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Critical Behavior of the Two-Dimensional First Passage Time

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We study the two-dimensional first passage problem in which bonds have zero and unit passage times with probability p and $1-p$, respectively. We prove that as the zero-time bonds approach the percolation threshold p_c, the first passage time exhibits the same critical behavior as the correlation function of the underlying percolation problem. In particular, if the correlation length obeys $\xi(p) \sim |p-p_c|^{-\nu}$, then the first passage time constant satisfies $\mu(p) \sim |p-p_c|^{\nu}$. At p_c, where it has been asserted that the first passage time from 0 to x scales as $|x|$ to a power ψ with $0<\psi<1$, we show that the passage times grow like $\log |x|$, i.e., the fluid spreads exponentially rapidly.

KEY WORDS: First passage time; critical behavior; two dimensions.

1. INTRODUCTION: DEFINITION OF THE MODEL AND SUMMARY OF RESULTS

First passage percolation was introduced in 1965 by Hammersley and Welsh[1] to describe certain features of the transport of fluids in a disordered medium. Since that time, the first passage model has been the subject of much rigorous work (see, e.g., the monograph of Smythe and Wierman[2] and the review of Kesten[3]). Recently, there has also been a good deal of theoretical and numerical analysis of first passage, with applications ranging from conduction in neural networks[4] to crack propagation in solids[5,6]; such work has focused primarily on the development of scaling theories and the calculation of critical exponents.

Here we present a (rigorous) characterization of the critical behavior of basic two-dimensional first passage percolation. Our analysis allows us

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to equate the critical exponent of the first passage time constant to that of the correlation length (should either exist) and to test the validity of certain scaling theories.

1.1. Description of the Model

The first passage model is defined by assigning independent and identically distributed variables ω_b to the bonds of a regular lattice. Henceforth, we will take this lattice to be $\mathbb{B}_d$, which is the set of all bonds between nearest neighbor points of the hypercubic lattice $\mathbb{Z}^d$. In the fluid transport interpretation, ω_b is regarded as the time it takes fluid to traverse the bond b, and hence is chosen to be nonnegative.[3] For a given realization of time coordinates, $\omega = \{\omega_b \mid b \in \mathbb{B}_d\}$, and a path $\not{p}: x \to y$ from site x to y of the lattice $\mathbb{Z}^d$, the path time is given by the sum

$$t_{\not{p}}(\omega) = \sum_{b \in \not{p}} \omega_b \tag{1.1}$$

One can then calculate the *first passage time*

$$t_{xy}(\omega) = \inf_{\not{p}: x \to y} t_{\not{p}}(\omega) \tag{1.2}$$

where the infimum is over all paths from x to y on $\mathbb{Z}^d$ in configuration ω. The description of the model is completed by declaring that if a set $S \subset \mathbb{Z}^d$ is "wet" at time t, then the wet set at time $t + \sigma$ includes the points $y \in \mathbb{Z}^d$ for which $t_{xy}(\omega) < \sigma$ for some $x \in S$.

The above description should be appropriate for systems with macroscopic flow where the pressure of the wet set is maintained at a constant value. This should be contrasted with other fluid transport problems, such as invasion percolation,[7,9,30] where it is the flow rate that is held constant: one bond is crossed by the fluid per unit time.

In the first passage problem, one is converned with how the first passage time scales with distance for asymptotically separated points. Let n denote both the point $(n, 0, \ldots, 0) \in \mathbb{Z}^d$ and the distance of this point from the origin. For a large class of distributions of time coordinates ρ (see Section 2 for precise statements), it follows from the theorem of Kingman[10] that with probability one

$$\lim_{n \to \infty} n^{-1} t_{0n}(\omega) \equiv \mu(\rho) \tag{1.3}$$

where $\mu(\rho)$ is a constant (i.e., independent of realization) called the time constant. The simplest distribution that produces interesting behavior is the

[3] There has also been some work on models with negative ω_b (see, e.g., Ref. 2, p. 75).

Bernoulli $\{0, 1\}$ distribution[4]: $\omega_b = 0$ with probability p and $\omega_b = 1$ with probability $1 - p$. This distribution is expected to provide a good description of binary systems in which the passage times of the two constituents are separated by many orders of magnitude. For this distribution, one can define the percolation threshold p_c as the point above which zero-time bonds form an infinite connected cluster with probability one. It can be shown[3,12 14,17] that the first passage model undergoes a transition at the percolation threshold in the sense that the time constant [which is here denoted by $\mu(p)$] is zero for $p > p_c$ and positive for $p < p_c$,

1.2. Main Results

This paper concerns the behavior of the two-dimensional first passage time in the critical regime. Using rigorous finite-size scaling arguments, we relate the behavior of the first passage time to that of the connectivity function in the underlying Bernoulli percolation model. Bernoulli percolation is defined simply by declaring the zero-time bonds to be "occupied" and those of unit time to be "vacant." The connectivity $\tau_{0n}(p)$ is

$$\tau_{0n}(p) = \text{Prob}(0 \text{ connected to } n \text{ by occupied bonds}) \tag{1.4}$$

and the correlation length is defined by

$$1/\xi(p) \equiv -\lim_{n \to \infty} n^{-1} \log \tau_{0n}(p) \tag{1.5}$$

We prove that if $\xi(p)$ diverges with critical exponent ν,

$$\xi(p) \sim |p - p_c|^{-\nu} \qquad \text{as} \quad p \uparrow p_c \tag{1.6}$$

(in a sense to be made precise in Section 2), then the first passage time constant exhibits the critical scaling

$$\mu(p) \sim |p - p_c|^{\nu} \qquad \text{as} \quad p \uparrow p_c \tag{1.7}$$

The behavior (1.7) is rather surprising in light of general scaling theories,[4] which relate the *approach* to criticality to the behavior of the first passage time *at* the critical point. It was assumed that at p_c the first passage time t_{0n} scales as a sublinear power ψ of the distance n. The scaling theories then relate this assumed power to the deviation of the time constant exponent from ν. The result (1.7) suggests that either the scaling theories are incomplete or that the behavior of t_{0n} at threshold has an

