

Large Deviations for the Contact Process and Two Dimensional Percolation

R. Durrett^{1,*} and R.H. Schonmann^{1,2,**}

¹ Department of Mathematics, Cornell University, White Hall, Ithaca, NY 14853, USA

² Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 20570, 01000 São Paulo, SP, BRASIL

Summary. The following results are proved: 1) For the upper invariant measure of the basic one-dimensional supercritical contact process the density of 1's has the usual large deviation behavior: the probability of a large deviation decays exponentially with the number of sites considered. 2) For supercritical two-dimensional nearest neighbor site (or bond) percolation the density Y_A of sites inside a square A which belong to the infinite cluster has the following large deviation properties. The probability that Y_A deviates from its expected value by a positive amount decays exponentially with the area of A , while the probability that it deviates from its expected value by a negative amount decays exponentially with the perimeter of A . These two problems are treated together in this paper because similar techniques (renormalization) are used for both.

1. Introduction

In a previous paper, [LS], Joel Lebowitz and one of the authors obtained some large deviation properties for FKG random fields and more specific results when the random field corresponds to an invariant measure of a translation invariant attractive spin system. In this paper we will obtain more information for the upper invariant measure of the one-dimensional basic contact process, solving a question which was left open in [LS]. We will show that for this measure the density of 1's has the usual large deviation behavior (the probability of a large deviation decays exponentially with the number of sites considered).

We will also consider large deviation for a field which appears in connection with two-dimensional site or (unoriented) bond percolation in the supercritical

* Partially supported by the National Science Foundation and the U.S. Army Research Office through the Mathematical Sciences Institute at Cornell

** Partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq (Brazil) and the U.S. Army Research Office through the Mathematical Sciences Institute at Cornell

regime: $\sigma(x)=1$ if x is in the infinite cluster and $\sigma(x)=0$ otherwise. For this field we will show that the probability that the density of 1's inside a large square deviates from its expected value by a positive amount decays exponentially with the area of the square, but the probability of the deviation by a negative amount decays only as an exponential of the perimeter of this square.

While the two problems considered in this paper at first sight may seem very different, they are in fact closely related. This fact is not so surprising when one remembers the similarity between the contact process and oriented percolation [Dur]. The strategy that we will use to prove the surface behavior (exponential decay with the perimeter) for percolation will be to first consider large deviations for one-dimensional embedded processes which are very similar to the upper invariant measure of the contact process.

These techniques are (rigorous) renormalization group type arguments, which in fact were first used for percolation [Rus2] and later adapted to the contact process in [DG]. This technique permits us to transform the original problem into a similar one for a slightly more complicated model – one-dependent oriented site percolation – but with the probability of occupancy very close to one. For this latter process one can use contour estimates to establish the desired results.

In Sect. 2 we introduce the basic terminology and notation and review the previous results. In Sect. 3 we recall some basic facts about the contact process and state our new result for this model, which will be proven in Sect. 5. In Sect. 4 we list some basic facts about percolation that we need and state our results, which will be proven in Sect. 6.

2. Terminology, Notation and Previous Results

Let Ω_d be the configuration space $\{0, 1\}^{\mathbb{Z}^d}$ endowed with the product topology corresponding to the discrete topology on $\{0, 1\}$. The random fields on Ω_d correspond to probability measures on its Borel σ -field Σ_d . The points of \mathbb{Z}^d are called sites and given a configuration $\eta \in \Omega_d$, $\eta(i) \in \{0, 1\}$ is called the spin at site i .

Let \mathcal{C} be the set of continuous functions from Ω_d to \mathbb{R} and \mathcal{M} be the subset of these functions which are coordinatewise non-decreasing. A measure μ on Σ_d is said to be FKG or to have positive correlations if for any pair of functions $f, g \in \mathcal{M}$,

$$\int fg d\mu \geq \int f d\mu \int g d\mu.$$

A measure μ is said to be stochastically greater than another one ν if for any $f \in \mathcal{M}$,

$$\int f d\mu \geq \int f d\nu.$$

In this case we write $\mu \geq \nu$ or $\nu \leq \mu$. A measure μ is translation invariant if for any $f \in \mathcal{C}$ and any $i \in \mathbb{Z}^d$

$$\int f(\eta) d\mu(\eta) = \int f(\eta - i) d\mu(\eta),$$

where $(\eta - i)(j) = \eta(j + i)$. For $\eta = 0, 1$, δ_c will represent the measure concentrated on the configuration which is identically η .

We will consider large deviations for the spin per site. For this purpose, for any $\Gamma \subset \mathbb{Z}^d$ and $\eta \in \Omega_d$ write $|\Gamma|$ for the cardinality for Γ and set

$$X_\Gamma(\eta) = |\Gamma|^{-1} \sum_{i \in \Gamma} \eta(i).$$

Set also

$$A_n^{(d)} = \{i \in \mathbb{Z}^d : 1 \leq i_r \leq n, r = 1, \dots, d\},$$

and

$$X_n^{(d)} = X_{A_n}^{(d)}.$$

We will use the abbreviations $X_n = X_n^{(1)}$, $Y_n = X_n^{(2)}$.

Theorem 1 in [LS] states that if μ is a translation invariant and FKG measure on Σ_d , which is not δ_0 or δ_1 then for any $x \in [0, 1]$,

$$(2.1a) \quad \lim_n n^{-d} \log \mu \{X_n^{(d)} \geq x\} = -\varphi_+(x),$$

$$(2.1b) \quad \lim_n n^{-d} \log \mu \{X_n^{(d)} \leq x\} = -\varphi_-(x)$$

where $\varphi_+(\cdot)$ and $\varphi_-(\cdot)$ are convex and bounded functions. Moreover, defining

$$(2.1c) \quad \varphi(x) = \max(\varphi_-(x), \varphi_+(x))$$

it is clear that $\varphi: [0, 1] \rightarrow [0, \infty]$ is a convex function and it follows that for any $0 \leq a < b \leq 1$ such that

$$(2.2) \quad \max(\varphi(a), \varphi(b)) > 0,$$

the following holds:

$$(2.3) \quad \lim_n n^{-1} \log \mu \{X_n^{(d)} \in [a, b]\} = - \inf_{a \leq x \leq b} \varphi(x)$$

Except for the annoying restriction (2.2) on a and b this statement is a large deviation property for $X_n^{(d)}$. Simple examples, such as mixtures of product measures with different densities show that this restriction cannot be eliminated only with the hypothesis above and more information about μ is needed to improve this result.

Further information about φ was obtained in [LS] when μ is an invariant measure of a translation invariant attractive spin system (TIASS). Our next task is to explain the last five words.

A spin system is a particular type of continuous time Markov process with states on Ω_d . Each spin $\eta(i)$ flips to the value $1 - \eta(i)$ according to a rate $c(i, \eta)$. Each spin does this independently of the others, but since the rates depend on the configuration η , the spins do interact in general. In order for the rates $c(i, \eta)$ to define a unique process there are some restrictions that they must

satisfy. For our purposes in this paper it is enough to know that these conditions are satisfied when the rates have finite range.

