Crabgrass, Measles, and Gypsy Moths: An Introduction to Interacting Particle Systems*

Richard Durrett

Our aim in this paper is to explain some of the results about interacting particle systems to someone who has no knowledge of probability theory but understands what it means to flip a coin with a probability p of heads. In what follows we will discuss five models. The last three are hinted at in the title. The first two are simpler systems that will be useful in explaining the others.

1. Oriented Percolation This model takes place on a graph with vertices $\{(x,n):x,n \in \mathbf{Z}$ and $x + n$ is even$\}$ and an oriented bond connecting (x,n) to $(x + 1, n + 1)$ and to $(x - 1, n + 1)$. It will be convenient to reverse the usual orientation of the vertical axis, so n increases as we move down:

$$\begin{array}{ccc} & (x,n) & \\ \swarrow & & \searrow \\ (x-1, n+1) & & (x+1, n+1). \end{array}$$

Each bond is independently open with probability p and closed with probability $1 - p$. Open bonds are thought of as air spaces that are large enough to permit the passage of a fluid, so if we let

$$\xi_n = \{y\text{: there is an open path from } (0,0) \text{ to } (y,n)\},$$

then ξ_n is the set of wet sites on level n when there is a source of fluid at (0,0). The model above is appropriate (if somewhat oversimplified) if one is thinking about the movement of oil in the ground pulled down by the force of gravity. With the last interpretation in mind, it is natural to ask if the material is porous enough for the fluid to penetrate it, or (letting P denote probability) "Is $P(\xi_n \neq \emptyset$ for all $n) > 0$?" When $\xi_n \neq \emptyset$ for all

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Figure 1. Oriented Percolation. $p = .55$.

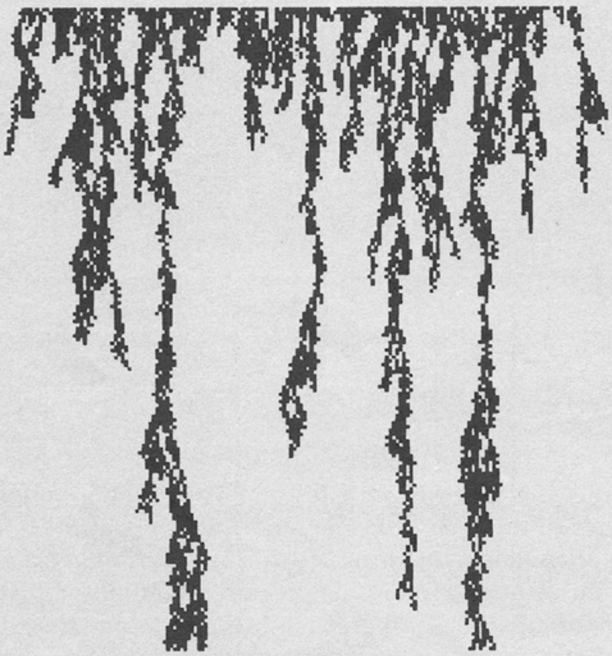

Figure 2. Oriented Percolation. $p = .60$.

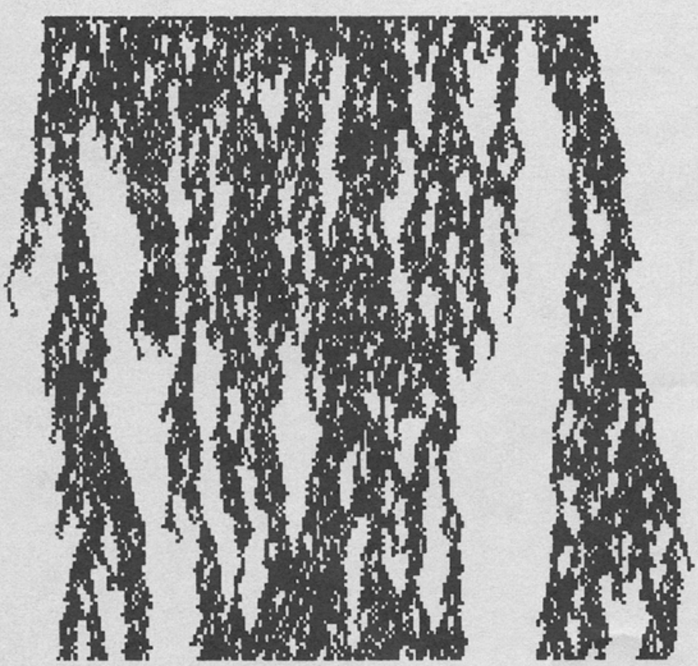

Figure 3. Oriented Percolation. $p = .65$.

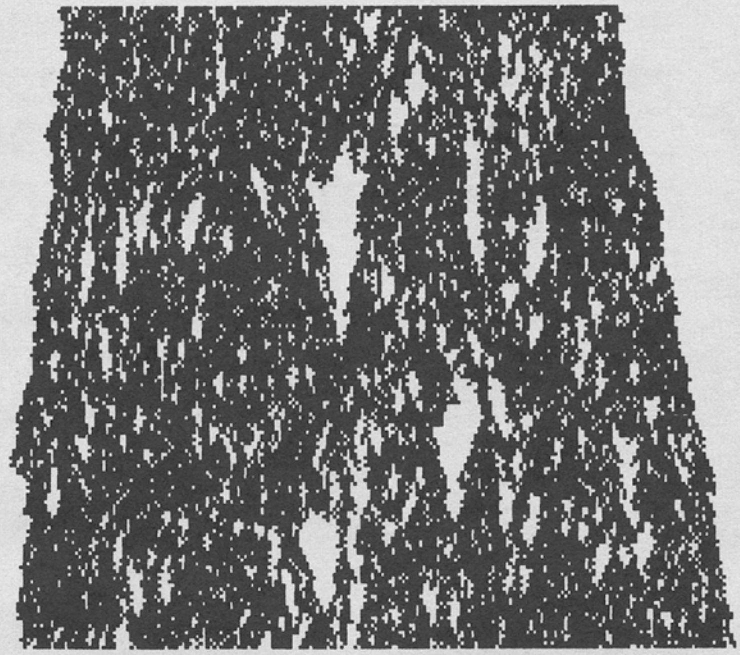

Figure 4. Oriented Percolation. $p = .70$.

n, we say that percolation occurs, and because the probability of percolation is a nondecreasing function of p, we define the critical value of p by

$$p_c = \inf \{p : P(\xi_n \neq \emptyset \text{ for all } n) > 0\}.$$

Here and throughout the article, we do not know the exact value of p_c. Rigorous bounds are not very informative, so to get a feel for what p_c is and what the process looks like, we will resort to simulation. Before rushing off to the computer, we should think for a minute. Each wet site gives rise to $2p$ wet sites on the average. A simple argument shows that the average number of wet sites at time n is at most $(2p)^n$. If $p < 1/2$, it follows that the probability of percolation is 0. With this in mind we start our investigations with $p = .55$. All the pictures show what happens when we start with $\{80, 82, 84, \ldots, 240\}$ occupied at $n = 0$ and run the system until $n = 175$. We will comment on the pictures separately.

