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LIMIT THEOREMS FOR THE SPREAD OF EPIDEMICS AND FOREST FIRES

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We prove that the "spatial epidemic with removal" grows linearly and has an asymptotic shape on the set of nonextinction.

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1. Introduction

In this paper we will prove theorems which describe the asymptotic behavior of two closely related processes: one which has been used to model the spread of epidemics (Bailey (1965), Mollison (1977), Cardy (1983), Grassberger (1983), Cardy and Grassberger (1985), Kuulasmaa (1982)) and forest fires (McKay and Jan (1984), von Niessen and Blumen (1986), Ohtsuki and Keyes (1986)), and a second which can be used to study path lengths in percolation clusters.

In the first model each site $z \in \mathbb{Z}^2$ can be in one of three states: 1, i, or 0. In the epidemic interpretation 1 = healthy, i = infected, and 0 = immune, while for a forest fire 1 = a live tree, i = on fire, and 0 = burned. Let η_t denote the process, $\eta_t(x)$ is the state of site x at time t. We will generally favor the epidemic formulation of the model, but occasionally we will talk about forest fires. We begin by describing the model in epidemic language following Mollison (1977).

An infected individual emits germs according to a Poisson process with rate α. A germ emitted from x goes to one of the four nearest neighbors $x+(1,0)$, $x+(0,1)$, $x+(-1,0)$, or $x+(0,-1)$ chosen at random (with equal probabilities). If the germ

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goes to the site of a healthy individual then that individual immediately becomes infected and begins to emit germs. It stays infected for a random time with distribution function F, then recovers and is immune to further infection (e.g. consider measles). We assume that F is concentrated on the nonnegative half line and is not the unit point mass at zero. To complete the description we declare that the infection periods and Poisson processes of germs associated with different sites are independent.

Given the description in the last paragraph the reader can undoubtedly construct the process in question, but for what follows it will be useful to have a special construction which we will give now. Let T_x, $x \in Z^2$ be independent identically distributed random variables with distribution F and for x, $y \in Z^2$ with $|x-y|=1$ let $e(x, y)$ be independent identically distributed random variables with $P(e(x, y) > t) = \exp(-\alpha t/4)$. T_x is the amount of time x will stay infected (if it ever becomes infected) and $e(x, y)$ is the time lag from the infection of x until the first germ from x is sent to y. We let

$$\tau(x, y) = \begin{cases} e(x, y) & \text{if } T_x > e(x, y), \\ \infty & \text{if } T_x \leq e(x, y), \end{cases} \tag{1.1}$$

and say the oriented bound (x, y) is *open* if $\tau(x, y) < \infty$ and *closed* otherwise. Given the definition of T_x and $e(x, y)$ it should be clear that bond (x, y) is open if x tries to infect y during its "lifetime," and $\tau(x, y)$ gives the time lag from the infection of x until it tries to infect y, with $\tau(x, y) = \infty$ if this never happens.

Let $C_0 = \{x$: x can be reached from 0 by a path of open bonds$\}$. In percolation language C_0 is the open cluster containing the origin 0. The relationship of C_0 to the epidemic model is explained by:

(1.2) C_0 = the set of sites that will ever become infected if initially the origin is infected and all other sites are healthy.

Proof. Clearly if y becomes infected at some time then $y \in C_0$ (because y was infected by a neighbor, who was infected by a neighbor, ... who was infected by 0). We argue the other inclusion by induction on the length of the shortest path to x. If the path has length 1 this is clear because when 0 tries to infect it either (a) the site is not infected and it succeeds or (b) the site is already infected and hence it is in C_0. If $x \in C_0$ and $0, x_1, \ldots, x_{n-1}, x_n = x$ is a shortest path to x then we can apply the last argument with 0 replaced by x_{n-1}. The induction hypothesis implies $x_{n-1} \in C_0$ and when x_{n-1} tries to infect x_n either (a) or (b) above occurs and in either case we can conclude $x_n \in C_0$.

The relationship of the general epidemic model to the percolation process described above was first noticed by Mollison (1977) (see p. 322) and developed by Kuulasmaa (1982), who called the resulting structures locally dependent random graphs. By applying the clutter percolation theorem of McDiarmid (1980) he was able to prove a useful comparison theorem. To state this result we need some notation: If $A \subset \{z: |z|=1\}$ let $\phi(A) = P(\text{all edges } (0, z)\ z \in A \text{ are closed})$. ϕ is called the zero-function of the epidemic model.