[4] Another interesting, but somewhat more difficult model is the Bernoulli $\{1,00\}$ distribution,[11] which has a variety of applications.[4]

unanticipated form. Our next result demonstrates that the scaling theories are consistent in the sense that $\psi = 0$. In fact, we prove the (stronger) result that the first passage time scales *logarithmically* with the distance:

$$c_1 \log n \leqslant E(t_{0n}) \leqslant c_2 \log n \tag{1.8}$$

where c_1 and c_2 are positive, finite constants. Alternatively, Eq. (1.8) means that the wet set spreads exponentially rapidly at the critical point.

The organization of this paper is as follows. In Section 2, we review the relevant backround material from first passage and Bernoulli percolation. In Section 3, we present the proofs of our results and contrast these with theoretical predictions.

2. PRELIMINARIES

2.1. The Time Constant of First Passage Percolation

We consider a nonnegative distribution ρ of time coordinates ω_b, which are assigned independently to each bond $b \in \mathbb{B}_d$.

One of the fundamental results of first passage percolation is the existence of a *time constant* $\mu(\rho)$ for sufficiently well-behaved distributions. The first results along these lines were proved by Hammersley and Welsh.[1] The strong result below follows from the Kingman subadditive ergodic theorem.[10]

Theorem 2.1. Let t_{0n} denote the first passage time between the origin and the point $(n, 0, \ldots, 0)$. If the (nonnegative) distribution ρ has finite first moment, then

$$\lim_{n \to \infty} n^{-1} t_{0n}$$

exists and is equal to the (finite constant)

$$\lim_{n \to \infty} E(n^{-1} t_{0n}) \equiv \mu(\rho) \tag{2.1}$$

with probability one (w.p.1) and in L^1.

The reader is referred to Kingman[10] or Smythe and Wierman[2] for a proof of the subadditive ergodic theorem.

Remark. It is worth noting that the condition of Theorem 2.1 can be weakened,[15] but some moment condition on ρ is actually required.[16]

The quantity $\mu(\rho)$ will be a focus of our attention. That its behavior is nontrivial is a consequence of the following result of Hammersley and Welsh.[1]

Proposition 2.2. Unless ρ is a degenerate distribution, $\mu(\rho) < E(\omega_b)$.

It turns out that the problem is of particular interest if the distribution ρ has an atom at zero. Of course, the simplest such example is the Bernoulli distribution with $\rho(0) = p > 0$ and $\rho(1) = 1 - p$; in this case, we denote the time constant by $\mu(p)$. For the two-dimensional system with this distribution, Kesten[12] proved that $\mu(p) > 0$ if and only if $p < p_c$. The analogous result for higher dimensions was recently proved by Kesten[13,14]; the statement $\mu(p) > 0$ if $p < p_c$ was also derived in Ref. 17.

In this paper, we will be examining the critical behavior of the time constant $\mu(p)$ in two dimensions. One of our principal results is that $\mu(p)$ is closely related to the correlation length $\xi(p)$ of Bernoulli percolation; hence, we review some basic properties of the latter.

2.2. The Correlation Length of Bernoulli Percolation

The Bernoulli bond percolation model[18] is defined by independently taking each bond $b \in \mathbb{B}_d$ to be *occupied* with probability p or *vacant* with probability $1 - p$; thus, there is a strict analogy with the first passage model if we say that the occupied and vacant bonds have zero and unit passage times, respectively. In any dimension $d > 1$, the percolation model is known[18] to undergo a transition at a nontrivial value of p, denoted by p_c, called the percolation theshold: if $p < p_c$, then there are only finite connected clusters of occupied bonds w.p.1, while for $p > p_c$, there is an infinite connected cluster of occupied bonds w.p.1.

Another characterization of the percolation threshold is in terms of the asymptotic behavior of a quantity called the connectivity: the connectivity between the origin and the point n, denoted by $\tau_{0n}(p)$, is the probability, at occupation density p, that these two points are connected by a path of occupied bonds. It is not hard to show that

$$\lim_{n \to \infty} n^{-1} \log \tau_{0n}(p) = \sup_{n \geq 1} n^{-1} \log \tau_{0n}(p) \tag{2.2}$$

we denote this limit by $-1/\xi(p)$, and call $\xi(p)$ the percolation *correlation length.* From the right-hand side, we can see that $1/\xi$ provides the *a priori* bound:

$$\tau_{0n}(p) \leqslant e^{-n/\xi(p)} \tag{2.3}$$

uniformly in n. From this, it is clear that whenever $\xi(p) < \infty$, $p < p_c$. The converse of this statement (which is not nearly as obvious) follows from the work of Hammersley,[19] combined with a "coincidence of critical points

result," the latter having been proved by Kesten[20] for $d=2$ and by Aizenman and Barsky[21] in general dimension. The properties of $\xi(p)$ are summarized in the following.