$p = .55$: The process "dies out." The reason for this is that even though an isolated particle will have $2p = 1.1 > 1$ particles on the average, two adjacent particles will on the average have $4p - p^2 = 1.8975 < 2$ children.

$p = .60$: Even though the process survived to level 175 in this realization, it can be shown that the system will eventually die out. By exploiting special properties of the model, one can show easily that $p_c > .618$. With a lot of work and a small computer, the last result can be improved to $p_c > .6298$.

$p = .65$: If numerical results can be trusted, and I be-

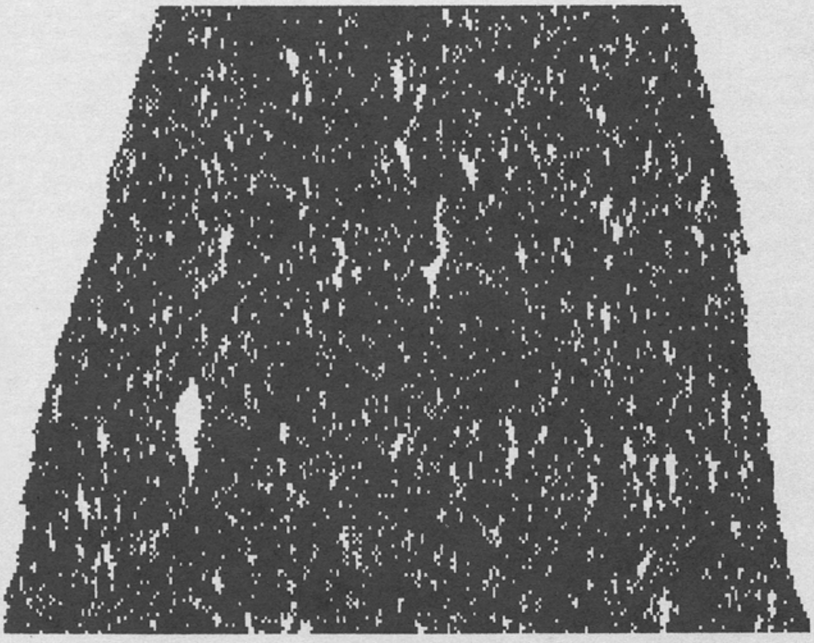

Figure 5. Oriented Percolation. $p = .75$.

lieve they can, we are just above the critical value ($p_c \approx .645$). Comparing this picture with the ones before and after it should explain why we say a "phase transition" occurs at p_c.

$p = .70$: The process is growing well at this point. The white fjords of the previous picture have become lakes. In this and the next picture we can see the phenomenon that is used to characterize the critical value: if we let r_n be the rightmost wet site on level n when initially $\{0, -2, -4, \ldots\}$ is wet, then r_n/n has a limit, denoted $\alpha(p)$, as $n \to \infty$ and $p_c = \inf\{p:\alpha(p) > 0\}$.

$p = .75$: Looking at the picture it is hard to believe that we cannot prove $p_c < .75$, but the best known result is $p_c < .84$. It is surprisingly nontrivial to prove that $p_c < 1$, and the reader is invited to do so. For the answer to this problem and more about oriented percolation (including all the facts cited above), see Durrett (1984).

2. Richardson's Model In this model the state at time n is a subset ξ_n of $\mathbf{Z}^d$. We think of each point in ξ_n as being occupied by an object that we call a "particle" and that you should think of as being a plant or an immobile animal (e.g., barnacle or mussel). The set of occupied points evolves according to very simple rules:

if $x \in \xi_n$, then $x \in \xi_{n+1}$;

if $x \notin \xi_n$, then
$$P(x \notin \xi_{n+1}|\xi_n) = (1-p)^{\#\text{ of occupied neighbors}}$$

The first rule says there are no deaths. To explain the second rule we begin with the left-hand side. It says: "The probability x is not in ξ_{n+1} given that ξ_n is the state at time n." On the right-hand side, the neighbors of x are the $2d$ points with $\|x - y\|_1 = 1$ (where $\|x - y\|_1 = |x_1 - y_1| + \ldots + |x_d - y_d|$). In words, the rule says each occupied neighbor independently sends a particle to x with probability p, so the probability they all fail is the right-hand side. The reader should note that the state at time n is a subset of $\mathbf{Z}^d$; i.e., each site is occupied by 1 or 0 particles, so if two neighbors simultaneously make the site occupied, only one particle results.

Our attention focuses on how ξ_n grows. The main result says that if ξ_n is the set of occupied sites when $\xi_0 = \{0\}$, then ξ_n has a limiting shape:

There is a convex set D so that for any $\epsilon > 0$ we have

$$n(1-\epsilon)D \cap \mathbf{Z}^d \subset \xi_n \subset n(1+\epsilon)D$$

for all n sufficiently large.

Loosely speaking, ξ_n looks like $nD \cap \mathbf{Z}^d$ when n is large. In one dimension $D = [-p,p]$, but we don't know much about D when $d > 1$, except for the trivial observations that it has the same symmetry as $\mathbf{Z}^d$, it contains $\{x:\|x\|_1 \leq p\}$, and it is contained in $\{x:\|x\|_1 \leq 1\}$, which is the limit when $p = 1$. This state of affairs exists because the result is proved by using the subadditive ergodic theorem to show that if

$$t_k = \inf\{n:(k,0,\ldots,0) \in \xi_n\}$$

then

$$t_k/k \to \inf_{j\geq 1} E(t_j/j) \text{ as } k \to \infty,$$

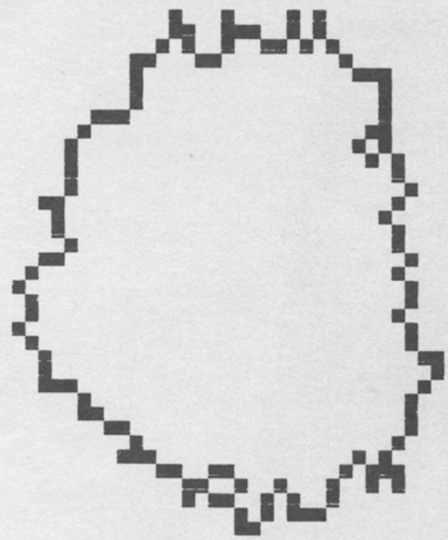

Figure 6. Richardson's Model.
$p = .25$, time 39.

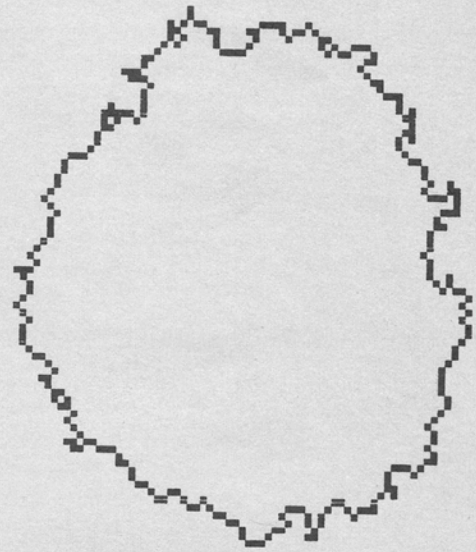

Figure 7. Richardson's Model.
$p = .25$, time 81.

where $E(t_j/j)$ stands for the average value of the random variable in parentheses.