(1.3) Let B be a collection of paths in $\mathbb{Z}^2$ and let $\mathcal{B}$ be the event that some path in B is open. If two epidemic models have $\phi_1(A) \geq \phi_2(A)$ for all A then $P_1(\mathcal{B}) \leq P_2(\mathcal{B})$.

The proof is in Kuulasmaa (1982), see pages 749-750.

If we consider the density of open bonds

$$p = P((0, (0, 1)) \text{ is open}) = 1 - \int_0^\infty e^{-\alpha s/4}\, dF(s)$$

as a parameter then the last result allows us to identify two extreme cases:

$$\phi_{\text{bond}}(A) = (1-p)^{|A|} \quad \text{(all bonds independent)}$$

where $|A|$ = the cardinality of A and

$$\phi_{\text{site}}(A) = (1-p) \text{ if } |A| \geq 1 \quad \text{(perfect correlation)}.$$

It is clear that any epidemic model has

$$\phi_{\text{site}}(A) = (1-p) \geq \phi(A).$$

Harris' (1960) inequality implies that the random variables $1_{\{(x,x+z) \text{open}\}}$ are positively correlated (see (2.1) in Section 2 for a proof) so

$$\phi(A) \geq (1-p)^{|A|} = \phi_{\text{bond}}(A).$$

Combining the last two observations with (1.3) gives

$$P_{\text{site}}(\mathcal{B}) \leq P(\mathcal{B}) \leq P_{\text{bond}}(\mathcal{B}).$$

To explain the notation we have used for the extreme cases, we begin by observing that in the "site" epidemic model either all the bonds $(x, x+z)$ with $|z| = 1$ are open or all are closed. A little thought reveals that the resulting percolation model is equivalent to the usual site percolation model in which sites $z \in Z^2$ are independently called open or closed with probabilities p and $1-p$ and two sites x and y belong to the same cluster if y can be reached from x by a path of open sites. See Kesten (1982) for a discussion of this model and for the other facts about percolation we will use below.

Having heard the term "site" explained the reader can probably guess what the "bond" model refers to, but this time things are not quite as simple. In the "bond" epidemic model each bond $(x, x+z)$ is independently open or closed but unlike the usual bond percolation model in which a bond is either open for passage in both directions or closed, the state of $(x, x+z)$ and $(x+z, x)$ are independent. This distinction, however, turns out to be a minor difference. Frisch and Hammersley (1963) showed that the two processes have the same critical value, and that for any two sets of sites S and T the probability there is an open path from S to T is the same in the two models. See Theorem 3.1 in McDiarmid (1980) for a simple proof of this result.

Combining the observations in the last three paragraphs it follows that if we let

$$\alpha_c^\circ(F)=\inf\{\alpha:\ P_{\alpha,F}(|C_0|=\infty)>0\}$$

(where the subscripts on P indicate the parameters of the model, the superscript $\circ$ is for "out from 0") and

$$p_c(F)=1-\int_0^\infty e^{-s\alpha_c^\circ(F)/4}\,ds$$

then

$$\tfrac{1}{2}\leq p_c(F)\leq p_c(\text{site})\approx 0.5927.$$

The number at the left end is a rigorous bound coming from Kesten's solution of the bond problem, the number on the right is not rigorous but is the consensus of a large number of numerical studies in the physics literature.

The results in the last paragraph (which are due to Kuulasmaa (1982)) show that $0<\alpha_c^\circ(F)<\infty$, and if one accepts the 0.5927, give reasonable bounds on the critical value. Having established the existence of a phase transition we come now to the question which is the main focus of our work: What does the epidemic look like when it lasts forever? Before answering this question we will start with a simpler problem which is the second model referred to in the first paragraph of the introduction.

Consider the usual bond percolation model in which bonds are open (in both directions) with probability p, closed with probability $1-p$, and distinct bonds are independent. If we think of the open bonds as pieces of wood which take one unit of time to burn, and if we set the origin on fire at time 0, then at time n all sites in C_0 a distance less than n from 0 will be burnt and those at distance n will be on fire. If we let $\zeta_n=$ the set of sites burnt at time n, $\xi_n=$ the set of sites on fire at time n, and for any set S use θS to denote $\{\theta y:\ \theta\in S\}$, then our result about the asymptotic behavior of this model can be stated as:

(1.4) If $\alpha>\alpha_c^\circ(F)$ then there is a convex set D such that for any $\varepsilon>0$,

$$P(C_0\cap n(1-\varepsilon)D\subset\zeta_n\subset n(1+\varepsilon)D \text{ for all sufficiently large } n)=1$$

and

$$P(\xi_n\subset n(1+\varepsilon)D-n(1-\varepsilon)D \text{ for all sufficiently large } n)=1.$$

Given results in Cox and Durrett (1981) (hereafter abbreviated as CD), and what is now known about percolation, it is easy to make (1.4) seem plausible. The paper just cited considered first passage percolation, a model in which undirected bonds $[x, y]$ with $|x-y|=1$ are assigned independent nonnegative random variables $\tau[x, y]$ and one studies the first passage times

$$t[x,y]=\inf\left\{\sum_{i=1}^{m}\tau[x_{i-1},x_i]:\ x_0,\ldots,x_m \text{ is a path from } x \text{ to } y\right\}.$$

(Notice the square brackets. Here and in what follows they indicate we are dealing with the ordinary unoriented situation.) If we let

$$\tau[x, y] = \begin{cases} 1 & \text{if } [x, y] \text{ is open,} \\ \infty & \text{if } [x, y] \text{ is closed,} \end{cases}$$

then it is easy to see that $\zeta_n = \{n\colon t[0, x] < n\}$ and $\xi_n = \{x\colon t[0, x] = n\}$. So the burning percolation cluster fits into the first passage percolation set-up.

The last paragraph as the old joke goes is the good news. The bad news is that results of CD cannot be applied since they assume $t[x, y] < \infty$. The last problem however is a minor difficulty. The proof of the main result in CD (see p. 584) starts by picking M so that $P(\tau[x, y] \leq M) \geq \frac{3}{4}$, declaring bonds $[x, y]$ to be *open* if $\tau[x, y] \leq M$ and *closed* otherwise, and then bounding passage times by constructing paths which consist only of open bounds. We will say more about the details of this proof in Section 3 when we have to generalize it to the current setting, but since the general strategy is to ignore closed bonds it should not be hard for the reader to believe that the proof given in CD generalizes immediately to the case $P(\tau[x, y] = \infty) < \frac{1}{4}$.

The last restriction comes from the fact that when the density of closed bonds $< \frac{1}{4}$, it is very easy to prove that every point is surrounded by a circuit of open bonds and give bounds on the circuit's size. By working harder (which was unnecessary in CD) one can show that the same bounds hold whenever the density of closed bonds is less than $1 - p_c$, and the proofs of CD can be repeated to prove the result stated above.

We will not give the details of the proof of (1.4) for the burning percolation cluster. There are two reasons for this. The first is that Y. Zhang and Y.C. Zhang (1984) have already proved a closely related result—they considered the first passage percolation model with $P(\tau[0, y] = 0) = 1 - P(\tau[0, y] = 1) = p$, y one of the four neighbors of 0, and studied the asymptotic behavior of the length of the shortest path from 0 to x which achieves the passage time $t[0, x]$. The second is that we will give all the details of the more difficult proof of our shape theorem for the epidemic model.

Returning to the epidemic model, define ζ_t as the set of immune (or "removed") sites at time t and ξ_t as the set of infected sites at time t, that is,

$$\zeta_t = \{x\colon \eta_t(x) = 0\} \quad \text{and} \quad \xi_t = \{x\colon \eta_t(x) = i\}.$$

We assume that initially the origin is infected and all other sites healthy,

$$\eta_0(x) = \begin{cases} i & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

The shape theorem is

Theorem 1. *Assume that* $\int_0^\infty s^2 \, dF(s) < \infty$ *and* $\alpha > \alpha_c^o(F)$. *Then there is a convex set* D *such that for any* $\varepsilon > 0$,

(i) $P(C_0 \cap t(1-\varepsilon)D \subset \zeta_t \subset t(1+\varepsilon)D$ *for all sufficiently large* $t) = 1$,

(ii) $P(\xi_t \subset t(1+\varepsilon)D - t(1-\varepsilon)D$ *for all sufficiently large* $t) = 1$.

The second moment assumption is necessary for (ii) to hold. It is clear that if F has a very fat tail then some individuals will remain infected a very long time, thus preventing the boundary type behavior of (ii). A simple Borel-Cantelli argument shows that "fat" means infinite second moment.