Proposition 2.3. The quantity ξ^{-1}: $[0, p_c] \to [0, \infty]$ is a monotone decreasing, continuous function of p which is zero at p_c.

The continuity cited above was established by Grimmett.[22]

The above results imply that $\xi(p)$ exhibits critical behavior as $p \uparrow p_c$ in the sense that it diverges continuously. Critical *scaling* is a much more subtle issue; it includes the assumption that $\xi(p)$ diverges with a power law:

$$\xi(p) \sim |p - p_c|^{-\nu} \qquad \text{as} \quad p \uparrow p_c \tag{2.4}$$

No statement along the lines of (2.4) has been rigorously established; nevertheless, one hopes that (2.4) is true, at least in the sense that

$$\lim_{p \uparrow p_c} \log \xi(p) / \log |p - p_c| \equiv -\nu \tag{2.5}$$

exists. Ultimately, we will obtain upper and lower bounds on $\mu(p)$ such that if either $\mu(p)$ or $\xi(p)$ exhibits critical scaling in the sense just described, then the other quantity will also exhibit this scaling.

Although the correlation length as described above is well-defined, it is also useful to have at one's disposal a physical realization of a "relevant length scale." It turns out, in a sense which will be made precise below, that the following notion is equivalent to the decay rate of correlations:

Let $\Gamma_{3,L} \equiv \{x \in \mathbb{Z}^d \mid 0 \leqslant x_1 \leqslant L, 0 \leqslant x_2, \ldots, x_d \leqslant 3L\}$ be a $1 + 3^{d-1}$ rectangle of scale L, and define $K_{3,L}(p)$ to be the probability, at density p, that the $x_1 = 0$ and $x_1 = L$ faces of $\Gamma_{3,L}$ are connected by a path of occupied bonds that lies inside $r_{3,L}$. Obviously, if $\xi(p) < \infty$, then as $L \to \infty$, $K_{3,L}(p) \to 0$. However, for L's that are of order unity (and p's that are not too small), $K_{3,L}(p)$ will be of order one. Evidently, then, we can locate a first scale $L_1(p)$ where $K_{3,L_1}(p)$ has become "small" by some fixed criterion.

The connection between these two versions of a length scale has been demonstrated in Ref. 23.

Proposition 2.4. In any dimension $d \geqslant 2$, there is a constant $c(d)$ such that if $L_1(p)$ is the largest length scale satisfying $K_{3,L_1}(p) \geqslant c(d)$, then $L_1(p)$ diverges as $p \uparrow p_c$. In particular, there are finite, positive constants $a_1(d)$ and $a_2(d)$ such that

$$a_1 L_1(p) \geqslant \xi(p) \geqslant [a_2 L_1(p)] / [1 + \log L_1(p)] \tag{2.6}$$

For a proof of Proposition 2.4, see Ref. 17 or Ref. 23.

An immediate consequence of this proposition is that if $L_1(p)$ exhibits critical scaling [in the sense of Eq. (2.5)] with critical exponent ν, so does $\xi(p)$ and vice versa. Other notions of a correlation length in terms of the moments of cluster sizes have been described in Ref. 24, and, for the case of two dimensions, these have recently been shown[25] to be equivalent to $\xi(p)$ in the sense of Proposition 2.4.

Many of the above results are established by considering the models that are *dual* to bond percolation. These are defined as follows: consider the lattice $\mathbb{Z}^{d*}$ that is obtained by shifting $\mathbb{Z}^d$ half a unit in each coordinate direction. Elementary cells (bonds, plaquettes, cubes, etc.) are defined on $\mathbb{Z}^{d*}$ in the usual fashion. Observe that the $(d-1)$-cells of $\mathbb{Z}^{d*}$ are in one-to-one correspondence with the bonds of $\mathbb{Z}^d$; the $(d-1)$-cell that is transverse to a given bond b is called the dual cell of b. We declare the dual cell of b to be occupied whenever b is vacant. Connectedness on $\mathbb{Z}^{d*}$ occurs whenever occupied $(d-1)$-cells share a $(d-2)$-dimensional element of their boundary.

The crossing events described above can be conveniently reformulated from the point of view of the dual models. In particular, when there is no left–right crossing of the $r_{3,L}$ by occupied bonds, then the left and right faces of the rectangle are separated by a connected sheet of occupied dual cells. Of special interest is the case of two dimensions, where the dual of bond percolation is again bond percolation. Here, the above statement amounts to the fact that either there is a left–right crossing of $r_{3,L}$ by occupied bonds or there is a top–bottom crossing by occupied dual bonds.

Some final results concerning box crossings, which will prove useful later, can be derived on the basis of the bounds of Russo[26] and Seymour and Welsh.[27] The proof and our subsequent applications are, however, strictly two-dimensional.

Proposition 2.5. For all L and p, define $R_L(p)$ to be the probability of a left-right crossing of the square $r_L \equiv \{x \in \mathbb{Z}^d | \, |x_i| \leqslant L/2\}$. Then, in $d=2$, there is a continuous and monotone increasing function $F_{\mathrm{RSW}}(x)$ on $[0, 1]$ satisfying $F_{\mathrm{RSW}}(0)=0$ and $F_{\mathrm{RSW}}(1)=1$ such that

$$K_{3,L}(p) \geqslant F_{\mathrm{RSW}}(R_L(p))$$

For a proof of Proposition 2.5, see either Ref. 26 or Ref. 27.

