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FUNCTIONALS OF BROWNIAN MEANDER AND BROWNIAN EXCURSION

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The primary concern of this paper is to calculate the distributions and/or means of the maxima, first entrance times, and occupation times of Brownian meander and Brownian excursion. The method employed is to develop conditioned families of random functions which have Brownian meander (or Brownian excursion) as their weak limit and then use the continuous mapping theorem.

1. Introduction and summary. This paper is a sequel to Durrett, Iglehart and Miller (1977). We use the notation of that paper, often without mention. In the above paper a number of functional central limit theorems (f.c.l.t.'s) are proved in which either Brownian meander, $\{W^+(t): 0 \leqq t \leqq 1\}$, or Brownian excursion, $\{W_0^+(t): 0 \leqq t \leqq 1\}$, is the limit process. The most important feature of f.c.l.t.'s is that a great variety of other limit theorems can be obtained immediately by employing the continuous mapping theorem. Such results are of little use if we do not know the distribution of functionals of the limit process. In the case of Donsker's theorem (for which the limit process is Brownian motion) we know the distribution of many functionals of interest; see Iglehart (1974), page 237, for a listing. Our concern in this paper is to calculate the distributions and/or means of the maxima, first entrance times, and occupation times for Brownian meander and Brownian excursion. Functionals of Brownian meander are studied in Section 2 and functionals of Brownian excursion in Section 3.

2. Functionals of Brownian meander. In this section we shall exploit the fact that $(W \mid m > -\varepsilon) \Rightarrow W^+$ as $\varepsilon \downarrow 0$ to obtain the distribution of a variety of functionals of W^+. The idea behind the method is to use the continuous mapping theorem; [1], Theorem 5.1. Suppose h is a measurable mapping from C to R^k and that $P\{W^+ \in D_h\} = 0$, where D_h is the set of discontinuities of h. Then $h(W^\varepsilon) \Rightarrow h(W^+)$.

For $h(x) = (\max_{0 \leqq s \leqq 1} x(s), x(1))$ the method mentioned above yields

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(2.1) PROPOSITION. *For* $x > 0$ *and* $0 < y \leqq x$

$$P\{M^+ \leqq x,\ W^+(1) \leqq y\} = \sum_{k=-\infty}^{\infty} [e^{-(2kx)^2/2} - e^{-(2kx+y)^2/2}]\,.$$

PROOF. The probability of the condition $P\{m > -\varepsilon\} \sim (2/\pi)^{\frac{1}{2}}\varepsilon$ as $\varepsilon \downarrow 0$. From [1], equation (11.10),

$$\begin{aligned} P\{-\varepsilon < m \leqq M \leqq x,\ -\varepsilon < W(1) \leqq y\} \\ = \sum_{k=-\infty}^{\infty} [N_1(-\varepsilon + 2k(x+\varepsilon),\ y + 2k(x+\varepsilon)) \\ - N_1(-y - 2\varepsilon + 2k(x+\varepsilon),\ -\varepsilon + 2k(x+\varepsilon))]\,. \end{aligned}$$

Now use dominated convergence to show that as $\varepsilon \downarrow 0$

$$P\{-\varepsilon < m \leqq M \leqq x,\ -\varepsilon < W(1) \leqq y\} \sim (2/\pi)^{\frac{1}{2}}\varepsilon \sum_{k=-\infty}^{\infty} [e^{-(2kx)^2/2} - e^{-(2kx+y)^2/2}]\,.$$

From this expression and the probability of the condition we obtain the desired result.

(2.2) COROLLARY. *For* $x > 0$,

$$P\{M^+ \leqq x\} = 1 + 2\sum_{k=1}^{\infty} (-1)^k \exp\{-(kx)^2/2\}\,. \tag{2.3}$$

Also

$$E\{M^+\} = (2\pi)^{\frac{1}{2}} \ln 2 = 1.7374 \cdots\,. \tag{2.4}$$

PROOF. For (2.3) just set $y = x$ in (2.1) and simplify. For (2.4) use (2.3) plus the relation $E\{M^+\} = \int_0^\infty P\{M^+ > x\}\,dx$.

