t) = 0$ for all λ.

Proof. If $[-L,L]^2$ contains A then no births can ever occur outside $[-L,L]^2$, so a simple argument shows there is an $\varepsilon_L > 0$ so that

$$P(\xi_n^A \neq \emptyset) \leq (1-\varepsilon_L)^n.$$

To sum things up if we let

$$\lambda_f = \inf\{y: P(\xi_t^A \neq \emptyset \text{ for all } t) > 0 \text{ for some finite set } A\}$$

then we have

Ex. 1 $\lambda_f \leq 4$

Ex. 2 $\lambda_f = \infty$.

Remark. Projecting onto $\{(x,-x): x \in Z\}$ and comparing with the two sided contact process shows $\lambda_f \leq 2$ in Ex. 1. It is trivial that $\lambda_f \geq \frac{1}{2}$ and as usual practically impossible to figure out exactly what λ_f is.

II. Having started from a finite set the next thing we want to contemplate is what happens when we start from $\xi_0^1 = Z^2$. In both cases it follows from general results about attractive spin systems (see Liggett (1985), Chapter 3) that we have:

(a) As $t \to \infty$ $\xi_t^1 \Rightarrow \xi_\infty^1$ a stationary distribution.

(b) If $\xi_\infty^1 = \delta_\emptyset$ (the point mass on the empty set) then there are not other stationary distributions.

From (a) and (b) it follows that if we let $\lambda_c = \inf\{\lambda: \xi_\infty^1 \neq \delta_\emptyset\}$ then the stationary distribution is $\delta_\emptyset$ for $\lambda<\lambda_c$ and is not unique for $\lambda>\lambda_c$. Comparing with the contact process again shows that in Ex. 1 $\lambda_c\leq 4$. The corresponding result for the other example is Theorem 1 in DG(1985) Ex. 2 $\lambda_c\leq 110$. The last result gives a ridiculous upper bound for λ_c (try to find a better one!). Simulations done by Tom Liggett suggest $\lambda_c\geq 12$ and a look at the over estimates in the proof in DG(1985) suggests $\lambda_c\leq 20$ but beyond this we have no idea what λ_c is.

Comparing the last result with the one in paragraph I. shows that in Ex. 2

$$\lambda_c\leq 110 \qquad \lambda_f=\infty$$

and hence $\lambda_c\neq\lambda_f$. (This result is somewhat surprising in view of the fact that $\lambda_c=\lambda_f$ in one dimensional attractive nearest neighbor (see Gray (1985)) and reversible nearest particle systems (see Liggett (1985), Chapter 7) and it was conjectured that $\lambda_c=\lambda_f$ for the contact process in any dimension (see Durrett and Griffeath (1982)).

Note: For the reader who thinks we have cheated by putting $\lambda_c<\infty$, $\lambda_f=\infty$ we would like to observe that if we let Ex. 3 = $(1-\varepsilon)$Ex.2 + εEx.1 and ε is small then $\lambda_c<\lambda_f<\infty$. This is Theorem 3 in DG(1985).

Problem. Once you see the proof of Theorem 1 it is easy to construct lots of examples with $\lambda_c<\infty, \lambda_f=\infty$. Since it is believed that for many models $\lambda_c=\lambda_f$ this brings up the problem of finding sufficient conditions for this to occur. It seems likely that this is true in Ex. 1 (although we do not know how to prove this) and we conjecture that this holds for all additive growth models but the latter question even in one dimension seems a very difficult problem.

III. Having started last time from $\xi_0^1=Z^2$ our next step is consider what happens starting from other simple initial distributions: ξ_0^p = product measure with density p, i.e. the events $\{x\in\xi_0^p\}$ are independent and each has probability p. With these initial distributions, we have

Ex. 1 $\quad \xi_t^p \Rightarrow \xi_\infty^1$ for any $p>0$.