From these so-called RSW bounds and the remarks about dual crossings, it is clear that *in two dimensions*, one may also define a correlation length in terms of square crossings, rather than "short-way" rectangle crossings. Indeed, there is a constant $\Delta > 0$, explicitly computable

in terms of $c(d=2)$ of Proposition 2.4 and F_{RSW} of Proposition 2.5, such that if we define

$$L_0(p) = \sup\{L \mid R_L(p) \geqslant \Delta\} \tag{2.7}$$

then $L_0(p)$ obeys the bound (2.6). The correlation length $L_0(p)$ will be used in a good deal of our subsequent analysis.

3. FIRST PASSAGE IN THE CRITICAL REGIME

In this section, we consider the critical behavior of two-dimensional first passage percolation. First, we focus on the *approach to criticality*, as characterized by the behavior of the time constant $\mu(p)$ as $p \uparrow p_c$. Thus, in Sections 3.1 and 3.2, we will take $p \lesssim p_c$ and derive lower and upper bounds on $\mu(p)$ in terms of the percolation correlation length. Assuming a power law divergence of $\xi(p)$, this gives us the critical exponent for the time constant.

Scaling theories of first passage percolation suggest that the time constant critical exponent is related to the exponent describing the first passage time as a function of distance *at the critical point.* In Section 3.4, we determine, exactly, the "power law" that describes this critical behavior. Although the value of both this and the time constant exponent are rather surprising, they do satisfy the relationship predicted by the scaling theory.

3.1. A Lower Bound

First, we will give a lower bound on $\mu(p)$ that holds in all dimensions. Essentially equivalent results have already appeared elsewhere[13,14,17]; however, the authors feel that the proof presented here has a certain esthetic appeal.

Theorem 3.1. Let $\mu(p)$ be the first passage time constant for the Bernoulli system on the bond lattice $\mathbb{B}_d$. Then $\forall p \in (0, 1)$,

$$\mu(p) \geqslant \sup_{\varepsilon > 0} [\xi^{-1}(p+\varepsilon)]/|\log(\varepsilon/1-p)|$$

Proof. The result follows easily if one observes that the Bernoulli distribution at density $p+\varepsilon$ can be generated in two independent steps: two sets of occupiers are independently distributed on the bonds of the lattice, the first at density p and the second at density $\varepsilon/(1-p)$, with the stipulation that if a $b \in \mathbb{B}_d$ is "hit" by at least one occupier, then it is occupied. Since the two steps take place independently, the overall occupation probability is $p + (1-p)\,\varepsilon/(1-p) = p+\varepsilon$; furthermore, the

events are independent from bond to bond. Evidently, then, this is the Bernoulli process at density $p+\varepsilon$.

The way we make use of the above observation is first to determine the value of t_{0n} in the density p configuration generated after step one. Then we have learned that there is a path from the origin to n that uses only t_{0n} vacant bonds. If all these t_{0n} bonds were to become occupied on the second stage—an event that occurs with probability $[\varepsilon/(1-p)]^{t_{0n}}$—there would be a zero-time path between the origin and n. Using this observation and then the Jensen inequality, we have

$$\begin{aligned}\text{Prob}_{p+\varepsilon}(t_{0n}=0) &\geqslant \sum_{k=0}^{\infty} \text{Prob}_p(t_{0n}=k)[\varepsilon/(1-p)]^k \\ &= E_p\{\exp[-|\log(\varepsilon/1-p)|\ t_{0n}]\} \\ &\geqslant \exp[-|\log(\varepsilon/1-p)|\ E_p(t_{0n})] \qquad (3.1)\end{aligned}$$

Noting that the left-hand side of (3.1) is just $\tau_{0n}(p+\varepsilon)$, and recalling (2.1) and (2.2), we obtain

$$-\xi(p+\varepsilon) \geqslant \log \tau_{0n}(p+\varepsilon) \geqslant -|\log(\varepsilon/1-p)|\ E_p(t_{0n})$$

which is the desired result. ∎

One expects that the time constant has power law behavior:

$$\mu(p) \sim |p-p_c|^{\theta} \qquad \text{as} \quad p \uparrow p_c \qquad (3.2)$$

again at least in the sense that

$$\lim_{p \uparrow p_c} \log \mu(p)/\log|p-p_c| \equiv \theta \qquad (3.3)$$

exists. Given this, Theorem 3.1 implies the following.

Corollary 3.1. If $\xi(p)$ and $\mu(p)$ exhibit scaling of the form (2.5) and (3.3), respectively, then $\theta \leqslant \nu$.

3.2. An Upper Bound for $d=2$

The question of upper bounds on the time constant is somewhat more delicate than that of lower bounds; as of yet, we can only analyze the problem in two dimensions. In the exponent language of Eq. (3.3), the result of this subsection is $\theta \geqslant \nu$ for $d=2$; combined with the result of the previous subsection, this determines the two-dimensional critical exponent θ (should it exist).

Theorem 3.2. Let $\mu(p)$ be the first passage time constant for the Bernoulli system on the bond lattice $\mathbb{B}_2$ and let $L_0(p)$ be the "correlation length" defined by Eq. (2.7). Then $\forall p \in [0, 1]$,

$$\mu(p) \leqslant \kappa L_0^{-1}(p)$$

where $\kappa < \infty$ is a constant independent of p.