The expression for the limiting constant is mathematically nice because the infimum exists, is nonnegative, and is finite. It does not lend itself well to computation (except of course for upper bounds), so we will continue our discussion by looking at some pictures. There are two sets of three. In each set the model is shown in $d = 2$ at the times it first reached the edge of $\{x:\|x\|_\infty \leq k\}$ when $k = 20, 40$, and 80. (Here $\|x\|_\infty = \sup_i|x_i|$.) One site is 4 pixels by 4 pixels in the first picture, 2 pixels by 2 pixels in the second, and a single pixel in the third; we have economized on ink by only coloring the sites in ξ_n that have a neighbor not in ξ_n.

In the first set of pictures $p = .25$. The reader should notice that the times roughly double going from one picture to the next, consistent with the linear growth stated above, and the set has very few holes. The last observation has proved difficult to make rigorous, and even nonrigorous studies cannot agree on the number of holes and how close they are to the boundary. An interesting unsolved problem related to this is to prove the central limit theorem for the passage times; i.e., find constants c_k so that

$$(t_k - \mu k)/c_k \to \text{a normal distribution.}$$

In $d = 1$ the result holds with $c_k = k^{1/2}$. In $d > 1$ smaller norming constants are probably needed, but there is little consensus about what to guess. The first studies suggested $c_k = \log k$, but $c_k = k^\alpha$ with $0 < \alpha < 1/2$ is probably correct.

The second set of pictures has $p = .75$ and demonstrates the one nontrivial fact that we can prove about D. The limiting shape has a "flat edge" if p is greater than the critical value of oriented percolation discussed earlier. The proof is simple: ξ_n is contained in $\{z \in Z^2:\|z\|_1 \leq n\}$, and if we look at $\xi_n \cap \{Z:z_1 + z_2 = n\}$, then we get a process equivalent to oriented percolation. From the last observation it follows that if $p > p_c$, then the intersection grows linearly with positive probability. With a little more work it can be shown that $D \cap \{x:\|x\|_1 \leq 1\}$ is an interval of length $\sqrt{2} \cdot \alpha(p)$, where $\alpha(p)$ is as in the previous section.

3. Measles The next process can be used to model the spread of a disease or a forest fire. The state of the process at time n is a function $\xi_n:\mathbf{Z}^2 \to \{1,i,0\}$. The states 1, i, and 0 have the following meanings in the two interpretations:

1	tree	healthy
i	on fire	infected
0	burned	immune

We have chosen measles for the disease because in that case once you have had the disease you cannot have it again. With this or a forest fire in mind, the dynamics of the model can be described as follows:

if $\xi_n(x) = 0$, then $\xi_{n+1}(x) = 0$;
if $\xi_n(x) = i$, then $\xi_{n+1}(x) = 0$;
if $\xi_n(x) = 1$, then
$$P(\xi_{n+1}(x) = 1|\xi_n) = (1 - p)^{\#\text{ of infected neighbors}}$$
$$P(\xi_{n+1}(x) = i|\xi_n) = 1 - P(\xi_{n+1}(x) = 1|\xi_n).$$

The reason for the first rule was explained above. In the second we have taken the unreasonable viewpoint that the disease always lasts for exactly one unit of time. This can be generalized considerably, but we only consider the simplest case here. The third rule should be familiar from the last model: each infected

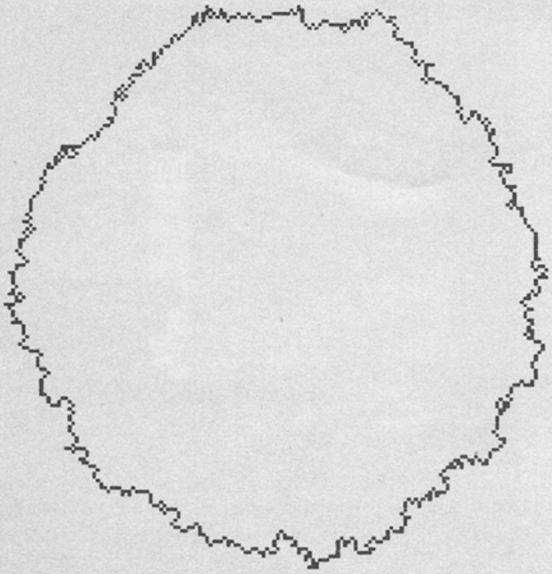

Figure 8. Richardson's Model.
$p = .25$, time 164.

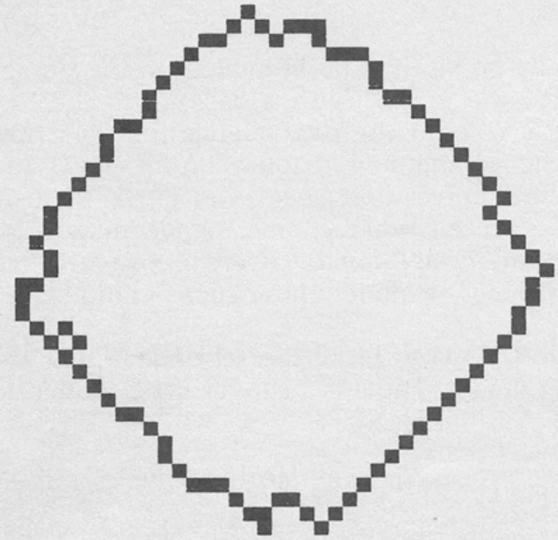

Figure 9. Richardson's Model. $p = .75$, time 19.

neighbor independently tries to infect x with probability p.

We will be interested in what happens when in the initial state one individual is infected and all others are healthy. This initial state is given by: $\xi_0(0) = i$ and $\xi_0(x) = 1$ for $x \neq 0$. Let $I_n = \{x:\xi_n(x) = i\}$ be the set of infected individuals at time n and let Ω_∞ be the event that the infection does not die out or "percolation occurs," meaning that $I_n \neq \emptyset$ for all n. As in the case of oriented percolation, the probability of Ω_∞ is a nondecreasing function of p, so we define

$$p_c = \inf\{p:P_p(\Omega_\infty) > 0\}.$$

For once in this paper we know exactly what p_c is! $p_c = ½$.

The last fact is the reason we have chosen to discuss the discrete time model here. To explain how we know $p_c = ½$, and to state what we know about the asymptotic behavior of ξ_n, we need to introduce a related percolation process. For each point x and each of its neighbors y, we flip a coin to see if x will try to infect y at the one time x is infected (if this ever occurs), and we draw an arrow from x to y if it will try. Let $C_0 = \{x:0 \to x\}$ where $0 \to x$ means there is a path of arrows from 0 to x. A little thought reveals that C_0 is the set of points that will ever be infected and $\Omega_\infty = \{C_0$ is infinite$\}$.