Ex. 2 $\quad$ If $p<p^*\equiv 1-$ (the critical probability for two dimensional oriented percolation) then

$$\xi_t^p \not\Rightarrow \delta_\emptyset.$$

Proof. Suppose there is an infinite sequence $x_n \in (\xi_0)^c$ so that for each $n \geq 0$, $x_{n+1} \in \{x_n + (1,0), x_n + (0,1)\}$ then $\{x_n : n \geq 0\} \subset (\xi_t)^c$ for all t and an easy argument shows $\xi_t \Rightarrow \delta_\emptyset$.

Looking back at the last proof we see that the only property of the product measures we needed was the existence of the x_n's. This suggests that we cannot have an equilibrium distribution in which the density is too low, so we

Conjecture: $\rho(\lambda) \equiv P(0 \in \xi_\infty^1)$ is discontinuous at λ_c. In support of the conjecture we would like to observe that if Z^2 is replaced by the binary tree a simple argument shows that $\lambda\rho(\lambda)(1-\rho(\lambda)) \geq 1$ (see DG(1985) Section 1 for details), so $\rho(\lambda)$ cannot $\to 0$ as $\lambda \downarrow \lambda_c$. The reader should note that "general nonsense" (see Griffeath (1981)) implies $\rho(\lambda)$ is right continuous and hence $\rho(\lambda_c) > 0$.

IV. Last but not least we want to consider what happens when we add spontaneous births at rate β, i.e. the new birth rates = old rates + β. Again there is a drastic difference in the two results.

Ex. 1. There is a unique stationary distribution and convergence to equilibrium occurs exponentially fast.

Ex. 2. If λ is such that $\xi_1^\infty \neq \delta_\emptyset$ when $\beta=0$, and β is chosen so that $6\beta^{1/4}\lambda^{3/4} < 1$ then the process with parameters β and λ has two translation invariant stationary distributions.

In view of the results in III i.e.

Ex. 1 $\qquad \xi_t^p \Rightarrow \xi_\infty^1$ for any $p>0$

Ex. 2 $\qquad \xi_t^p \Rightarrow \delta_\emptyset$ for $p<p^* \approx .345$

these conclusions should not be surprising. In Ex. 1 $\delta_\emptyset$ is an unstable equilibrium because points artibtarily close to it converge to another fixed point. In Ex. 2 it is an attracting fixed point and hence a stable equilibrium.

It would be nice to prove results which make the last two sentences precise, but failing this we are satisfied with the result given above because outside of the Ising model and some examples due to Toom, there are very few (if any) examples in which all the flip rates are positive and there are two stationary distributions, and furthermore the new example has two new properties the old ones don't. (i) There is an open set of nonergodic examples in the set of nearest neighbor translation invariant growth models and (ii) as $\lambda \downarrow \lambda_c$ the lower bound on the allowed β's approaches $1/6^4\lambda_c^3>0$ a phenomenon which suggests that $\xi_1^\infty \neq \delta_\emptyset$ at λ_c.

The statements of the results above are already quite lengthy so there is no time or space to do anything but to make some simple general remarks about the proofs.

A. The proofs for Ex. 1 are all based on the fact that it is an additive process and has a set valued dual process $\tilde{\xi}_t \subset Z^2$. For Ex. 2 no such dual exists but we can define a new type of dual process (which can be defined for any attractive process and reduces to the usual notion for additive processes) where the state at time t χ_t is a collection of subsets of Z^2 with the interpretation that $\{0\}$ is in ξ_t if and only if some $A \in \chi_t$ is completely occupied in ξ_0.

B. Theorems 1, 2, and 3 all concern the behavior when some parametric (i.e. $1/\lambda$, β, or ε respectively) is small so it should not come as a surprise that the results are proved by "perturbation arguments." We write an expansion for the quantity of interest in terms of the small parameter and prove that the series has a positive radius of convergence.

Proofs of the type referred to above are "contour arguments." A good example of an argument of this type is the proof given in Gray and Griffeath (1982) and in fact with hindsight our proofs of Theorems 1, 2, and 3 are straightforward generalizations of the argument obtained by rewriting the easy case of Gray and Griffeath's proof in terms of the dual process. Further details are left to the reader or see DG(1985).

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