Remark. By Proposition 2.4 and the remarks following it [cf. Eqs. (2.6) and (2.7)], Theorem 3.2 may also be expressed in terms of the conventional percolation correlation length $\xi(p)$:

$$\mu(p) \leqslant \kappa' \xi^{-1}(p) \tag{3.4}$$

Proof. Consider the first passage problem in the region $\{x \in \mathbb{Z}^2 | \ |x_1| \leqslant 1/2T\}$, which we will call the "time tunnel" of width T. Let us denote by $t_{0n}^T(\omega)$ the first passage time between the origin and n along bonds that lie exclusively in the tunnel in configuration ω. It is easy to see that

$$\mu^T(p) \equiv \lim_{n \to \infty} n^{-1} t_{0n}^T \tag{3.5}$$

exists and is a (T-dependent) constant w.p.1. Since the tunnel times are not exceeded by the full-space times, we have

$$\mu^T(p) \geqslant \mu(p) \tag{3.6}$$

In fact, it is straightforward to show that the above inequality is strict for all T, and that $\forall p \in [0, 1]$

$$\lim_{T \to \infty} \mu^T(p) = \mu(p) \tag{3.7}$$

It is also useful to consider the end-to-end tunnel times—that is, the minimum time from the line $x_2 = 0$ to the line $x_2 = n$ along paths inside the tunnel. We will denote this first passage time by t_{0n}^T. Note that t_{0n}^T can differ from t_{0n}^T by at most an n-independent constant. Thus, $\mu^T(p)$ is also the time constant for t_{0n}^T.

Observe that the end-to-end tunnel times have a geometric interpretation. Indeed, let us represent each unit time bond as a *barrier* on the dual lattice. Then, the time along each path is simply the number of barriers crossed by the path. Furthermore, the travel time between any two sets is just the number of *independent* surfaces (formed by barriers) that separate the sets. For our purposes, this means we can compute t_{0n}^T by counting the number of independent surfaces in the tunnel that separate $x_2 = 0$ from $x_2 = n$. See Fig. 1.

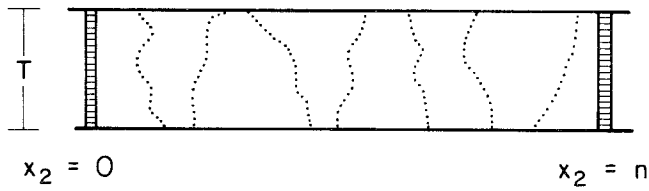

Fig. 1.

Given Eq. (3.7), one may wonder if there is some scale T at which $\mu^T(p)$ becomes a good approximation to $\mu(p)$. A partial answer is provided by considering the time tunnel of width $L_0(p)$ [i.e., the tunnel with $T = L_0(p)$], where $L_0(p)$ is the "correlation length" defined by Eq. (2.7). We begin with the special case in which n is also exactly $L_0(p)$, i.e., the case of a "square tunnel." In order to estimate $E(t_{0L_0}^{L_0})$, we first note that by the definition of $L_0(p)$, $t_{0L_0}^{L_0}$ is zero with probability at least as large as Δ:

$$\text{Prob}(t_{0L_0}^{L_0} \geqslant 1) \leqslant 1 - \Delta \tag{3.8}$$

To estimate the probability that $t_{0L_0}^{L_0}$ exceeds two, we use the identity

$$\text{Prob}(t_{0L_0}^{L_0} \geqslant 2) = \text{Prob}(t_{0L_0}^{L_0} \geqslant 1)\ \text{Prob}(t_{0L_0}^{L_0} \geqslant 2 \mid t_{0L_0}^{L_0} \geqslant 1) \tag{3.9}$$

Given that $t_{0L_0}^{L_0} \geqslant 1$, there must be a dual surface (of barriers) that separates the left side of the square from the right. The conditioning is "satisfied" when we further condition on the *leftmost* such surface. (That is, we enumerate all possible surfaces $\mathscr{S}$ and further condition on the event that some $s \in \mathscr{S}$ *is* the leftmost surface. These events form a disjoint partition of the event that $t_{0L_0}^{L_0} \geqslant 1$.) When the leftmost surface happens to be s, the situation to the left of s is quite complicated—life must be arranged in such a way that no surface lying to the left of s is fully formed. On the other hand, things to the right of s are relatively simple; indeed, they are virtually unconditioned. Now, regardless of which surface s is, it is evident that, conditioned on s being the leftmost surface, the probability of getting another barrier ($\equiv$ the probability of getting a barrier strictly to the right of s) is smaller than the probability of having found a barrier in the first place. All this means that

$$\text{Prob}(t_{0L_0}^{L_0} \geqslant 2 \mid t_{0L_0}^{L_0} \geqslant 1) \leqslant 1 - \Delta \tag{3.10}$$

or

$$\text{Prob}(t_{0L_0}^{L_0} \geqslant 2) \leqslant (1 - \Delta)^2 \tag{3.11}$$

The type of argument used to derive (3.11) has been formulated in a general context by van den Berg and Kesten.[31]

Repeating the above line of reasoning, we have

$$\text{Prob}(t_{0L_0}^{L_0} \geqslant n) \leqslant (1-\Delta)^n \tag{3.12}$$

so that

$$E(t_{0L_0}^{L_0}) = \sum_{n=1}^{\infty} \text{Prob}(t_{0L_0}^{L_0} \geqslant n) \leqslant (1-\Delta)/\Delta \tag{3.13}$$

The above argument can be applied with only slight modification to the ful tunnel. Indeed, let us consider $n = NL_0$ and estimate $t_{0NL_0}^{L_0}$. We will call the region $\{x \mid jL_0 \leqslant x_2 \leqslant (j+1)\, L_0,\ |x_1| \leqslant L_0/2\}$ the jth segment of the tunnel. Starting at the left end of the first segment, we must estimate the number of independent barrier surfaces that separate us from the right end of the Nth segment. Now, any surface that separates the left side of the tunnel from the right "starts" on the top of the tunnel ($x_1 = +L_0/2$) and "ends" on the bottom ($x_1 = -L_0/2$). The bottom point of a surface will be called its *anchor* point.