Next we turn our attention to the distribution of $(M^+(t),\ W^+(t),\ W^+(1))$. Using the same method employed in (2.1) we obtain

(2.5) PROPOSITION. *For* $0 < t < 1$, $x > 0$, *and* $0 < y \leqq x$, $z > 0$

$$\begin{aligned} P\{M^+(t) \leqq x,\ W^+(t) \in dy,\ W^+(1) \in dz\} \\ = t^{-\frac{3}{2}} \sum_{k=-\infty}^{\infty} (2kx + y) g(1-t, y, z) \exp\{-(2kx+y)^2/2t\}\,dy\,dz\,. \end{aligned}$$

(2.6) COROLLARY. *For* $0 < t < 1$ *and* $x > 0$

$$\begin{aligned} P\{M^+(t) \leqq x\} = 2\sum_{k=-\infty}^{\infty} [e^{-(2kx)^2/2} N_{t(1-t)}(2kx(1-t),\ 2kx(1-t) + 2x) \\ - e^{-[(2k+1)x]^2/2t} N_{t(1-t)}(0, x)]\,. \end{aligned}$$

First entrance times for Brownian meander are defined for $x > 0$ by $T_x^+ = \inf\{t \geqq 0 \colon W^+(t) = x\}$. From the usual duality relationship we have $P\{T_x^+ > t\} = P\{M^+(t) < x\}$ which in conjunction with (2.6) yields the first entrance law for Brownian meander to the level x.

As a last functional we consider occupation times. Let A be a Borel set of $[0, \infty)$. Then the expected amount of time in $[0, 1]$ that W^+ is in A is given by

$$E\{\textstyle\int_0^1 1_A(W^+(s))\,ds\} = \int_A \int_0^1 P\{W^+(s) \in dx\}\,ds\,.$$

A simple probabilistic argument yields

(2.7) PROPOSITION. *For* A *a Borel set of* $[0, \infty)$

$$E\{\textstyle\int_0^1 1_A(W^+(s))\,ds\} = \int_A m(x)\,dx\,,$$

where

$$m(x) = 2 \int_0^x \exp\{-(x+y)^2/2\}\, dy\,.$$

PROOF. Begin by writing

$$E\{\int_0^1 1_A(W^+(s))\, ds\} = \int_A \int_0^1 P\{W^+(s) \in dx\}\, ds$$
$$= \int_A dx \int_0^1 2s^{-\frac{3}{2}} x e^{-x^2/2s} N_{1-s}(0, x)\, ds\,.$$

Next write $N_{1-s}(0, x)$ as an integral and interchange order of integration to obtain

$$E\{\int_0^1 1_A(W^+(s))\, ds\} = \int_A \left[2(2\pi)^{\frac{1}{2}} \int_0^x dy \int_0^1 \frac{xe^{-x^2/2s}}{(2\pi s^3)^{\frac{1}{2}}} \cdot \frac{e^{-y^2/2(1-s)}}{[2\pi(1-s)]^{\frac{1}{2}}}\, ds\right] dx\,. \tag{2.8}$$

Let $T_x = \inf\{t > 0 : W(t) = x\}$, the passage time to the level x. Then for $x > 0$, $P\{T_x \in ds\} = (xe^{-x^2/2s}/(2\pi s^3)^{\frac{1}{2}})\, ds$; see Itô–McKean (1974), page 25. Furthermore,

$$P\{W(1) \in dy + x \mid T_x = s\} = P\{W(1) \in dy + x \mid W(s) = x\}$$
$$= \frac{e^{-y^2/2(1-s)}}{[2\pi(1-s)]^{\frac{1}{2}}}\, dy\,.$$

Using these facts together with the path continuity of W (for W to hit $x + y$ at $t = 1$ it must hit x at some time $s \in (0, 1)$), (2.8) yields

$$E\{\int_0^1 1_A(W^+(s))\, ds\} = \int_A [2(2\pi)^{\frac{1}{2}} \int_0^x P\{W(1) \in dy + x\}]\, dx$$
$$= \int_A [2 \int_0^x \exp\{-(x+y)^2/2\}\, dy]\, dx\,,$$

the desired result.

Observe that $m'(x) = 4\exp(-2x^2) - 2\exp(-x^2/2)$ so that m is unimodal and attains its maximum at $[(\frac{2}{3}) \ln 2]^{\frac{1}{2}} = 0.67978 \cdots$.