In [LS] a subclass of these systems was considered which satisfies:

- (a) translation invariance: $c(i, \eta) = c(i + j, \eta + j)$;
- b) attractiveness: $c(i, \cdot)$ is non-decreasing on $\{\eta: \eta(i) = 0\}$ and non-increasing on $\{\eta: \eta(i) = 1\}$.

The TFIASS have some nice properties (for proofs see [Lig]). In particular if $S(t)$ is the corresponding semigroup, then $\delta_1 S(t)$ (resp. $\delta_0 S(t)$) converges weakly as $t \rightarrow \infty$ to an invariant measure ν_+ (resp. ν_-). These measures, ν_+ and ν_- , are translation invariant FKG and ergodic with respect to translations. The set of invariant measures for the system is unique if and only if $\nu_- = \nu_+$, which is also equivalent to $\rho_- = \rho_+$, where

$$\rho_{\pm} = \nu_{\pm} \{ \eta(0) = 1 \}.$$

Theorem 2 in [LS] implies that if ν is an invariant measure for a TFIASS which is also translation invariant and FKG (for instance ν_- or ν_+), then the corresponding function ϕ defined by (2.1) satisfies

$$\phi(x) > 0 \quad \text{if } x < \rho_- \quad \text{or } x > \rho_+.$$

This result is most useful when the system has a unique invariant measure. Then $\phi(x) = 0$ if and only if $x = \rho_- = \rho_+$; (2.2) is empty and therefore (2.3) holds for any $0 \leq a < b \leq 1$. On the other hand the previous result does not tell us anything about $\phi(x)$ for $\rho_- < x < \rho_+$, when $\rho_- < \rho_+$, as in the case of the supercritical contact process.

3. The Contact Process

Contact processes, first studied by Harris [Har2], are TFIASS which can be thought of as models for the spread of an infection. Here we will restrict ourselves to the one dimensional basic contact process and refer to it simply as the contact process. This model can be described as follows. One individual is located at each site of the lattice Z . The individual at site i may be healthy ($\eta(i) = 0$) or infected ($\eta(i) = 1$). Sick individuals recover at a constant rate which will be chosen to be 1. A healthy individual is contaminated by its infected neighbors at a rate which is proportional to the number of neighbors who are infected. So the rates are

$$c(i, \eta) = \begin{cases} 1 & \text{if } \eta(i) = 1 \\ \lambda(\eta(i-1) + \eta(i+1)) & \text{if } \eta(i) = 0, \end{cases}$$

where $\lambda \in [0, \infty]$ is called the infection parameter. We refer the reader to Chap. 6 of [Lig] for a good review of the state of knowledge for this model by the end of 1984. For earlier reviews see [Gri1] and [Gri2]. The contact process is also closely related to oriented percolation in two dimensions; see [Dur] for a review of this latter model and its relation with the contact process. The results below have analogues for oriented percolation and, in particular, Theo-

rem 1 can be proven for this model with the same arguments used for the contact process.

The basic result for the contact process is the existence of two different behaviors for λ small and large: there is a $\lambda_c \in (0, \infty)$ such that

- a) if $\lambda < \lambda_c$, then $v_- = v_+ = \delta_0$,
- b) if $\lambda > \lambda_c$, then there are exactly two extremal invariant measures: $v_- = \delta_0$ and $v_+ = v$. (Of course, v depends on λ).

Moreover $\rho_+ = \rho = \rho(\lambda)$ is a continuous function on $[\lambda_c, \infty)$. It is still an open question whether $\rho(\lambda)$ is continuous on $[0, \infty)$, i.e., whether $\rho(\lambda_c) = 0$, or, in other words, whether $v_+ = \delta_0$ at λ_c or not.

If $\lambda > \lambda_c$, it follows from the results in [LS] quoted in Sect. 2 that for v , $\varphi(x) > 0$ if $x > \rho(\lambda)$. It was also proved in [LS] that for x small enough $\varphi(x) > 0$. In this paper we will prove that, in fact, $\varphi(x)$ is strictly positive except at $x = \rho(\lambda)$. It follows then that (2.2) is always satisfied and therefore,

Theorem 1. *If v is the upper invariant measure of the contact process for some $\lambda > \lambda_c$, then there exists a convex function $\varphi: [0, 1] \rightarrow [0, \infty)$ such that $\varphi(x) = 0$ if and only if $x = \rho(\lambda)$ and*

$$\lim_n n^{-1} \log v\{X_n \in [a, b]\} \rightarrow - \inf_{a \leq x \leq b} \varphi(x),$$

for any $0 \leq a < b \leq 1$.

It is interesting to contrast this result with the large deviation properties of the invariant measures of stochastic Ising models (see Chap. 4 of [Lig] for a description of these models). For the subclass of these models which are also TIASS, v_- and v_+ are both Gibbs measures which correspond to a same potential. Therefore the corresponding functions φ are the same and if $\rho_- < \rho_+$ (which is well known to occur for some potentials), then this function is identically zero on $[\rho_-, \rho_+]$ and strictly positive outside this interval. Large deviation properties for a larger class of Gibbs measures were first derived in [Lan] and more recently in a stronger sense (the Donsker-Varadhan formulation) in [Com], [Ell], [FO] and [Oll]. Their results show that the pattern above is universal for Gibbs measures: the function φ is the same for all the Gibbs measures corresponding to a given potential and it has a horizontal flat part if and only if there is more than one Gibbs measure for this potential (phase transition).

4. Percolation

Percolation models have been extensively studied by physicists and mathematicians (see [Kes] or [CC] for reviews of rigorous results). Here we consider site percolation on Z^2 , but it will be clear that everything will apply as well to bond percolation.

First we need a little more terminology. Say that $i, j \in Z^2$ are neighbors if they have one coordinate in common and the other differs by one unit. They will be said to be (*) neighbors if either they are neighbors or both their coordinates differ by one unit. A chain (resp. (*) chain) is a set of sites $\{i_1, i_2, \dots, i_n\}$

such that i_r and i_{r+1} are neighbors (resp. (*) neighbors) for $r=1, \dots, n-1$. In this case, i_1 and i_n are called the terminal points of the chain or (*) chain.

Site percolation is defined by associating to each site $i \in \mathbb{Z}^2$ an independent random variable $\alpha(i)$ which assumes the values $+1$ with probability p and -1 with probability $1-p$. If $\alpha(i)=1$ we say that i is occupied and if $\alpha(i)=-1$ that i is vacant. Each realization of the random field α induces a partition of \mathbb{Z}^2 defined by saying that i and j are in the same cluster if and only if there exists a chain of occupied sites with i and j as terminal points.

The basic result about this model is the existence of $p_c \in (0, 1)$ such that:

- a) if $p \leq p_c$ then all clusters are finite with probability 1;
- b) if $p > p_c$ there exists exactly one infinite cluster with probability 1.