We will denote by Q_j the number of independent barriers that are anchored in the jth segment—regardless of their dependence (or lack thereof) on surfaces anchored outside the jth segment. It is evident that

$$t_{0NL_0}^{L_0} \leqslant \sum_{j=1}^{N} Q_j \tag{3.14}$$

since the right-hand side may overcount the total number of independent barriers, as well as counting those barriers whose top endpoint lies to the right of the jth segment. Now, the Q_j are a stationary sequence and only mildly dependent; indeed, $\text{Prob}(Q_j \mid Q_k) \to \text{Prob}(Q_j)$ exponentially as $|j-k| \to \infty$. We thus have

$$\mu^{L_0} L_0 = \lim_{N \to \infty} N^{-1} t_{0NL_0}^{L_0} \leqslant E(Q_1) \tag{3.15}$$

Let us estimate $E(Q_1)$. To this end, let a_L denote a square annulus of inner diameter L and outer diameter $3L$, and let $A_L(p)$ denote the probability, at density p, that there is a circuit of zero-time (or "occupied") bonds in the annulus. It is easy to obtain a lower bound on $A_L(p)$ similar to the Russo–Seymour–Welsh bound (Proposition 2.5). Indeed,

$$A_L(p) \geqslant g(R_L(p)) \tag{3.16}$$

with $g = [F_{\text{RSW}}]$.[4] Note that g has the same properties as F_{RSW}; i.e., g is strictly increasing with $g(0) = 0$ and $g(1) = 1$. Using the definition (2.7) of L_0, we have

$$A_{L_0}(p) \geqslant g(\Delta) > 0 \tag{3.17}$$

Now imagine that the jth segment of the tunnel coincides with the top central square of the annulus a_{L_0} (see Fig. 2). Clearly, if there is a circuit of zero-time bonds in a_{L_0}, then there are no barriers anchored in the jth segment. Thus, $Q_j = 0$ with probability exceeding $A_{L_0}(p)$. From (3.17) and the reasoning used to derive (3.13), it is easy to see that

$$E(Q_1) \leqslant [1 - g(\Delta)]/g(\Delta) \tag{3.18}$$

Using this, as well as (3.15) and (3.6), we have

$$\mu(p) \leqslant \{[1 - g(\Delta)]/g(\Delta)\}\, L_0^{-1}(p) \tag{3.19}$$

as claimed. ∎

If one assumes power law behavior of μ and ξ with exponents θ and ν, Theorem 3.2 immediately implies $\theta \geqslant \nu$. However, together with Theorem 3.1, we have a stronger result.

Corollary 3.2. If either $\mu(p)$ or $\xi(p)$ has critical scaling of the form (3.3) or (2.5), then both quantities obey their respective scalings and $\theta = \nu$.

Corollary 3.2 should be contrasted with real space renormalization-group calculations on first passage[5] and Monte Carlo simulations on related models,[6] from which it was concluded that θ "$\gtrsim$" ν.

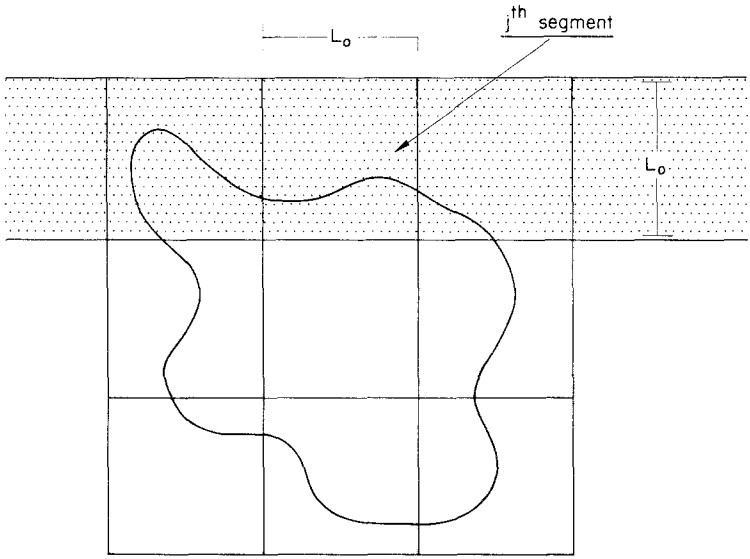

Fig. 2.

Corollary 3.2 is also somewhat unexpected in light of the predictions of general scaling theories for models of this sort (see, e.g., Ref. 4). The conclusions of these theories are summarized below.