The percolation process described in the previous paragraph is not the same as the usual bond percolation model in which we flip a coin for each pair of neighbors x and y to see if they are connected by a bond that can be traversed in either direction. Somewhat surprisingly it is equivalent in a very strong sense: if S and T are two subsets of $\mathbf{Z}^2$, then the probability of a path from S to T is the same in the two models. The last fact is surprising, but once guessed

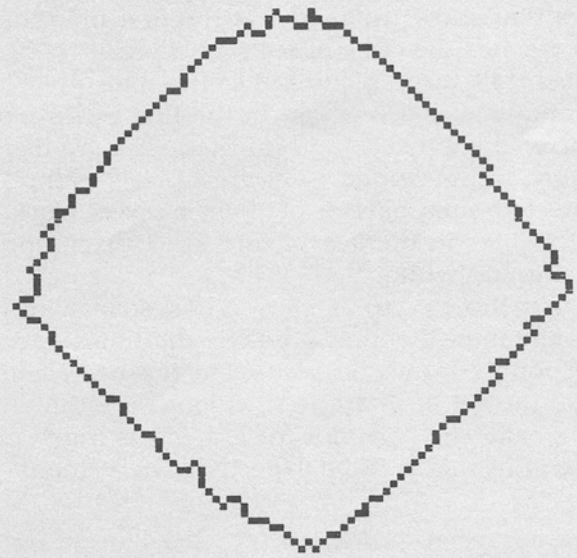

Figure 10. Richardson's Model. $p = .75$, time 41.

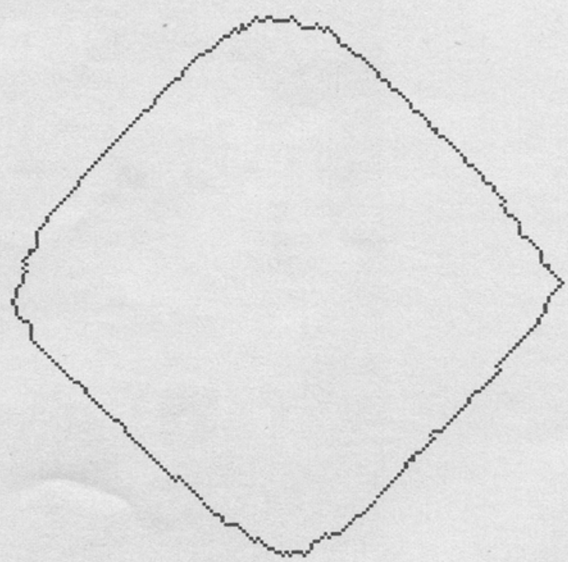

Figure 11. Richardson's Model. $p = .75$, time 86.

can easily be verified by induction on the number of bonds in the graph (with a passage to the limit to handle $\mathbf{Z}^2$). With the equivalence just described in hand, the fact that $p_c = \frac{1}{2}$ follows from what is known about bond percolation (see Kesten [1982] for a survey of results on this process), and the techniques used to study that model can be used to prove a "shape theorem" for the model under consideration:

Let $J_n = \{x:\xi_n(x) = 0\}$ be the set of sites that are immune at time n. There is a convex set D so that for all $\epsilon > 0$

$$n(1 - \epsilon)D \cap C_0 \subset J_n \subset n(1 + \epsilon)D$$

for all n sufficiently large.

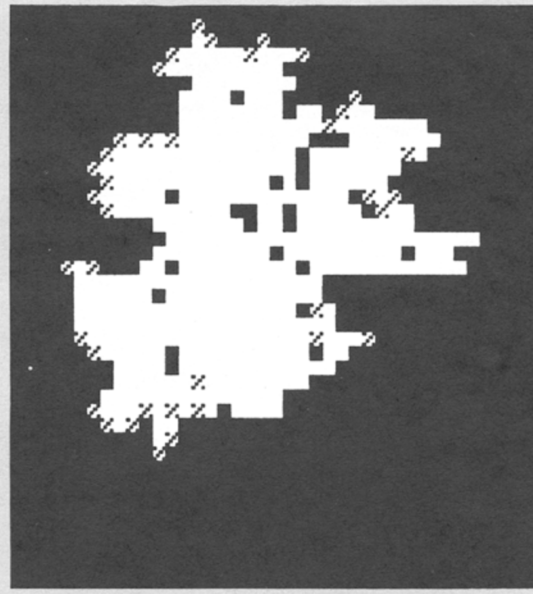

Figure 15. Forest Fire. $p = .51$, time 23.

As before, we will close the discussion by looking at some simulations. Again, we have two sets of three pictures that show the model at the first time the infection reaches the edge of $\{x:\|x\|_\infty \leq k\}$ for $k = 20$, 40, and 80. In all three pictures healthy sites are black, and immune sites are white. In the first picture, sites are 4 pixels by 4 pixels, and infected sites are marked with an X. In the second, sites are 2 pixels by 2 pixels, and infected sites have 2 of their 4 pixels black. Finally, the sites in the last picture are 1 pixel, and infected sites are white.

The first three pictures show the system with $p =$ .60. As the limit theorem predicts, the times approximately double from one picture to the next, and by time 100 the set of immune sites looks roughly like a growing ball. The behavior for $p = .51$ is much more irregular, but this is to be expected. The value of p is close to the critical value, and the correlation length—which measures the distance we have to go until the limiting shape starts to appear—is large. Nonrigorous results tell us that the correlation length diverges like $(p - \frac{1}{2})^{-v}$ as $p \downarrow \frac{1}{2}$, where $v \approx \frac{4}{3}$, and that if we look at the system inside the correlation length and let $p \downarrow$ 1/2, the limit will be a fractal object with Hausdorf dimension less than 2, but we are far from proving anything like this.

4. Gypsy Moths In this process (officially called the contact process) and the next one, time is continuous; i.e., the process is defined for all $t \geq 0$. Otherwise, these processes are much like a version of Richard-

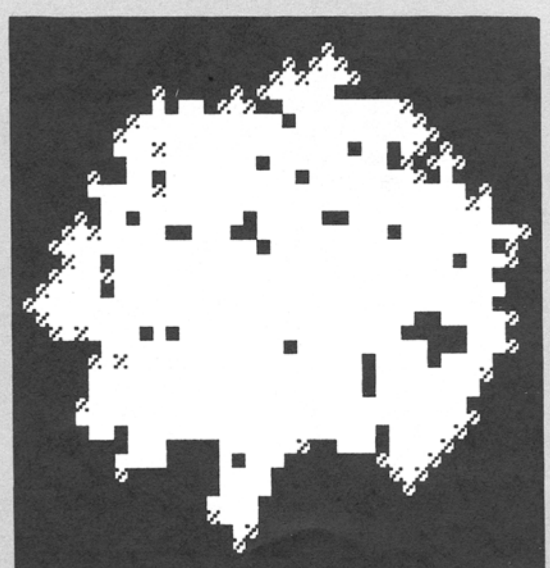

Figure 12. Forest Fire. $p = .60$, time 22.