3. Functionals of Brownian excursion. Using the fact that $(W^+ \mid W^+(1) \leqq \varepsilon) \Rightarrow W_0^+$ we shall compute the distribution of $M_0^+(t)$ for $0 < t \leqq 1$. In addition the expected occupation time of a set A by W_0^+ will be calculated. From [3], Theorem 6.2, we have

(3.1) PROPOSITION. *For* $x > 0$

$$P\{M_0^+ \leqq x\} = 1 + 2 \sum_{k=1}^\infty \exp\{-(2kx)^2/2\}[1 - (2kx)^2]\,.$$

For an analytic verification that the expression in (3.1) is a df and that its density is positive for $x > 0$ see Chung (1975), pages 24–26. In order to compare the df's $P\{M^+ \leqq x\}$ and $P\{M_0^+ \leqq x\}$ we have plotted them in Figure 1. Note that M^+ is stochastically larger than M_0^+, as one would guess.

(3.2) COROLLARY. $E\{M_0^+\} = (\pi/2)^{\frac{1}{2}} = 1.2533 \cdots$.

PROOF. Since $E\{M_0^+\} = \int_0^\infty P\{M_0^+ > x\}\, dx$, we have

$$E\{M_0^+\} = 2 \lim_{n\to\infty} \int_{n^{-1}}^\infty \sum_{k=1}^\infty e^{-(2kx)^2/2}[(2kx)^2 - 1]\, dx\,.$$

Now for $k \geqq n/2$ the integrand is nonnegative on $[n^{-1}, \infty)$ so monotone convergence and integration by parts give

$$E\{M_0^+\} = 2 \lim_{n\to\infty} \sum_{k=1}^\infty e^{-(2k/n)^2/2} n^{-1} = 2 \int_0^\infty e^{-(2x)^2/2}\, dx = (\pi/2)^{\frac{1}{2}}\,.$$

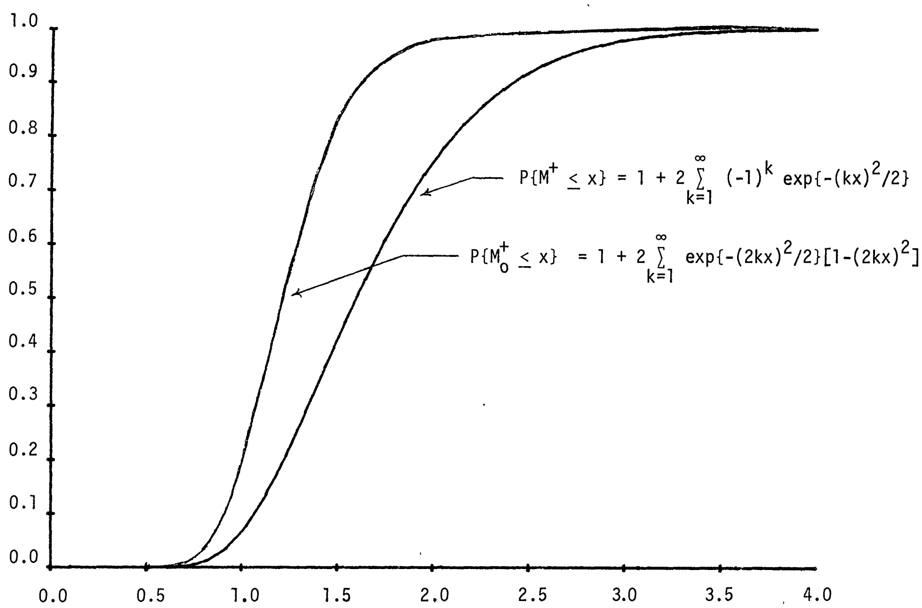

FIG. 1. Distributions of maxima.

The calculation above was first done by M. Bramson. Observe that justifying the interchange here is more than a technicality since a haphazard interchange of $\int_0^\infty$ and $\sum_{k=1}^\infty$ gives the absurdity $E\{M_0^+\} = 0$.

Next we obtain the distribution of $M_0^+(t)$ for $0 < t < 1$.

(3.3) PROPOSITION. *For $0 < t < 1$ and $x > 0$*

$$P\{M_0^+(t) \leqq x\}$$
$$= [2/\pi t^3(1-t)^3]^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} e^{-(2kx)^2/2} \int_0^x y(2kx+y)e^{-[y+2kx(1-t)]^2/2t(1-t)}\, dy\,.$$

PROOF. First use Proposition 2.5 to calculate $P\{M^+(t) \leqq x,\ W^+(t) \in dy,\ W^+(1) \leqq \varepsilon\}$ and integrate out y from 0 to x.

Define $T_0^+(x)$ to be the first entrance time of Brownian excursion to the level $x > 0$. Then again (3.3) yields $P\{T_0^+(x) > t\}$.