Define now

$$\sigma(i) = \begin{cases} 1 & \text{if } i \text{ belongs to the infinite cluster} \\ 0 & \text{otherwise.} \end{cases}$$

The random field σ is ergodic with respect to translations. This fact can be proved using a multidimensional analogue of Proposition 6.31 of [Bre]; for this purpose observe that σ can be expressed by $\sigma(i) = \Phi(\alpha - i)$, where $\Phi(\alpha) = 1$ (resp. 0) if the origin belongs (resp. does not belong) to the infinite cluster, when the sites are occupied or vacant according to the field α . It follows that in particular $Y_n(\sigma) = |A_n^{(2)}|^{-1} \sum_{i \in A_n^{(2)}} \sigma(i) \rightarrow \theta(p)$ a.s. as $n \rightarrow \infty$, where $\theta(p)$ is the prob-

ability that a given site, say the origin, belongs to an infinite cluster. $\theta(p)$ is known to be a continuous function of p and $\theta(p) > 0$ if and only if $p > p_c$.

The random field σ is FKG. This follows from the fact that for each $i \in \mathbb{Z}^2$ $\sigma(i)$ is a coordinatewise increasing function of the product random field α and this latter field is FKG, as was first proven in [Har 1]. Therefore Theorem 1 in [LS] applies to σ . Nevertheless we will prove that the two tails of the distribution of $Y_n(\sigma)$ have different large deviation properties. We will prove that if μ is the law of σ ,

Theorem 2. For $p > p_c$ there exists a convex function $\varphi: [\theta(p), 1] \rightarrow [0, \infty)$ s.t. $\varphi(x) = 0$ if and only if $x = \theta(p)$ and

$$(4.1) \quad \lim_n n^{-2} \log \mu \{ Y_n \in [a, b] \} \rightarrow - \inf_{a \leq x \leq b} \varphi(x),$$

for any $\theta(p) \leq a < b \leq 1$.

Theorem 3. For $p > p_c$ and any $0 \leq a < b \leq \theta(p)$ there exist $C_1, C_2, \gamma_1, \gamma_2 > 0$ (which depend on a and b) such that

$$(4.2) \quad C_1 e^{-\gamma_1 n} \leq \mu \{ Y_n \in [a, b] \} \leq C_2 e^{-\gamma_2 n}$$

Remark. From Theorem 3 it is clear that if we extend the function φ in Theorem 2 by defining $\varphi(x) = 0$ for $x \in [0, \theta(p))$, then (4.1) is satisfied for any $0 \leq a < b \leq 1$.

These results tell us that while the probability of deviations above $\theta(p)$ decay exponentially with the area of $A_n^{(2)}$ the probability of deviations below $\theta(p)$ decay only as an exponential of the perimeter of $A_n^{(2)}$. It is trivial to prove

the lower bound in (4.2) when $a=0$. Indeed, if all the sites on the exterior boundary of $A_n^{(2)}$ are vacant, then no site in $A_n^{(2)}$ can belong to the infinite cluster. It follows that

$$\mu\{Y_n \leq b\} \geq \mu\{Y_n = 0\} \geq (1-p)^{4(n+1)}.$$

We believe, but were not able to prove that the following result, sharper than Theorem 3, holds:

Conjecture 1. *There exists a function $\gamma: [0, \theta(p)] \rightarrow [0, \infty)$ such that $\gamma(x)=0$ if and only if $x=\theta(p)$ and*

$$\lim_n n^{-1} \log \mu\{Y_n \in [a, b]\} = - \inf_{a \leq x \leq b} \gamma(x),$$

for any $0 \leq a < b \leq \theta(p)$.

After thinking about this problem, we believe that when $Y_n(\eta) \leq b$ it does so because of the appearance of one bubble inside of which the sites do not percolate. For b close to $\theta(p)$ this bubble must be small and therefore completely contained in $A_n^{(2)}$. In this case the volume V of such a bubble must satisfy $\theta(n^2 - V) = bn^2$, from which $V = (1 - b/\theta)n^2$. Since the price to form such a bubble is proportional to its perimeter which grows like $CV^{1/2}$, we expect

Conjecture 2. *For x close to $\theta(p)$, $\gamma(x) = C(1 - x/\theta)^{1/2}$ for some constant C which may depend on p .*

The remarks above will clearly be difficult to make rigorous (see [MS] for related results about the Ising model). We have however been able to verify Conjecture 1 in the extreme case $Y_n=0$. To state this result we have first to introduce the correlation length $\xi(p)$ defined by

$$-1/\xi(p) = \lim_n n^{-1} \log P(C_{(0,0),(n,0)})$$

where $C_{i,j}$ is the event that there is a (*) chain of vacant sites from i to j . The limit exists by superadditivity (see Lemma 1 in Sect. 6) and is strictly negative if $p > p_c$.

Theorem 4. *For $p > p_c$,*

$$(4.3) \quad \lim_n n^{-1} \log \mu\{Y_n = 0\} = -4/\xi(p),$$

In order to prove the upper bound in (4.2) we will first prove a large deviation property for the random field σ restricted to one dimensional subsets of Z^2 . Set

$$Z_n = n^{-1} \sum_{i_1=1}^n \sigma(i_1, 0).$$

For these random variables we will prove

Theorem 5. For $p > p_c$ there exists a convex function $\kappa: [0, 1] \rightarrow [0, \infty)$ such that $\kappa(x) = 0$ if and only if $x = \theta(p)$ and

$$(4.4) \quad \lim_n n^{-1} \log \mu \{Z_n \in [a, b]\} \rightarrow - \inf_{a \leq x \leq b} \kappa(x)$$

for any $0 \leq a < b \leq 1$.

One of our motivations for studying the random field σ is the fact that it has the unusual large deviation behavior stated in Theorems 2 and 3, which contrast with the usual behavior for the upper invariant measure of the contact process or the one-dimensional restriction of σ considered in Theorem 5. Similar results were obtained for the Gibbs measures corresponding to nearest neighbor ferromagnetic interactions in [Sch], using some of the techniques in the present paper.

5. Proof of Theorem 1

Warning. In this and in the next section C and γ will represent positive finite constants, but their values may change from expression to expression.