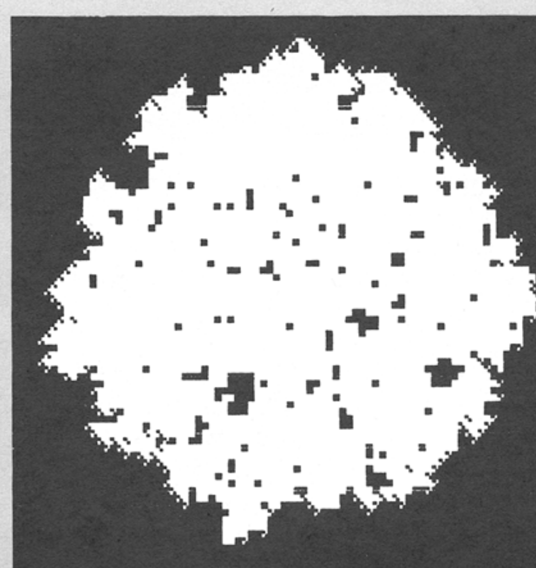

Figure 13. Forest Fire. $p = .60$, time 47.

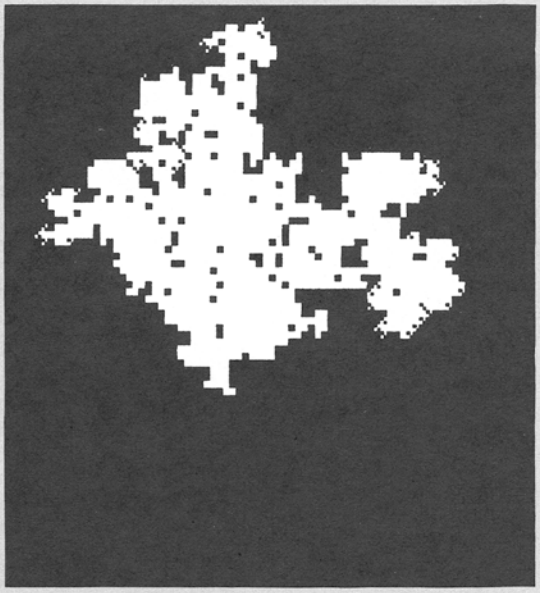

Figure 16. Forest Fire. $p = .51$, time 55.

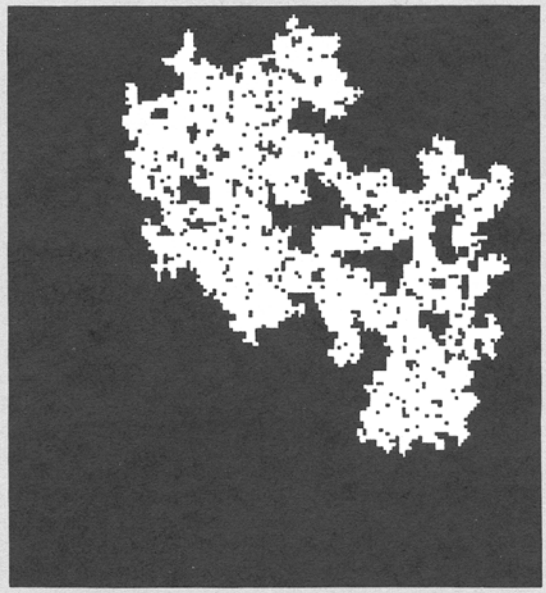

Figure 17. Forest Fire. $p = .51$, time 128.

son's model in which occupied sites become vacant with positive probability. The state at time t is $\xi_t \subset \mathbf{Z}^d$. As before, we could think of ξ_t as the set of "occupied" sites, but to be true to our title, we think of the points of $\mathbf{Z}^d$ being trees, and the ξ_t as indicating the trees infected by gypsy moths.

The system evolves according to the following rules:

if $x \in \xi_t$, then $P(x \notin \xi_{t+s}|\xi_t) = s + o(s)$;
if $x \notin \xi_t$, then $P(x \in \xi_{t+s}|\xi_t) =$
$\beta s(\#\text{ of occupied neighbors})/2d + o(s)$,

where $o(s)$ means that the missing terms when divided by s go to 0 as $s \to 0$. For readers familiar with Markov chains, the rules may be expressed as: particles die at rate one and are born at vacant sites at rate $\beta(\#$ of occupied neighbors). If you are not familiar with Markov chains, the best way to think about the system is in the way it is implemented in a computer. To do this when $\beta \geq 1$, we pick a site at random from the finite set of sites under consideration and call it x. If x is vacant, we pick one of its neighbors at random and make x occupied if the neighbor is. If x is occupied, we pick a number at random from the interval (0,1) (by calling the computer's random-number generator), and kill the particle if (and only if) the number is less than $1/\beta$. We repeat this procedure to simulate the system. If there are m sites, then $t\beta m$ cycles correspond roughly to t units of time.

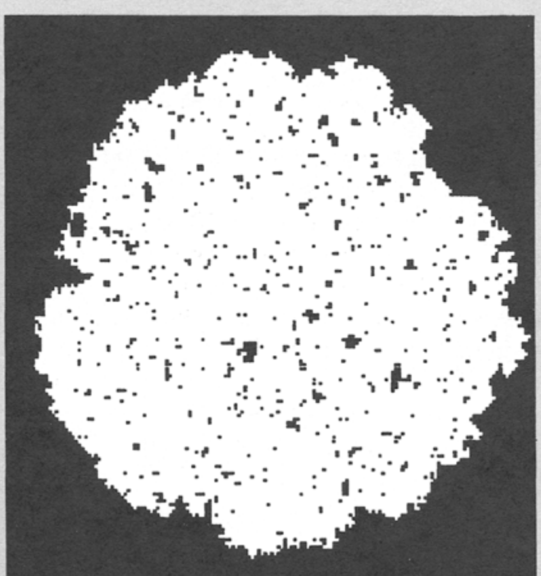

Figure 14. Forest Fire. $p = .60$, time 100.

From the descriptions above, it should be clear that the contact process is the continuous time analog of oriented percolation. With this analogy in mind, we let

$$\beta_c = \inf\{\beta : P(\xi_t^0 \neq \emptyset \text{ for all } t) > 0\},$$

where ξ_t^0 is the state at time t starting from $\xi_0^0 = \{0\}$.

By now the reader should not be surprised that we do not know the value of β_c. Two results worth mentioning are: (1) it is easy to show $\beta_c \geq 1$; and (2) a clever argument of Holley and Liggett shows $\beta_c \leq 4$. The second bound is good in low dimensions and the first in high dimensions. Numerically $\beta_c \approx 3.3$ for $d = 1$, $\beta_c \approx 1.65$ for $d = 2$, and it has been shown that $\beta_c \to 1$ as $d \to \infty$.

In the last two sections we have seen that starting from a finite set the system expands linearly and has an asymptotic shape. This is true again here, but we will have to introduce a few concepts to state the limit result. We begin by considering what happens in the process ξ_t^1 starting from all sites occupied; i.e., $\xi_0^1 = \mathbf{Z}^d$.