Finally we turn to the expected occupation time of a Borel set $A \subset [0, \infty)$ by Brownian excursion. This result was first obtained by Lévy (1939). The probabilistic evaluation of the integral is new and was also discovered independently by K. L. Chung ([2], page 30), who has also calculated higher moments of the occupation time.

(3.4) PROPOSITION. *For A a Borel set of $[0, \infty)$*

$$E\{\textstyle\int_0^1 1_A(W_0^+(s))\, ds\} = \int_A e(x)\, dx\,,$$

where

$$e(x) = 4x \exp(-2x^2)\,.$$

PROOF. Following the procedure used in proving (2.7) we see that

$$\begin{aligned} e(x) &= \int_0^1 p_0^+(0, 0, s, x)\, ds \\ &= (2/\pi)^{\frac{1}{2}} \int_0^1 [s(1-s)]^{-\frac{3}{2}} x^2 \exp\{-x^2/2s(1-s)\}\, ds\,. \end{aligned}$$

Next regroup the terms in the integrand to obtain

$$e(x) = (8\pi)^{\frac{1}{2}} \int_0^1 \frac{xe^{-x^2/2s}}{(2\pi s^3)^{\frac{1}{2}}} \cdot \frac{xe^{-x^2/2(1-s)}}{(2\pi(1-s)^3)^{\frac{1}{2}}}\, ds\,. \tag{3.5}$$

In the proof of (2.7) we noted that $f_x(s) = xe^{-x^2/2s}/(2\pi s^3)^{\frac{1}{2}}$ is the density at s of the passage time T_x for Brownian motion. Thus (3.5) can be written as

$$e(x) = (8\pi)^{\frac{1}{2}} \int_0^1 f_x(s) f_x(1-s)\, ds\,.$$

However, the integral above can be viewed as a first entrance decomposition of $f_{2x}(1)$; if Brownian motion is to hit level $2x$ for the first time at $t = 1$, it must have previously hit level x for the first time at some time $s \in (0, 1)$. Hence

$$e(x) = (8\pi)^{\frac{1}{2}} f_{2x}(1)$$

which is the desired result.

As we saw in the meander case the function e is unimodal, since $e'(x) = 4\exp(-2x^2)[1 - 4x^2]$. Its maximum is attained at $x = \frac{1}{2}$.

In order to compare the occupation time densities m and e for Brownian meander and Brownian excursion we have plotted them in Figure 2. Note that m has a heavier tail and larger mode than does e. This is in keeping with our intuition that the meander should assume larger values than the excursion.

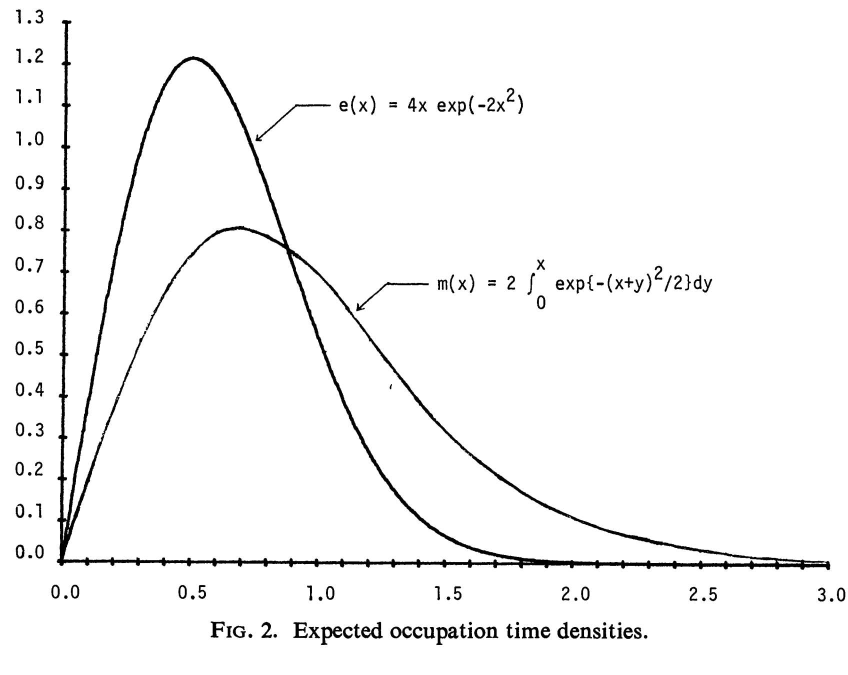

FIG. 2. Expected occupation time densities.

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