From the results reviewed in Sect. 2, all we have to prove is that $\varphi(x) > 0$ if $x < \rho$. In other words, given $x < \rho$ we have to prove that there are $C, \gamma > 0$ such that

$$(5.1) \quad \nu \{X_n \leq X\} \leq C e^{-\gamma n}.$$

In order to prove (5.1) we construct ν using a percolation substructure on $Z \times \mathbb{R}_+$ as in [Gri1], [Gri2] or Sect. III.6 of [Lig]. To each $i \in Z$ associate three independent Poisson process with rates $1, \lambda$ and λ respectively. Let $\{T_i^{(k)}(n): n = 1, 2, \dots\}, k = 1, 2, 3$, be the arrival times for each one of these Poisson processes. Write a δ at each point $(i, T_i^{(1)}(n)), i \in Z, n = 1, 2, \dots$. Draw an arrow from $(i, T_i^{(2)}(n))$ to $(i + 1, T_i^{(2)}(n))$, for each $i \in Z, n = 1, 2, \dots$. Draw an arrow from $(i, T_i^{(3)}(n))$ to $(i - 1, T_i^{(3)}(n))$ for each $i \in Z, n = 1, 2, \dots$. The effect of the δ 's is to heal the infected individuals and the effect of each arrow is to contaminate a healthy individual which is at its tip by an infected one which is at its origin. Say that there is a path from (i, s) to (j, t) if it is possible to go from the former to the latter following straight lines up (with the first coordinate kept fixed and the second increasing) and across arrows in the direction of their orientation, without crossing any δ 's.

Define now the random field

$$\tau(i) = \begin{cases} 1 & \text{if for any } t \geq 0 \text{ there is a path from } (i, 0) \text{ to some } (j, t), j \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

From the basic properties of the contact process (see the reviews) it follows that τ is distributed according to the law of ν .

Let $\alpha = \alpha(\lambda)$ be the edge speed for the contact process defined by

$$\alpha = \lim_{t \rightarrow \infty} t^{-1} E r_t,$$

where

$$r_t = \max \{i \in Z : \text{there is a path from some } (j, 0), j \leq 0 \text{ to } (i, t)\}.$$

We will now introduce the renormalized bond construction; for details see Sect. VI.3 in [Lig] or Sects. 9 and 10 in [Dur]. We adopt the notation of the second paper and choose the quantity called δ there to be .1. Set

$$\mathcal{L} = \{(m, n) : m + n \text{ is even, } n \geq 0\},$$

with the norm $\|(m, n)\| = (|m| + |n|)/2$. For each $(m, n) \in \mathcal{L}$ define

$$C_{m,n} = (0.9\alpha Lm, Ln)$$

where L will be chosen later. The $C_{m,n}$ are the sites of the renormalized lattice. Let $A_{0,0}$ be the parallelogram with vertices $u_0 = (-0.15\alpha L, 0)$, $v_0 = (-0.05\alpha L, 0)$, $u_1 = u_0 + 1.1L(\alpha, 1)$, $v_1 = v_0 + 1.1L(\alpha, 1)$. Let $B_{0,0}$ be the parallelogram whose vertices are obtained from those of $A_{0,0}$ by inverting the sign of the first coordinate. Set $A_{m,n} = A_{0,0} + C_{m,n}$, $B_{m,n} = B_{0,0} + C_{m,n}$.

Using the construction above we define a random field $\eta(z)$, $z \in \mathcal{L}$ by setting $\eta(z) = 1$ if the following good event $G_{m,n}$ happens and $\eta(z) = 0$ otherwise.

$$G_{m,n} = \{\text{there is a path inside } A_{m,n}, \text{ which joins its two horizontal sides and there is a path inside } B_{m,n} \text{ which joins its two horizontal sides}\}.$$

The random field η has some nice properties. The first two are:

- (5.2) the random variables $\eta(z)$ are 1-dependent, i.e., if z_1, \dots, z_n are points with $\|z_i - z_j\| > 1$ for $i \neq j$ then $\eta(z_1), \dots, \eta(z_n)$ are independent.
- (5.3) If $\varepsilon > 0$ and $\lambda > \lambda_c$ then we can pick L large enough so that $p := P(\eta(z) = 1) > 1 - \varepsilon$.

The next property relates oriented percolation in the original substructure to oriented percolation for the η field. For this purpose write $z \xrightarrow{R} \infty$ (R stands for renormalization) if there is an infinite sequence of sites in \mathcal{L} , $z = (m_1, n_1), (m_2, n_2), \dots$, such that for $i = 1, 2, \dots, (m_{i+1}, n_{i+1}) \in \{(m_i - 1, n_i + 1), (m_i + 1, n_i + 1)\}$ and $\eta((m_i, n_i)) = 1$. Define now the following subsets of $Z \times \mathbb{R}_+$

$$\begin{aligned} R_0 &= \{i \in Z; 0 \leq i < 0.05\alpha L\} \times \{0\}, \\ R_m &= R_0 + C_{m,0} \quad \text{for } m \in 2Z, \\ T_0 &= \text{triangle with vertices } (-0.15\alpha L, 0), (0.15\alpha L, 0) \text{ and } (0, 0.15L), \\ T_m &= T_0 + C_{m,0} \quad \text{for } m \in 2Z. \end{aligned}$$

The reason for introducing these definitions is that (see Fig. 1)

- (5.4) for $(i, 0) \in R_m$, if $(m, 0) \xrightarrow{R} \infty$ and there is a path from $(i, 0)$ to $(T_m)^c$ in the original percolation substructure, then $\tau(i) = 1$.

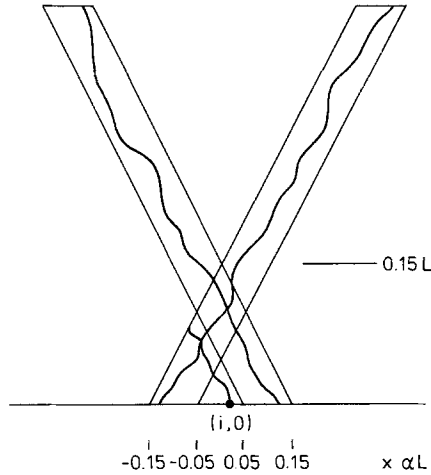


Fig. 1. Occurrence of a path from $(i, 0)$ to $(T_0)^r$ and of the event $G_{(0,0)}$

We will use the construction above to show that given $x < \rho$ we can choose L such that there exist $C, \gamma > 0$ which satisfy

$$(5.5) \quad P(|S_r|^{-1} \sum_{i \in S_r} \tau(i) \leq x) \leq C e^{-\gamma r},$$

where

$$S_r = \bigcup_{k=0}^{r-1} R_{2k}.$$

It is clear that (5.5) implies (5.1) since 36 translations of S_r cover the set $\{i \in \mathbb{Z} : 0 \leq i \leq 1.8\alpha Lr\}$ and therefore by translation invariance, if $n = \lceil 1.8\alpha Lr \rceil$

$$P\left(n^{-1} \sum_{i=1}^n \tau(i) \leq x\right) \leq 36 P(|S_r|^{-1} \sum_{i \in S_r} \tau(i) \leq x) \leq C e^{-\gamma n}.$$

So (5.1) holds when $n = \lceil 1.8\alpha Lr \rceil$ for some integer r . This result is enough for our purposes since we know already that the limit (2.1.b) exists. (Using this argument it follows also that (5.1) holds for any n .)