3.3. An Interlude on Scaling Theory

Scaling theory relates the approach to criticality, as here described by the presumed behavior

$$\mu(p) \sim |p - p_c|^{\theta} \qquad \text{as} \quad p \uparrow p_c \tag{3.20}$$

to the behavior at p_c, as here characterized by the scaling of t_{0n} with distance. On the basis of the fact that $E_p(t_{0n})$ scales linearly with distance for $p < p_c$ and tends to a constant for $p > p_c$, the behavior at p_c is assumed to be of the form

$$E_p(t_{0n}) \sim n^{\psi} \qquad \text{at} \quad p = p_c \tag{3.21}$$

with some power $\psi < 1$. The scaling theory also assumes a particular *scaling form* of $E_p(t_{0n})$ away from p_c, from which it is predicted[4] that

$$\theta = \nu(1 - \psi) \tag{3.22}$$

We emphasize that three separate assumptions are required for the above conclusion, namely the two power law behaviors (3.20) and (3.21), and the scaling form of $E_p(t_{0n})$ from which (3.22) follows.

If one accepts this theory, then our Corollary 3.2 implies $\psi = 0$. This is initially rather disconcerting, since one anticipated a power law behavior of $E_p(t_{0n})$ at threshold.

3.4. Behavior at the Critical Point

In this subsection, we prove that $E_p(t_{0n})$ does *not* have power law behavior at p_c; indeed, it scales as a logarithm. Note that this implies that the fluid spreads *exponentially fast* at threshold. Given the result of the previous subsection, this exponential spreading, though mildly alarming, shows that the scaling theory is at least self-consistent.

Theorem 3.3. Let t_{0n} be the first passage time between the origin and n in the Bernoulli system on $\mathbb{B}_2$. There exist finite, positive constants k_1 and k_2 such that at the critical density p_c,

$$k_1 \log n \leqslant E_{p_c}(t_{0n}) \leqslant k_2 \log n$$

for all n large enough.

Proof. We will first establish a roughly equivalent result, which is worthwhile in its own right, and from which the above statement can be readily obtained. Consider the box $\Gamma_n \equiv \{x \in \mathbb{Z}^2 \mid 0 \leqslant |x_1|, |x_2| \leqslant n\}$ of width $2n$, centered at the origin (see Fig. 3). Let us define $\mathscr{T}_n$ to be the minimum time required for fluid that starts at the origin to wet the outside of Γ_n. We will show that there are nontrivial constants b_1 and b_2 such that

$$b_1 \log n \leqslant E_{p_c}[\mathscr{T}_n] \leqslant b_2 \log n \tag{3.23}$$

It is obvious that the lower bound in (3.23) implies the lower bound in the statement of the theorem. Not surprisingly, this is the bound that is more easily obtained.

The crucial ingredient in our analysis is that, at the critical point, there is no relevant length scale left in the problem. In particular, the occurrence of events that characterize the system at distances smaller than the correlation length are now typical at all scales. To illustrate this point, observe that (essentially by definition) the probabilities $R_L(p_c)$, $K_{3,L}(p_c)$, and $A_L(p_c)$ are of order one, independent of L. (By this, we mean that there are nontrivial upper and lower bounds on these probabilities that are uniform in L. Of course, for the self-dual model, $R_L(p_c) = 1/2$, but this fact plays no role in our analysis.)

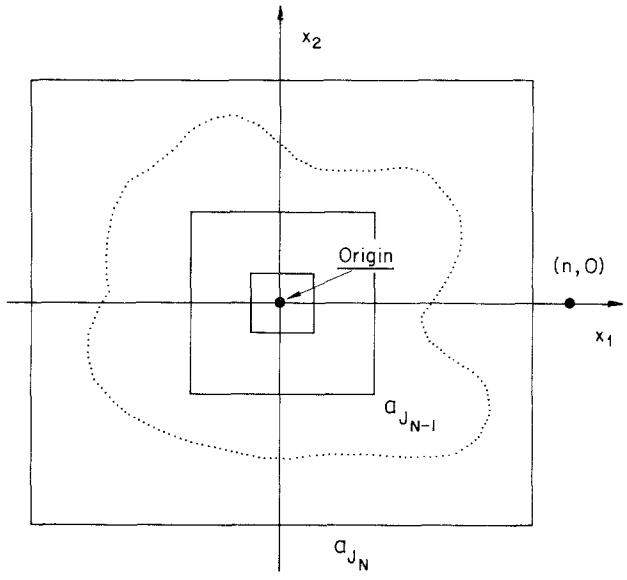

Fig. 3.

From this, and a bit of elementary construction, it is possible to show that, on all scales, the event of a barrier surface separating the inner and outer boundaries of the annulus a_L (i.e., the event that there is no zero-time path from Γ_L to Γ_{3L}) occurs typically, but is not terribly likely (see, e.g., Ref. 17 or Ref. 28). Indeed, denoting the probability of this event by $A_L^*(p_c)$, we have

$$\alpha_1^* \leqslant A_L^*(p_c) \leqslant \alpha_2^* \tag{3.24}$$

with the α_i^* nontrivial. This almost immediately implies the lower bound of Eq. (3.23).

To see this, divide Γ_n into N disjoint annular regions $a_{J_1},..., a_{J_N}$, each one enclosing the one before. In particular, take $J_k = 3^k$ and choose N so that $3^{N+1} \geqslant n \geqslant 3^N$; the setup is illustrated in Fig. 3. The time to the boundary of L_n is not less than the number of these annuli that possess a circuit of dual barriers. Hence,

$$E[\mathscr{T}_n] \geqslant N\alpha_1^* \geqslant (\alpha_1^*/\log 3) \log n \tag{3.25}$$

which implies the lower bound of (3.23) with $b_1 = \alpha_1^*/\log 3$.