In Section 2 we will show that the percolation estimates needed for the arguments of CD can be generalized to our epidemic model. The key to the developments in that section is (2.2), whose proof is a generalization of an argument of Russo. Once this is proved modern percolation technology, in particular (2.5) the "Rescaling Lemma" of Aizenman, Chayes, Chayes, Frohlich, and Russo takes over and allows us to prove the estimates we need. An interesting consequence of the developments in Section 2 is that we can prove that *at the critical value* $\alpha_c^\circ(F)$ the epidemic dies out with probability one (see Theorem 2). This is the behavior expected in models of this type (e.g. critical branching processes die out) but it is rare that one can prove it.

In Section 3 we begin the proof of the shape theorem by investigating the limiting behavior of the first passage times. Define the *directed* first passage process $t(x, y)$ using the $\tau(x, y)$ defined in (1.1) just as $t[x, y]$ was defined using the $\tau[x, y]$. We prove

Theorem 3. *If* $\theta \in \mathbb{Z}^2$ *then there is a constant* $\mu(\theta)$ *so that, as* $n \to \infty$,

$$\left[\frac{1}{n} t(0, n\theta) - \mu(\theta)\right] 1_{(n\theta \in C_0)} \to 0 \quad a.s.$$

Given the percolation estimates developed in Section 2 the proof of Theorem 3 is almost a word for word repetition of the proof of Theorem 1 in CD. The only new difficulty is that we are dealing with oriented percolation, so that we have to exercise some care in tying open paths together.

In Section 4 we complete the proof of the shape theorem. The first step is to extend C_0 and $t(0, x)$ by assuming that when $z \in \mathbb{Z}^2$ becomes infected then so do all the points in $z + (-\frac{1}{2}, \frac{1}{2}]^2$, and to prove that Theorem 3 holds for all $\theta \in \mathbb{R}^2$. With this done we can identify the limiting set as $D = \{\theta \in \mathbb{R}^2: \mu(\theta) \leq 1\}$. The limit theorem for first passage times allows us to prove that if we pick a finite set $\{\theta_1, \ldots, \theta_k\} \subset \{\theta: \mu(\theta) \leq 1\}$ and points $x_i \in C_0$ near $\theta_i t$ then all the x_i will be in ζ_t with high probability. The last observation gives us a fairly dense set of points in D which are in $t^{-1}\zeta_t$. Then to complete the proof it suffices as it has several times in the past (Richardson (1973), Schurger (1979, 1980), Bramson and Griffeath (1980), Cox and Durrett (1981), Durrett and Griffeath (1982)) to prove there is $\delta > 0$ so that with probability one for all large enough t

$$\zeta_t \supset \{x: |x| < \delta\} \cap C_0.$$

This last result can be proved by imitating proofs in Sections 3 and 4 of CD. The shape theorem then follows easily.

The last three paragraphs can be summarized in one sentence: by combining ideas from percolation with results of CD on first passage percolation we are able to treat first passage percolation on infinite clusters and analyze a model for epidemics and forest fires. This state of affairs makes it seem that things are well understood, but the very opposite is true. Minor changes in the model lead to (hard) open problems:

(i) Suppose we consider (as Savit and Ziff (1985) did) the site version of the model: an individual becomes infected at rate α if at least one of its neighbors is on fire. The construction used above breaks down (try it) and we can say almost nothing.

(ii) Suppose we consider (as Bailey (1965) did) the discrete time model in which the disease lasts one unit of time and an infected individual can infect his four nearest neighbors $x+(1,0),\dots,x+(0,-1)$ *and* the four diagonally adjacent ones $x+(1,1),\dots,x+(1,-1)$, flipping 8 independent coins to see which sites become infected. Bailey observed (see p. 255) that "the range between $p=0.2$ and $p=0.4$ obviously deserves close attention" because a "steady shift in distribution with changes in p is immediately apparent." Kuulasmaa's theorem applies to Bailey's model so we can identify the shift in distribution with the onset of percolation. Thanks to Frisch and Hammersley we can identify the threshold with the critical value of ordinary bond percolation on the obvious graph (but we do not know what this value is). What is worse the percolation machinery used above breaks down on this non-planar graph, so we cannot analyze this epidemic model. This problem should not be too far out of reach, but will take some serious work to solve.