Our strategy to prove (5.5) can be divided into two steps. First let

$$U_r = r^{-1} \sum_{k=0}^{r-1} (1 - I_{\{(2k,0) \xrightarrow{R} \infty\}}),$$

where $I_{\{\cdot\}}$ is the indicator function of the event $\{\cdot\}$. U_r is the fraction of the sites $\{(0, 0), \dots, (2(r-1), 0)\}$ of \mathcal{L} which do not percolate. We will use (5.2) and (5.3) to verify that choosing L large enough there are $C, \gamma > 0$ such that

$$(5.6) \quad P(U_r \geq \rho - x) \leq C e^{-\gamma r}.$$

The second step is to use (5.4). For $(i, 0) \in R_m$ set

$$\tilde{\tau}(i) = \begin{cases} 1 & \text{if there is a path from } (i, 0) \text{ to } (T_m)^c, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$V_m = |R_m|^{-1} \sum_{i \in R_m} \tilde{\tau}(i),$$

$$W_r = r^{-1} \sum_{k=0}^{r-1} V_{2k} = |S_r|^{-1} \sum_{i \in S_r} \tilde{\tau}(i).$$

By (5.4) and the fact that $V_m \leq 1$ it follows that if $W_r > \rho$ and $U_r < \rho - x$, then

$$\sum_{i \in S_r} \tau(i) \geq |S_r| W_r - |S_r| U_r > |S_r| x.$$

Hence

$$(5.7) \quad P(|S_r|^{-1} \sum_{i \in S_r} \tau(i) \leq x) \leq P(W_r \leq \rho) + P(U_r \geq \rho - x).$$

But W_r is the average of the i.i.d. random variables V_m . Since $EV_m > \rho$ it follows from the large deviation theorem for bounded i.i.d. random variables that there exist $C, \gamma > 0$ (which depend on L) such that

$$(5.8) \quad P(W_r \leq \rho) \leq C e^{-\gamma r}.$$

From (5.7) and (5.8) the proof will be complete once we verify (5.6). This will be done using contour methods as in Sect. 10 of [Dur].

Given a finite $T \subset 2Z$, define the event

$$E_T = \{ \text{there is no } z \in T \times \{0\} \text{ such that } z \xrightarrow{R} \infty \}.$$

In order for E_T to happen there must be a contour Γ which “isolates T from infinity” and which may be constructed as follows. Let

$$C = \{ z \in \mathcal{L} : \text{there is } m \in T \text{ with } (m, 0) \xrightarrow{R} z \},$$

$$D = \{ (a, b) \in \mathbb{R}^2 : |a| + |b| \leq 1 \},$$

$$H = \bigcup_{z \in C} (z + D).$$

If $|C| < \infty$ let Γ be the boundary of the unbounded component of $(\mathbb{R} \times (-1, \infty)) \setminus H$. Set $a_1 = \max T$. The segment from $(a_1, -1)$ to $(a_1 + 1, 0)$ is always present in Γ and can be considered as its first segment. Give to this segment the orientation indicated and continue to follow Γ in this direction until it hits the line $\mathbb{R} \times \{-1\}$ again. At that moment there are two possibilities: either we walked along the whole contour Γ , and in this case it is connected or it has more than one connected component and we just walked along one of them – let

us call it Γ_1 . In the second case let T_1 be the subset of T which Γ_1 isolates from infinity and set $a_2 = \max(T \setminus T_1)$. Then the segment from $(a_2, -1)$ to $(a_2 + 1, 0)$ is present in Γ and we can proceed as above and define recursively Γ_i and $a_i, i = 1, \dots, k$, where k is the number of connected components of Γ .

It follows from the discussion above that there are at most 3^{l-1} contours of length l . If Γ is connected this is the usual estimate, which follows from the fact that a contour never passes through an arc twice. And if Γ is not connected then each time it reaches the line $\mathbb{R} \times \{-1\}$ it must restart at a precise arc.

An argument in Sect. 10 of [Dur] shows that the probability that Γ above is a given contour with length l is small then $(1-p)^{l/36}$, where p was defined in (5.3). A contour which isolates T from infinity must have at least length $2|T|$. Therefore

$$P(E_T) \leq \sum_{l=2|T|}^{\infty} 3^l (1-p)^{l/36}.$$

By (5.3) we can choose L large enough so that $p > 1 - 3^{-36}$. Then

$$P(E_T) \leq C e^{-\gamma_p |T|},$$

where C is a positive constant and

$$\gamma_p = -\log(9(1-p)^{1/18}).$$

Note that

$$(5.9) \quad \lim_{p \rightarrow 1} \gamma_p = \infty.$$

Now,

$$\begin{aligned} P(U_r \geq \rho - x) &\leq \sum_{\substack{T = \{0, 2, \dots, 2r-2\} \\ |T| \geq (\rho-x)r}} P(E_T) \leq 2^r C e^{-\gamma_p(\rho-x)r} \\ &= C e^{(\log 2 - (\rho-x)\gamma_p)r}. \end{aligned}$$

From (5.3) and (5.9) it is possible to choose L large enough so that $\log 2 - (\rho - x)\gamma_p < 0$. This finishes the proof. \square

6. Proof of the Results About Percolation

We will need some more notation. A circuit ($(*)$ circuit) is a chain ($(*)$ chain) whose terminal points are neighbors ($(*)$ neighbors). A set $S \subset Z^2$ is said to be connected ($(*)$ connected) if every pair of points in S are terminal points of a chain ($(*)$ chain) contained in S . A set S surrounds a set R if any infinite connected set which intersects R intersects also S . Given $S \subset Z^2$, ∂S will denote its (interior) boundary, i.e., the set of sites in S which are neighbors to sites in S^c .

Connectivity and $(*)$ connectivity are related by duality relations (see [Kes]). For instance a point belongs to an infinite cluster of occupied sites if and only if it is not surrounded by a $(*)$ circuit of vacant sites.

Proof of Theorem 2. We will prove that given $x > \theta(p)$ there exists $C, \gamma > 0$ such that

$$(6.1) \quad \mu(Y_n \geq x) \leq C e^{-\gamma n}.$$

This implies that $\varphi_+(x)$ defined by (2.1.a) is strictly positive in this interval and therefore so is $\varphi(x)$. If $\theta(p) < b \leq 1$, then (2.2) is satisfied and (4.1) holds in this case. Observe that since the limit (2.1.a) exists it is enough to prove that (6.1) holds for a subsequence of (Y_n) .