The spirit of the upper bound in (3.23) is similar, but the estimate is somewhat more intricate. Since we are after an upper bound on $\mathscr{T}_n$, we will focus on the slightly longer times $\mathscr{T}_{J_{N+1}}$. Geometrically speaking, we are trying to estimate the number of independent rings of barriers that separate the origin from the outside of Γ_n (or $\Gamma_{J_{N+1}}$). Any such ring may visit several of the annuli $a_{J_1},..., a_{J_{N+1}}$. Let us start a classification scheme for rings according to which annulus is the largest one visited. Therefore, we will call the *outer reach* of a ring the largest of our annuli visited by the ring, and denote by V_k the number of independent rings surrounding the origin whose outer reach is the annulus a_{J_k}. Evidently,

$$\mathscr{T}_{J_{N+1}} \leqslant \sum_{k=1}^{N+1} V_k \tag{3.26}$$

All that is needed is a bound on $E_{p_c}[V_k]$ that is independent of k. To this end, we define in an analogous fashion the *inner reach* of a ring, and call V_k^j, $0 \leqslant j \leqslant k$, the number of independent rings with outer reach k and inner reach $k-j$. (Thus, for example, V_k^0 is the number of independent rings that stay completely in the kth annulus.) It should be clear that

$$V_k \leqslant \sum_{j=0}^{k-1} V_k^j \tag{3.27}$$

Now it is not especially difficult to show that

$$\text{Prob}(V_k^0 > n) \leqslant [\alpha_2^*]^n \tag{3.28}$$

and, in fact, to derive a similar result for the V_k^1. The total estimate, however, is begining to deteriorate, so we need to control the V_k^j systematically for large values of j. We will prove that, uniformly in k, there is an $\eta < 1$ and an $H < \infty$ such that

$$E(V_k^j) \leqslant H\eta^j \tag{3.29}$$

Asserting that this has already been done for $j = 0$ and 1, suppose then that $j \geqslant 2$. In order to form a barrier surface encircling the origin that starts at the kth layer and penetrates inward to the $(k-j)$th layer, it has to be the case that none of the intervening annuli contain a complete circuit consisting of the opposite type (zero-time) bonds. The probabilities of these "zero-time circuits" are uniformly bounded below by $\alpha \equiv g(\Delta)$, where Δ is some uniform lower bound on $R_L(p_c)$, and g is the function defined in the proof of Theorem 3.2. Furthermore, the events in different annuli are independent; we thus have

$$\text{Prob}(V_k^j > 0) \leqslant (1-\alpha)^{j-1} \tag{3.30}$$

from which we obtain

$$E(\mathscr{T}_k^j) \leqslant [\alpha(1-\alpha)]^{-1}\alpha^j \tag{3.31}$$

for $j \geqslant 2$. This implies the estimate promised in Eq. (3.29).

Adding everything up, we find

$$\begin{aligned} E(V_k) &\leqslant \sum_{j=0}^{k-1} E(V_k^j) \\ &\leqslant H \sum_j \eta^j = H[1-\eta]^{-1} \end{aligned} \tag{3.32}$$

Thus, if n is larger than one, we have that $E_{p_c}[\mathscr{T}_n]$ is bounded by a constant times $\log n$:

$$E_{p_c}[\mathscr{T}_n] \leqslant b_2 \log n \tag{3.33}$$

Already (3.33) gives a reasonably complete picture of the critical point behavior. To obtain the result in the statement of the theorem, we need only one more ingredient. The first passage time t_{0n} is, of course, the number of barrier rings that separate the origin from n. Since there cannot be

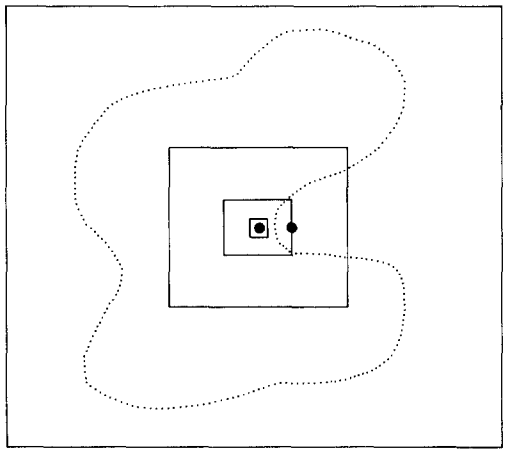

Fig. 4.

any infinite surfaces at the critical point,[29] these rings fall into one of two categories: (1) those rings that surround the origin but do not enclose n; (2) those rings that surround n, but do not enclose the origin. By translation symmetry, the *expected* number of rings in category 1) equals that of category 2). Thus, we will go to work on 1) and double the answer when we are through.

Now, the time $\mathscr{T}_n$ is already a good estimate on the number of rings in either category. However, we have so far failed to count the rings with an outer reach extending beyond the box Γ_n (or $\Gamma_{J_{N+1}}$) (see Fig. 4). These rings can be handled by more or less the same procedure that was used earlier. First, it is noted that any such ring with an outer reach of $A_{J_{N+k}}$ must cross of order k scales in order to *avoid* surrounding n. Thus, both the probability of observing such a ring and the expected number of such rings is exponentially small in k. Summing over all k, this yields only a finite (i.e., n-independent) contribution. When twice this is added to twice the estimate of Eq. (3.33), one obtains the desired result. ▮

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Note added in proof: H. Kesten has recently informed us that he can now prove a stronger form of Theorem 3.1, namely that $\forall p \in (0, 1)$, there is a $c(p) > 0$ such that $\mu(p) \geqslant c(p)\, \xi^{-1}(p)$.

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