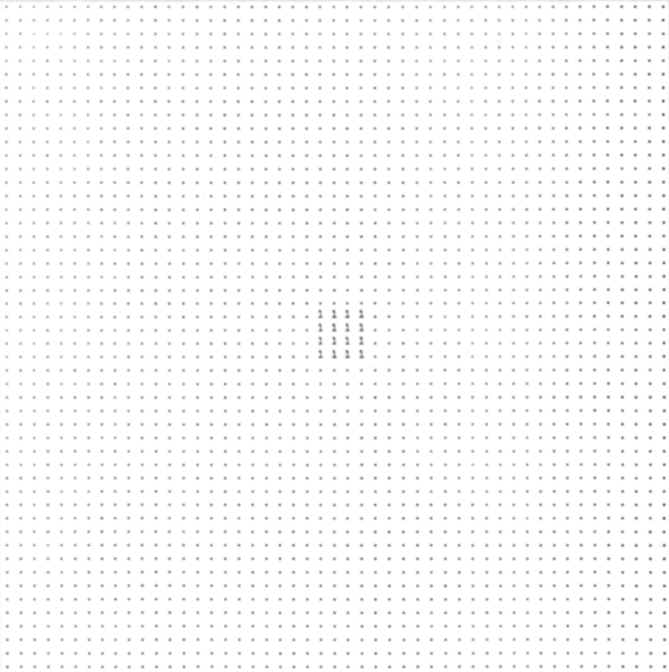

Figure 18. Gypsy Moths. Time 0.

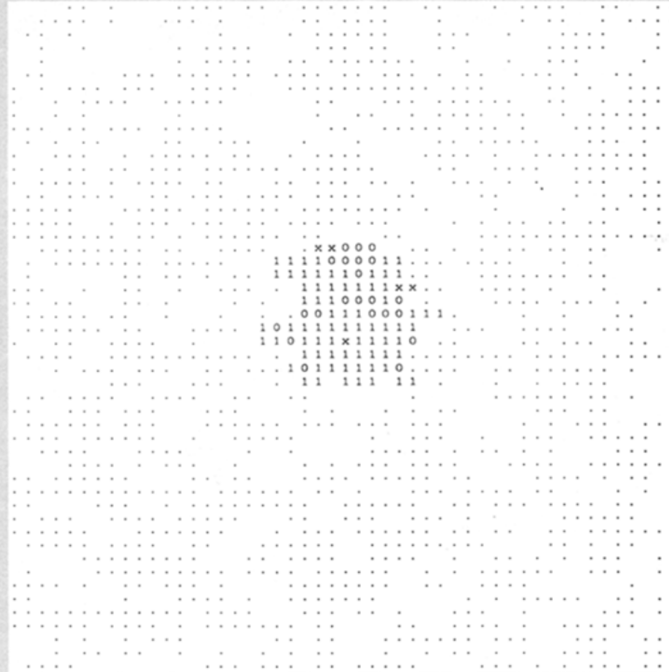

Figure 19. Gypsy Moths. Time 20, density .5812.

Now $\mathbf{Z}^d$ is the largest state (in the partial order $\supset$), and the computer implementation of the model described above has the property that if we use it to run two versions of the process ξ_t^A and ξ_t^B starting from initial states $A \supset B$, then we will have $\xi_t^A \supset \xi_t^B$ for all t. If we let $A = \mathbf{Z}^d$ and $B = \xi_s^1$ in the last observation, then we see that ξ_t^1 is larger than ξ_{t+s}^1 in the sense that the two random sets can be constructed on the same space with $\xi_t^1 \supset \xi_{t+s}^1$. Once one understands the last sentence, a simple argument shows that as $t \to \infty$, ξ_t^1 decreases to a limit we call ξ_∞^1, where the convergence occurs in the sense that

$$P(\xi_t^1 \cap C \neq \emptyset) \downarrow P(\xi_\infty^1 \cap C \neq \emptyset) \text{ for all finite sets } C.$$

It follows from Markov chain theory that ξ_∞^1 is an equilibrium distribution for the process; i.e., if the initial state has this distribution, then this will be the distribution at all $t \geqslant 0$. If $\beta < \beta_c$ then ξ_∞^1 is not interesting —it is $\emptyset$ with probability 1—but if $\beta > \beta_c$ it is a nontrivial equilibrium distribution. The reader will note that $\beta = \beta_c$ has been left out in the last statement. Presumably this value falls under the first case, but this is a very difficult open problem.

With the equilibrium distribution introduced, we are now in a position to describe the limiting behavior starting from a finite set. Suppose we use the computer implementation to run two versions of the process, one starting from a finite set A and the other starting from all of $\mathbf{Z}^d$ occupied, and we call the two resulting processes ξ_t^A and ξ_t^1. The "shape theorem" in this setting is:

There is a convex set D so that if $\xi_t^A \neq \emptyset$ for all t, then for any $\epsilon > 0$ we have

$$t(1 - \epsilon)D \cap \xi_t^1 \subset \xi_t^A \subset t(1 + \epsilon)D$$

for all t sufficiently large.

The statement of the result is contorted by the fact that ξ_t^A may become $\emptyset$, in which case it stays $\emptyset$ for all time. The theorem tells us that when this does not occur ξ_t^A look roughly like $\xi_\infty^1 \cap tD$. In words, it is a linearly growing "blob in equilibrium"; more poetically, it is an "expanding gray disk." The disk is called gray because the equilibrium state has correlations that are exponentially decaying, and hence if we look at the configuration of occupied (white) and vacant (black) cells from a distance, all we will see is the average value (a shade of gray).

To illustrate the last theorem, we have included pictures of a simulation of the process in $d = 2$ with $\beta = 3$ and $A = \{0,1,2,3\}^2$, viewed at times 0, 20, 40, and 60. Ones mark the points in ξ_t^A. Periods mark points of ξ_t^1 that are not in $H_t = \bigcup_{s<t} \xi_s^A$ (the set of points hit by time t), and blanks mark the points in the complement of H_t that are not in ξ_t^1. Finally zeros mark the points in H_t that are not in ξ_t^1 (and hence not in ξ_t^A), and x's mark the points in H_t that are in ξ_t^1 but not in ξ_t^A. The point of this labeling scheme is that ones and zeros mark the points in H_t where ξ_t^A and ξ_t^1 agree, and x's mark the points where they disagree, so one sees that the two processes agree across most of H_t.

In addition to illustrating the shape theorem, the pictures are supposed to give you a feel for what the

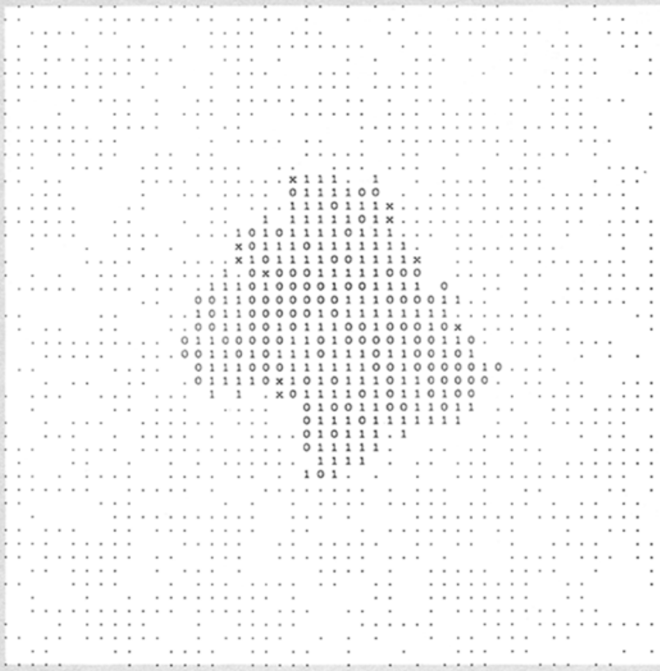

Figure 20. Gypsy Moths. Time 40, density .5952.