The proof of (6.1) will be very similar to the proof of Theorem 2 in [LS] (for the invariant measures of TIASS). Given a positive integer N , to be chosen later, and $k \in Z^2$ let

$$\Gamma(k) = \{i \in Z^2 : i - kN \in \Gamma\},$$

where

$$\Gamma = \{i, \dots, N\}^2.$$

Given $i \in Z^2$ let k_i be defined by $i \in \Gamma(k_i)$. Define now

$$\zeta(i) = \begin{cases} 1 & \text{if the cluster of } i \text{ reaches } \partial\Gamma(k_i), \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that

$$\mu(Y_n \geq x) \leq P(n^{-2} \sum_{i \in A_n^{(2)}} \zeta(i) \geq x).$$

Furthermore, the restrictions of ζ to different boxes $\Gamma(k)$ are i.i.d. If

$$(6.2) \quad x > \theta_N(p) := E(|\Gamma|^{-1} \sum_{i \in \Gamma} \zeta(i))$$

and if $n = mN$ for some integer m , then, from the large deviation theorem for bounded i.i.d. random variables

$$P(n^{-2} \sum_{i \in A_n^{(2)}} \zeta(i) \geq x) \leq C e^{-\gamma m} = C e^{-(\gamma/N)m},$$

with $C, \gamma > 0$. Therefore (6.1) will be proven once we show that given $x > \theta(p)$ there exist N such that (6.2) is satisfied, i.e.,

$$(6.3) \quad \lim_{N \rightarrow \infty} \theta_N = \theta(p)$$

For each N let $\tilde{\Gamma} = \Gamma \setminus [N^{1/2}, N - N^{1/2}]^2$. Now, for any $i \in \tilde{\Gamma}$, by translation invariance $P(\zeta(i) = 1) \leq P(\text{the cluster of the origin contains at least } N^{1/2} \text{ sites}) := a_N$.

Hence

$$\theta(p) \leq \theta_N \leq a_N + 4N^{-1/2},$$

from which (6.3) follows since clearly $\lim_{N \rightarrow \infty} a_N = \theta(p)$. \square

The rest of this section is organized as follows: first we prove Theorem 4 since it is simpler than the other theorems and the arguments used to prove it will illustrate some techniques that we use to prove the lower bound in Theorem 3. Then we prove Theorem 5 which is the key step for our proof of the upper bound in Theorem 3. Finally we prove Theorem 3. We start with some lemmas.

Write $C_{i,j}^a$ for the event that there is a (*) chain of vacant sites connecting i and j and contained in the strip $Z \times \{-a, \dots, a\}$. As in the introduction, omit a in the notation above if the strip is replaced by the whole space Z^2 , or use the notation C_{ij}^∞ .

Lemma 1. *For any $0 \leq p \leq 1$ and $0 < a \leq \infty$ the following limit exists*

$$\lim_{n \rightarrow \infty} n^{-1} \log P(C_{(0,0),(n,0)}^a) := -\Delta_a(p).$$

Furthermore

$$P(C_{(0,0),(n,0)}^a) \leq e^{-\Delta_a(p) \cdot n}.$$

Lemma 2. *For any $0 \leq p \leq 1$*

$$\lim_{a \rightarrow \infty} \Delta_a = \Delta_\infty := 1/\xi(p).$$

For $i, j \in Z^2$, let $d(i, j) = \max(|i_1 - j_1|, |i_2 - j_2|)$.

Lemma 3. *For any $0 \leq p \leq 1$*

$$P(C_{i,j}) \leq e^{-d(i,j)/\xi(p)}$$

The proofs of these lemmas appeared in several places (for instance, they appeared partly in [CC] and [Ng]), but they are short, so for the reader's convenience we repeat them here.

Proof of Lemma 1. By the FKG property and translation invariance

$$\begin{aligned} P(C_{(0,0),(n+m,0)}^a) &\geq P(C_{(0,0),(n,0)}^a \cap C_{(n,0),(n+m,0)}^a) \\ &\geq P(C_{(0,0),(n,0)}^a) P(C_{(n,0),(n+m,0)}^a) \\ &= P(C_{(0,0),(n,0)}^a) P(C_{(0,0),(m,0)}^a). \end{aligned}$$

From this the result follows by standard superadditivity arguments (see the proof of (1) in [Dur] or of Theorem VI.2.6 in [Lig]). \square

Proof of Lemma 2. It is clear that $\Delta_a(p) \geq \Delta_\infty(p)$. On the other hand, by Lemma 1, for any $\varepsilon > 0$ there is an n such that

$$\begin{aligned} \exp(-n(\Delta_\infty(p) + \varepsilon)) &\leq P(C_{(0,0),(n,0)}^\infty) \\ &= \lim_{a \rightarrow \infty} P(C_{(0,0),(n,0)}^a) \leq \lim_{a \rightarrow \infty} \exp(-n\Delta_a(p)). \end{aligned}$$

Hence

$$\Delta_\infty(p) + \varepsilon \geq \Delta_a(p)$$

and since ε is arbitrary, the result follows. \square

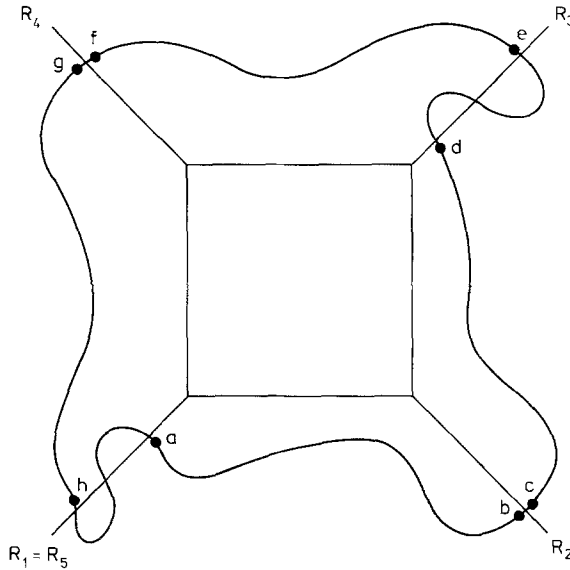


Fig. 2. The four * chains ab, cd, ef and gh are those mentioned in the proof of the upper bound in Theorem 4

Proof of Lemma 3. Without loss of generality assume that $i = (0, 0)$, $d(i, j) = |i_1 - j_1| = |j_1|$. Define $k = (2j_1, 0)$. Then by FKG

$$P(C_{i,k}) \geq P(C_{i,j}) P(C_{j,k}).$$

But by symmetry

$$P(C_{i,j}) = P(C_{j,k}).$$

Hence by Lemma 1

$$\begin{aligned} P(C_{ij}) &\leq (P(C_{i,k}))^{1/2} \\ &\leq (e^{-2|j_1|/\xi(p)})^{1/2} = e^{-d(i,j)/\xi(p)} \quad \square \end{aligned}$$

What does not follow from the arguments above is the finiteness of $\xi(p)$ for $p > p_c$. This fact, which will be fundamental for us, follows from the results in [Rus 1] (see [CC] or [Kes]).