(iii) A third variation of the model which has been considered is "forest fire with wind" (Ohtsuki and Keyes (1986)). In this discrete time model a burning tree at x can ignite the trees at $x+(1,0)$ and $x+(0,1)$ with probability p but will ignite the trees at $x+(-1,0)$ and $x+(0,-1)$ with probability $q<p$. When $q=0$ this reduces to oriented percolation and if $p>p_c$ the forest fire will burn a cone with opening angle $\theta(p)$ in the first quadrant (see Durrett (1984)). It is easy to see that this can happen when $q>0$. Mapping out the "phase diagram" of this system is a difficult problem, but there is some hope because we are dealing with a planar graph and there is a duality for these models (see the end of Section 2).

(iv) The results mentioned above show that if we consider ordinary bond percolation with $p>p_c$ $(=\frac{1}{2})$ and let l_n be the length of the shortest open path from 0 to $(n,0)$ then there is a constant $\nu(p)$ so that

$$\left[\frac{1}{n}l_n-\nu(p)\right]1_{\{(n,0)\in C_0\}}\to 0$$

as $n\to\infty$. Many papers in the physics literature (Ritzenberg and Cohen (1984), Barnes (1985), Grassberger (1985), Edwards and Kerstein (1985)) have addressed the question: How does $\nu(p)\to\infty$ as $p\downarrow p_c$? To indicate what type of answer is desired we recall that Chayes, Chayes, and Durrett (1987) have studied the problem in which $\tau[x, y]=0$ with probability p and $\tau[x, y]=1$ with probability $1-p$, and have shown that as $p \uparrow p_c$ the time constant $\mu(p) \to 0$ like $\xi(p)^{-1}$, $\xi(p)=$ the correlation length. Leaving the last term undefined (see the paper cited for further explanation) we can at least state our last problem: show that, as $p \downarrow p_c$, $\nu(p) \to \infty$ like $(\xi(p))^{\theta}$ where $\theta > 1$ is related to the connectivity properties of the "incipient infinite cluster". See Kesten (1986) for a rigorous definition. Even the physicists cannot decide if θ is a new exponent or related to the other ones. Rigorous results on this question will be difficult to come by.

(v) Last but not least (as the referee pointed out) is the problem of what happens in the epidemic when recovery is possible: e.g. $0 \to 1$ at rate β (independent of the state of its neighbors). We have considered the case $\beta=0$. If $\beta=\infty$ then we never see the 0's, and if F is exponential then the 1's and i's form a contact process. A shape result for this process has been proved by Durrett and Griffeath (1982). Generalizing their proof to $0<\beta<\infty$ seems to be a difficult problem. One complication (absent when $\beta=0$) is the possibility for the origin to be infected infinitely many times.

2. Percolation theory

In this section we will show that although our model has dependent directed bonds it is enough like "ordinary" two dimensional percolation so that the results needed for the arguments in CD can be proved. As in the case of ordinary percolation the study of sponge crossings is the key. Let $R_{J,K}=R(J, K)$ be the probability that there is a right-left crossing of the "sponge" $(0, J) \times (0, K)$ by open bonds. We will drive inequalities relating the $R_{kL,L}$ for various values of k. These results will imply that if we define the sponge crossing critical value α_s by

$$\alpha_s=\inf\{\alpha: \limsup_{L\to\infty} R_{L,L}=1\},$$

then $\alpha_c^\circ=\alpha_s$, and the epidemic dies out at the critical value.

Our first step is to prove the $\tau(x, y)$ are "associated" in the sense of Newman and Wright (1981).

(2.1) Lemma. *If f and g are bounded coordinatewise increasing functions,*

$$X=f(\tau(x_1, y_1), \ldots, \tau(x_m, y_m))$$

and

$$Y=g(\tau(x_1', y_1'), \ldots, \tau(x_n', y_n')),$$

then $EXY \geq EXEY$.

Proof. $\tau(x, y)=h(e(x, y), -T_x)$ where

$$h(a, b)=\begin{cases} a & \text{if } a<-b, \\ \infty & \text{if } a>-b, \end{cases}$$

is an increasing function in each of its two arguments. Thus X and Y are increasing functions of independent random variables and it follows from the original result of Harris (1960) that $EXY \geq EXEY$.

With this preliminary established we can begin the real work of this section. The hardest step of what follows is to prove the "RSW" (for Russo (1978) and (1981), and Seymour and Welsh (1978)) lemma.

(2.2) Lemma (RSW). $R_{3L/2,L} \geq (1-(1-R_{L,L})^{1/2})^3$.