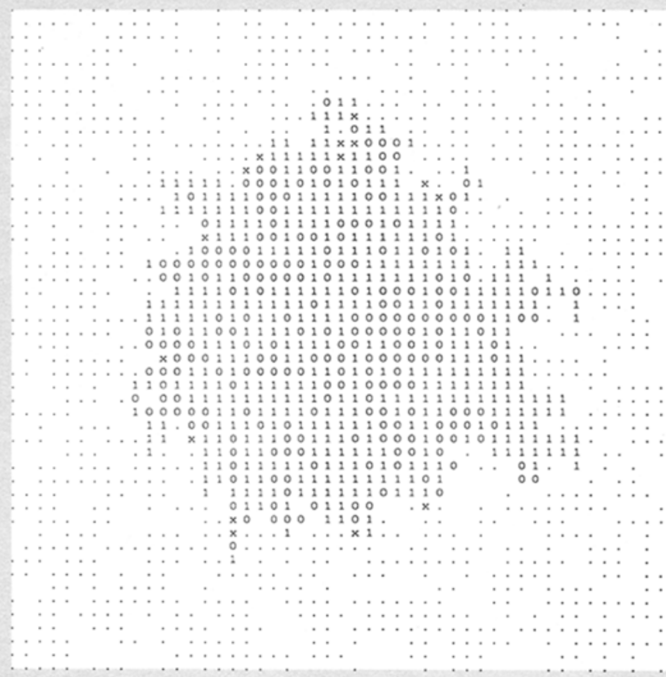

Figure 21. Gypsy Moths. Time 60, density .6316.

equilibrium state looks like. When $\beta = 3$ the density of occupied sites in equilibrium is about .6, and the system is very close to equilibrium at time 20. The reader should note that while the discussion above stated that ξ_t^1 decreases to a limit, the densities at times 20, 40, and 60 are .5812, 5952, and .6316!

5. Crabgrass In this process the state at time t is $\xi_t \subset \mathbf{Z}^d/M = \{z/M : z \in \mathbf{Z}^d\}$, where M is a large integer. Think of a lawn consisting of a lot of plants with a small spacing between them. With this (and the contact process) in mind, we say two points x and y are neighbors if $\|x - y\|_\infty \leq 1$ and formulate the dynamics as follows:

if $x \in \xi_t$, then $P(x \notin \xi_{t+s} | \xi_t) = s + o(s)$;
if $x \notin \xi_t$, then $P(x \in \xi_{t+s} | \xi_t) =$
$\beta s(\#\text{ of occupied neighbors})/v(M) + o(s)$,

where the $o(s)$ was explained in the last section, and $v(M) = (2M + 1)^d - 1 =$ the number of neighbors a point has (v is for volume).

The normalization above is chosen so that the birth rate from an isolated particle is β, and hence the critical value $\beta_c(M)$ (defined in the last section but now recording the dependence of M) satisfies $\beta_c(M) \geq 1$. At first glance, increasing the range of the interaction makes the process more complicated, but in fact as $M \to \infty$ things get much simpler: $\beta_c(M) \to 1$ and if we fix $\beta > 1$ then

$$P(\xi_t^0 \neq \emptyset \text{ for all } t) \to (\beta - 1)/\beta.$$

The right-hand side is the probability of survival for a branching process—a system in which particles die at rate 1, reproduce at rate β, and are not limited by the restriction of at most one particle per site.

Intuitively the results above say that the contact process behaves like the branching process when M is large. The difficulty in proving this is that for any fixed M, differences appear when t is about $(\log M)/(\beta - 1)$, and in the last two statements we are letting $t \to \infty$ before $M \to \infty$. If we keep β and t fixed as $M \to \infty$, we get a situation that is much easier to analyze but still says something about the time evolution of the process. Consider a sequence of initial states in which sites are independently designated as occupied or vacant and with $P(x \in \xi_0^M) \to u(x,0)$ uniformly on compact sets. In this case, the states of various sites at time t are asymptotically independent as $M \to \infty$ ("propagation of chaos"), and the probability of occupancy $u_M(x,t) = P(x \in \xi_t^M)$ converges uniformly on compact sets to a limit $u(x,t)$ that satisfies

$$\frac{\partial u}{\partial t} = -u + \beta(1 - u)(u * \psi) \qquad (*)$$

where

$$(u * \psi)(x) = \int u(y)\psi(x - y)dy$$

and

$$\psi(y) = \begin{cases} (1/2)^d & \text{if } \|y\|_\infty \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The last assertion is easy to explain (and not much harder to prove). The $-u$ comes from the fact that

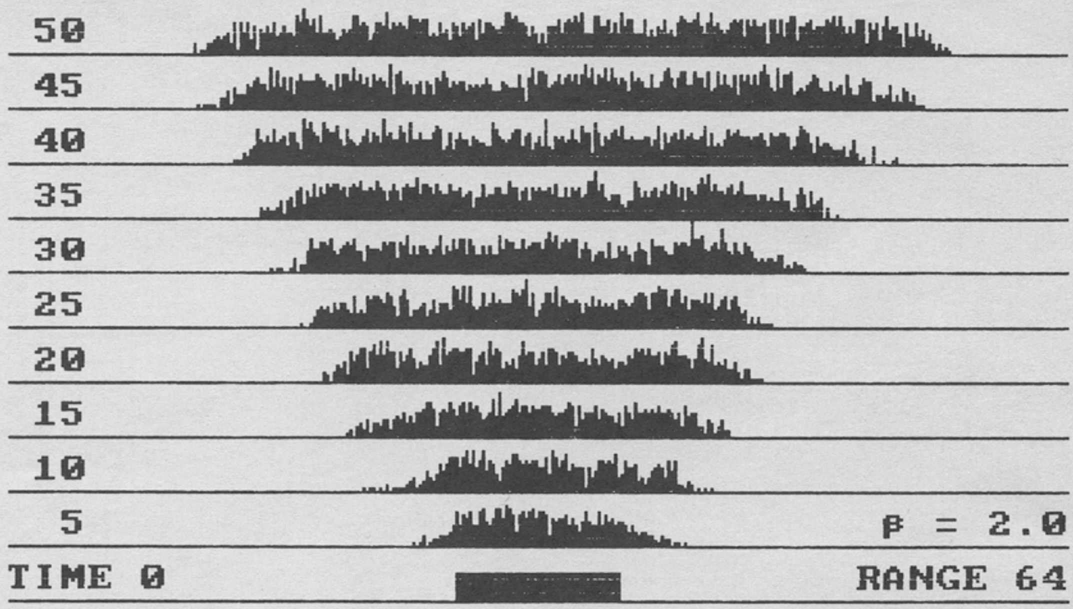

Figure 22. Crabgrass. Range = 64.