Proof of Theorem 4. Consider the following four straight half lines (see Fig. 2):

$$\begin{aligned} R_1 = R_5 &= \{i \in \mathbb{Z}^2 : i_1 = i_2 \leq 0\}, \\ R_2 &= \{i \in \mathbb{Z}^2 : i_1 = -i_2 + n \geq n\}, \\ R_3 &= \{i \in \mathbb{Z}^2 : i_1 = i_2 \geq n\}, \\ R_4 &= \{i \in \mathbb{Z}^2 : i_1 = -i_2 + n \leq 0\}. \end{aligned}$$

If no site in $A_n^{(2)}$ belongs to the infinite cluster, then $A_n^{(2)}$ must be surrounded by a (*) circuit of vacant sites. Therefore there must be four (*) chains of vacant

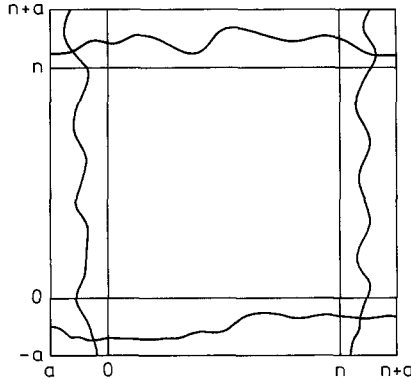


Fig. 3. Occurrence of the events $T_i, i = 1, \dots, 4$

sites connecting \bar{R}_i to $\bar{R}_{i+1}, i = 1, \dots, 4$, where for $S \subset Z^2$ we define $\bar{S} = \{i \in Z^2 : i \in S \text{ or } i \text{ is } * \text{ neighbor to a point in } S\}$. Furthermore each one of these four paths must be in a different region. Therefore

$$\begin{aligned} \mu(Y_n = 0) &\leq [P(\bar{R}_1 \text{ is connected to } \bar{R}_2 \text{ by a vacant } * \text{ chain})]^4. \end{aligned}$$

But using Lemma 3 the r.h.s. above is bounded above by $(f(n) e^{-n/\zeta(p)})^4$, where $f(n)$ grows only polynomially with n . Hence

$$\limsup_{n \rightarrow \infty} n^{-1} \log \mu(Y_n = 0) \leq -4/\zeta(p).$$

To prove a bound in the other direction consider for a fixed a the following rectangles (see Fig. 3)

$$\begin{aligned} S_1 &= \{-a, \dots, n+a\} \times \{-a, \dots, 0\}, \\ S_2 &= \{n, \dots, n+a\} \times \{-a, \dots, n+a\}, \\ S_3 &= \{-a, \dots, n+a\} \times \{n, \dots, n+a\}, \\ S_4 &= \{-a, \dots, 0\} \times \{-a, \dots, n+a\}. \end{aligned}$$

Let $T_i, i = 1, \dots, 4$, be the event that there is a (*) chain of vacant sites in S_i , joining its parallel sides of length $a + 1$. The occurrence of $\bigcap_{i=1}^4 T_i$ implies that no site in $A_n^{(2)}$ belongs to the infinite cluster.

By FKG

$$\mu(Y_n = 0) \geq \prod_{i=1}^4 P(T_i).$$

Using Lemma 1

$$\liminf_{n \rightarrow \infty} n^{-1} \log \mu(Y_n = 0) \geq -4 \Delta_a(p).$$

Since a is arbitrary, Lemma 2 implies the desired lower bound. \square

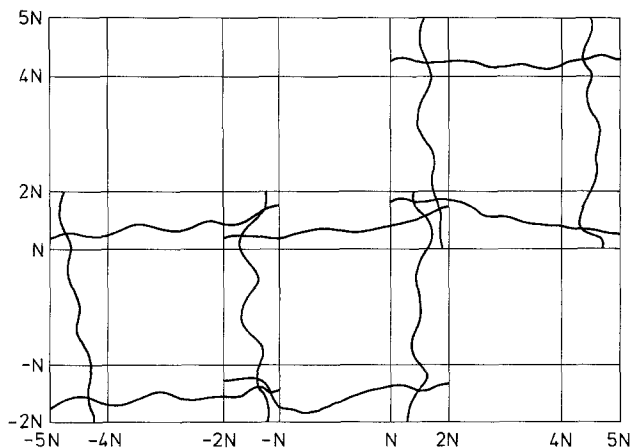


Fig. 4. Occurrence of events $H_{(-1,0)}$, $H_{(0,0)}$ and $H_{(1,1)}$

Before proving Theorem 5 we need one more lemma. Let a and b be two positive real numbers and consider the event E_n that there is a chain of occupied sites connecting the two vertical sides of the rectangle $\{i \in Z^2: 0 < i_1 < an, 0 < i_2 < bn\}$.

Lemma 4. For any $p > p_c$,

$$\lim_{n \rightarrow \infty} P(E_n) = 1.$$

Proof. If E_n fails then there is a (*) chain of vacant sites connecting the two horizontal sides of the rectangle. Using Lemma 3 and the finiteness of $\zeta(p)$ it is easy to see that the probability of such an event vanishes as $n \rightarrow \infty$. \square

We define now a “renormalized site” construction which will play for percolation the same role as the “renormalized bond” construction played for the contact process (see Fig. 4). Consider the following rectangle, where N is an integer

$$\begin{aligned} A &= \{-2N, \dots, 2N\} \times \{-2N, \dots, -N\}, \\ B &= \{N, \dots, 2N\} \times \{-2N, \dots, 2N\}, \\ C &= \{-2N, \dots, 2N\} \times \{N, \dots, 2N\}, \\ D &= \{-2N, \dots, -N\} \times \{-2N, \dots, 2N\}. \end{aligned}$$

For $k \in Z^2$ define $A_k = A + 3Nk$ and analogously for B_k, C_k and D_k . Say that the good event $H_k^{(A)}$ happens if there is a chain of occupied sites inside A_k connecting its two smaller sides. Define $H_k^{(B)}, H_k^{(C)}$ and $H_k^{(D)}$ analogously and set $H_k = H_k^{(A)} \cap \dots \cap H_k^{(D)}$. Define now a random field $\zeta(k), k \in Z^2$ by setting $\zeta(k) = 1$ if H_k happens and $\zeta(k) = 0$ otherwise. For $k \in Z^2$ write $k \xrightarrow{R} \infty$ if k belongs to an infinite (*) cluster of ζ . The field ζ has all the nice properties of the “renormalized bond” field defined for the contact process:

- i) ζ has a finite range of dependence
- ii) $\lim_{N \rightarrow \infty} P(\zeta(k)=1)=1$

iii) If the site i belongs to the square $\{-N, \dots, N\}^2 + 3Nk$ and its cluster (in the original field) reaches the boundary of the larger square $\{-2N, \dots, 2N\}^2 + 3Nk$, then $\sigma(i)=1$, provided that $k \xrightarrow{R} \infty$.

Furthermore percolation for the field ζ on Z^2 obviously dominates oriented percolation for this same field restricted to $\mathcal{L} = \{(m, n) : n \geq 0, m+n \text{ is even}\}$ as defined in Sect. 5. But when restricted to \mathcal{L} , ζ is 1-dependent, therefore the very same contour estimates used in Sect. 5 can be used here.

Proof of Theorem 5. We have only to prove that there are constants C and γ which depend on x such that

$$(6.4) \quad \mu(Z_n \geq x) \leq C e^{-\gamma n} \quad \text{if } x > \theta(p),$$

$$(6.5) \quad \mu(Z_n \leq x) \leq C e^{-\gamma n} \quad \text{if } x < \theta(p).$$

(6.4) can be proven with the same argument used to prove (6.1). The proof of (6.5) using the “renormalized site” construction is analogous to the proof of (5.1). \square

Proof of Theorem 3.

a) *Proof of the upper bound.* By translation invariance and (6.5)

$$\begin{aligned} \mu(Y_n \in [a, b]) &\leq \mu(Y_n \leq b) \\ &\leq P\left(\bigcup_{i_2=1}^n \left\{n^{-1} \sum_{i_1=1}^n \sigma(i) \leq b\right\}\right) \\ &\leq n \cdot \mu\{Z_n \leq b\} \leq C e^{-\gamma n}. \end{aligned}$$

b) *Proof of the lower bound.* The strategy of the proof will be to force the sites inside a box contained in $A_n^{(2)}$ not to percolate by constraining the sites on its boundary to be vacant. This has a cost which grows only with the length of the boundary of the box. Assuming that the field σ outside this box is not affected very much by this conditioning, it follows that the density of percolating sites in $A_n^{(2)}$ can be forced by this procedure to be, with large probability, between a and b for n large. The technical part will be to show that the assumption in the reasoning above is in some sense correct.

If we choose a square box, its side l can be taken such that $\theta(p) \cdot (n^2 - l^2) \approx [(a+b)/2]n^2$. Accordingly, solving for l in the last equation, let $\Omega_n \subset A_n^{(2)}$ be a square whose side l_n is the integer part of $n(1 - (a+b)/2\theta(p))^{1/2}$. Now consider the event A_n that all the sites in $\partial\Omega_n$ are vacant. Then

$$P(A_n) = (1-p)^{4l_n} = C e^{-\gamma n}$$

and

$$\mu\{Y_n \in [a, b]\} \geq P(Y_n(\sigma) \in [a, b] | A_n) \cdot P(A_n).$$

The proof will be finished once we prove that

$$\lim_{n \rightarrow \infty} P(Y_n(\sigma) \in [a, b] | A_n) = 1.$$

Now,

$$(6.6) \quad P(Y_n(\sigma) \in [a, b] | A_n) = P(Y_n(\sigma) \leq b | A_n) - P(Y_n(\sigma) < a | A_n).$$

Clearly, using FKG

$$\begin{aligned} P(Y_n(\sigma) \leq b | A_n) &= P(n^{-2} \sum_{i \in A_n^{(2)} \setminus \Omega_n} \sigma(i) \leq b | A_n) \\ &\geq P(n^{-2} \sum_{i \in A_n^{(2)} \setminus \Omega_n} \sigma(i) \leq b) \\ &= P(|A_n^{(2)} \setminus \Omega_n|^{-1} \sum_{i \in A_n^{(2)} \setminus \Omega_n} \sigma(i) \leq bn^2 / (n^2 - l_n^2)), \end{aligned}$$

which goes to 1 by the ergodicity of σ and the choice of l_n .

We will show now that the last term on the r.h.s. of (6.6) goes to zero. For $\delta > 0$ let Ω_n^δ be a square with the same center as Ω_n and side $l_n + 2[n\delta]$. Let B_n be the event that Ω_n is surrounded by a circuit of occupied sites contained in the annulus $\Omega_n^\delta \setminus \Omega_n$. Using Lemma 4 it is easy to prove that for any $\delta > 0$

$$(6.7) \quad \lim_{n \rightarrow \infty} P(B_n | A_n) = \lim_{n \rightarrow \infty} P(B_n) = 1$$

Now,

$$(6.8) \quad \begin{aligned} P(Y_n(\sigma) < a | A_n) &\leq P(Y_n(\sigma) < a | A_n \cap B_n) \cdot P(B_n | A_n) \\ &\quad + P((B_n)^c | A_n). \end{aligned}$$

But

$$\begin{aligned} P(Y_n(\sigma) < a | A_n \cap B_n) &\leq P(n^{-2} \sum_{i \in A_n^{(2)} \setminus \Omega_n^\delta} \sigma(i) < a | A_n \cap B_n) \\ &= P(n^{-2} \sum_{i \in A_n^{(2)} \setminus \Omega_n^\delta} \sigma(i) < a | B_n). \end{aligned}$$

And using FKG the r.h.s. above is less than or equal to

$$\begin{aligned} P(n^{-2} \sum_{i \in A_n^{(2)} \setminus \Omega_n^\delta} \sigma(i) < a) &= P(|A_n^{(2)} \setminus \Omega_n^\delta|^{-1} \sum_{i \in A_n^{(2)} \setminus \Omega_n^\delta} \sigma(i) < a) \\ &< an^2 / (n^2 - (l_n + 2[\delta n])^2), \end{aligned}$$

which goes to zero by the ergodicity of σ and the choice of l_n , provided that δ is small enough. The combination of this fact with (6.7) and (6.8) completes the proof. \square

The lower bound obtained above is of the form

$$\mu(Y_n \in [a, b]) \geq C(1 - p)^{4n(1 - (a+b)/2\theta(p))^{1/2}}$$

Choosing l_n as the integer part of $n(1-(b-\varepsilon)/\theta(p))^{1/2}$ for positive $\varepsilon < b-a$, the proof would still be correct and after taking the limit $\varepsilon \rightarrow 0$ we would get

$$\liminf_{n \rightarrow \infty} n^{-1} \log \mu(Y_n \in [a, b]) \geq 4(1-b/\theta(p))^{1/2} \log(1-p).$$

Observe that the r.h.s. goes to zero as $b \rightarrow \theta(p)$ and it has the form proposed in Conjecture 2. Nevertheless, as we know from Theorem 4, the correct price to “kill the sites inside Ω_n ” is not $(1-p)^{4l_n}$ but $Ce^{-4l_n/\xi(p)}$. Using the techniques of the proof of Theorem 4 it is easy to modify the proof above to get then

$$\liminf_{n \rightarrow \infty} n^{-1} \log \mu(Y_n \in [a, b]) \geq -4(1-b/\theta(p))^{1/2}/\xi(p).$$

Still, it is not clear that this is a sharp estimate, since bubbles with a different shape than a square may have a smaller cost per volume. The estimate above can be reproduced for other shapes of bubbles. But the comparison between the results, needed to decide which one is the correct shape, depends on having sharp relations between correlation lengths in different directions, which to our knowledge is still an open problem. Even if the approach above could produce a sharp lower bound, the corresponding upper bound would have to be proven by a method different from the one we used.

Acknowledgements. We benefited from discussions with Jennifer Chayes, Lincoln Chayes and Harry Kesten.

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Received April 5, 1987; received in revised form November 3, 1987