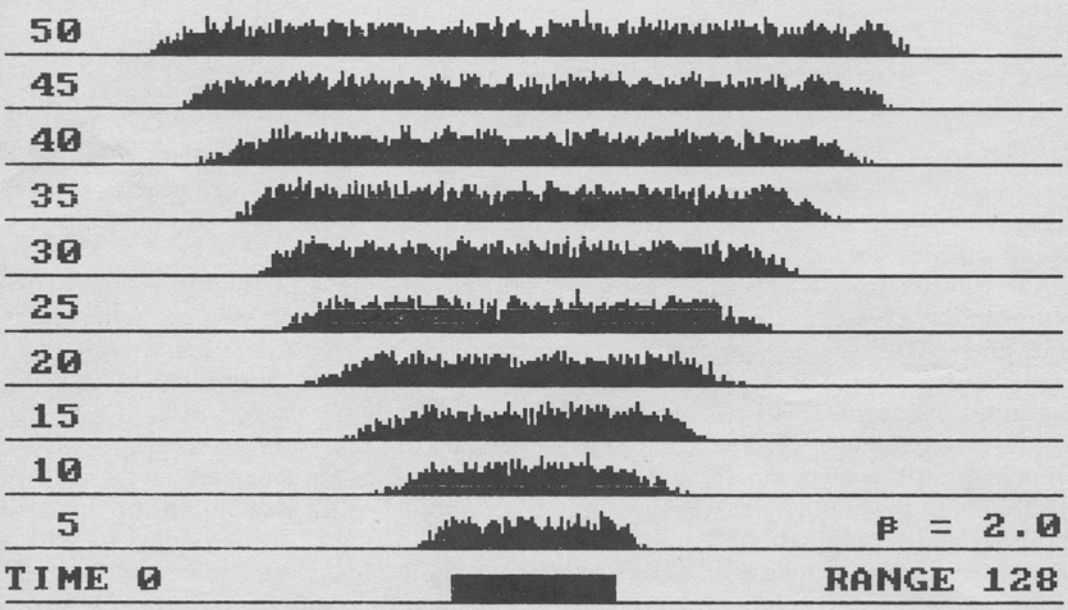

Figure 23. Crabgrass. Range = 128.

deaths occur at rate 1; the other term from the facts that (1) births from y to x occur at rate $\beta/v(M)$ if x is vacant, y is occupied, and $\|x - y\|_\infty \leq 1$, and (2) in the limit as $M \to \infty$ the occupancy of y and the vacancy of x are independent.

The last result is useful because it tells us something concrete about the time evolution of the process when M is large, and although the information is not very explicit, it is much better than what we know about the case $M = 1$. It is interesting to note that by using some facts about the particle system we can derive a representation for the solution of (*) that allows us to show that there is a convex set D (which can be described explicitly) so that starting from compactly supported nonzero initial data

$$\begin{aligned} u(x,t) &\approx (\beta - 1)/\beta \quad \text{if} \quad x \in (1 - \epsilon)tD \\ &\approx 0 \quad \text{if} \quad x \notin (1 + \epsilon)tD. \end{aligned}$$

Finally, of course, we have some computer pictures to illustrate the last theorem. The pictures show the system in $d = 1$ with $\beta = 2$ when the range M is 64, 128, and 512, starting from an interval in which every other site is occupied. Each pixel across represents $M/4$ sites, and the height of the black column is the number of occupied sites divided by 1, 2, or 4, respectively. The reader should note that while the linear spread is clearly visible in all three cases, the picture is quite ragged when $M = 64$, and smoothes out as M increases, but there are still significant fluctuations when $M = 512$.

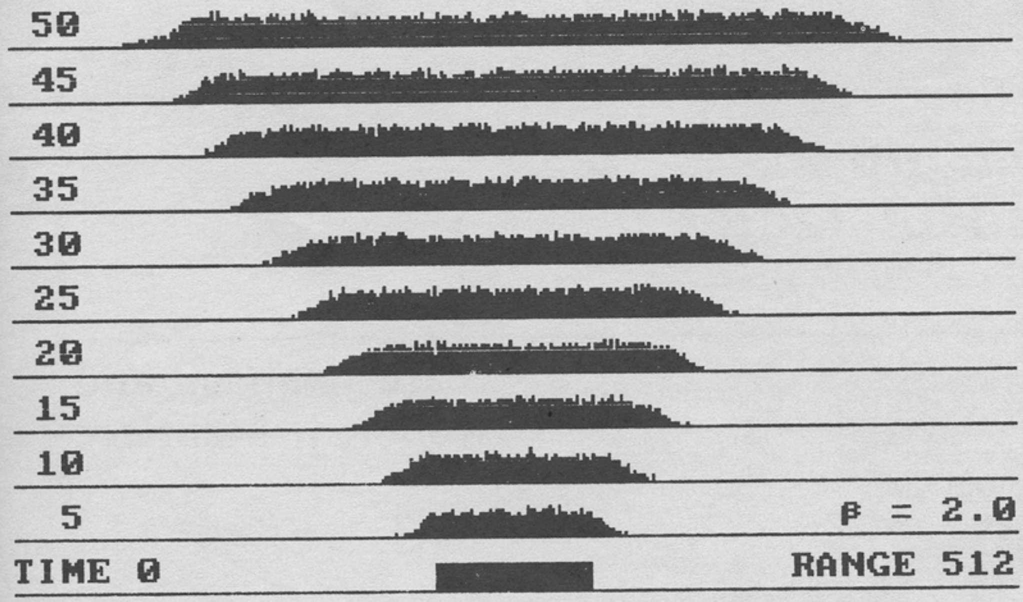

Figure 24. Crabgrass. Range = 512.

References

For readers who would like to learn more about the subject, we will now give some references—first for the specific results given and then for the subject in general.

Oriented percolation is surveyed in Durrett (1984). For Richardson's model, his original paper (1973) is a good source. For a more recent treatment with minimal assumptions, see Cox and Durrett (1981). That paper contains the key ideas for the analysis of the epidemic model in Cox and Durrett (198?). The contact process is surveyed in Chapter 6 of Liggett (1985), but he does not prove the shape theorem. This is done in Durrett and Griffeath (1982) for β greater than the one-dimensional critical value and in Durrett and Schonmann (1987) for $\beta > \beta_c^*$, a value greater than or equal to β_c and conjectured to be equal to β_c. See the paper cited for more on this point. Finally, the results in Section 5 are in unpublished work by Bramson, Durrett, and Swindle.

For an introduction to interacting particle systems as a whole, Liggett's (1985) book is a highly recommended source. Durrett (1985), Tautu (1986), and the volume containing Durrett and Schonmann are conference proceedings that will fill you in on recent developments. Last but not least, for a "soft" introduction to the subject, you should get IPSmovies, a collection of 33 Pascal programs (written for Turbo Pascal and available from Wadsworth Publishing Co.) that simulate most of the systems discussed and a number of processes we have not talked about.

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Truth is much too complicated to allow anything but approximations.

John von Neumann