Note. When we are done with the model under consideration we will need to claim that the proof of (2.2) works for a slightly different model with positively correlated bonds. To prepare for this we ask the reader to check as he goes along that in all steps but one the argument uses only positive correlations. We need independence only when the primed variables are introduced in the proof of (2.2) and used to conclude

$$P(F'_s|E_s) \leq 1-(1-R_{L,L})^{1/2}.$$

We will start the proof of (2.2) with a lemma which explains the unusual formula in the answer. An event A is called *increasing* if whenever $\omega \in A$ and every open bond in ω is open in ω', then $\omega' \in A$.

(2.3) Lemma (The square root trick). *Let A_1 and A_2 be increasing events. If $A = A_1 \cup A_2$ and $P(A_1) = P(A_2)$ then*

$$P(A_1) \geq 1-(1-P(A))^{1/2}.$$

Proof. From set theory and Harris' inequality, we get

$$(1-P(A_1))^2 = P(A_1^c)^2 = P(A_1^c)P(A_2^c) \leq P(A_1^c \cap A_2^c) = 1-P(A)$$

so

$$P(A_1) \geq 1-(1-P(A))^{1/2}.$$

The lemma above allows us to have paths begin or end in one half of an $L \times L$ square without dividing the probability by 2. With this and a little geometric trickery, we can tie three paths together to cross a $3L/2 \times L$ rectangle. In this part of the proof we follow Russo (1981, p. 230–231) very closely. Here and in what follows all paths are assumed to be *self-avoiding*.

Let s be a right-left crossing of $(0, L) \times (0, L)$ such that the first point of intersection of s and $\{L/2\} \times (0, L)$ has y-coordinate $\leq L/2$. Let E_s be the event that s is *open* and is the *lowest* open right-left crossing of $(0, L) \times (0, L)$. (We should prove as Kesten (1980) did that there *is* a lowest crossing but we omit this tedious detail. At the end of the section, when we discuss duality, we will sketch a proof.) Let s_r be the portion of this path from $\{L\} \times (0, L)$ until the first time it hits $\{L/2\} \times (0, L)$, and let s_{rr} = the reflections of s_r through $\{L\} \times (0, L)$. Let $\mathscr{A}(s_r \cup s_{rr})$ = the points in $(L/2, 3L/2) \times (0, L)$ above $s_r \cup s_{rr}$ and let $\mathscr{B}(s)$ = the points in $(0, L) \times (0, L)$ on or below s.

Up to this point everything has been the same as in Russo's proof, but for reasons we will explain in a minute, the next step must be different. For all the $x \in \mathscr{B}(s)$ define new variables T'_x, $e'(x, y)$, $\tau'(x, y)$ which are independent of the original random variables and have the same distribution, and extend the definition of the primed variables to $(0, 3L/2) \times (0, L)$ by setting $T'_x = T_x$, $e'(x, y) = e(x, y)$, $\tau'(x, y) = \tau(x, y)$ for $x \notin \mathscr{B}(s)$. For convenience of exposition we will refer to the variables with the primes as *new* variables and call the original variables *old*.

Since E_s is measurable with respect to the σ-field of the bonds which begin in $\mathscr{B}(s)$, the new variables have the same distribution as the old ones. Let F'_s be the event that there is an open path t in the new system starting from $(L/2, 3L/2) \times \{L\}$ and connected to s_r in $\mathscr{A}(s_r \cup s_{rr})$. In Russo's original proof this path could be combined with s to make a path to s_r in the old system but this time the orientation is wrong (see Fig. 1). Fortunately there is a way around this.

Let t' consist of the bonds in t which begin in $\mathscr{A}(s_r \cup s_{rr}) \setminus \mathscr{B}(s)$. We claim that the union of s and t' protects $\{L/2\} \times [L/2, L]$ (see Fig. 1). That is, if u is a right-left crossing of $(L/2, 3L/2) \times (0, L)$ which ends in $\{L/2\} \times [L/2, L)$ then s, t', and u can be combined to give a right-left crossing of $(0, 3L/2) \times (0, L)$. To prove this, we begin by observing that the Jordan curve theorem implies that u must intersect s or t. If u intersects s then following u until it first hits s and then continuing along s gives the desired path. If u intersects t, and if the first intersection (along u) is in $\mathscr{B}(s)$ then we are done, since u must have intersected s at an earlier ($\leq$) time. If, on the other hand, the first intersection is in $\mathscr{B}(s)^c$, we follow t' from this point until the next time it hits s and continue with s to cross the rectangle.

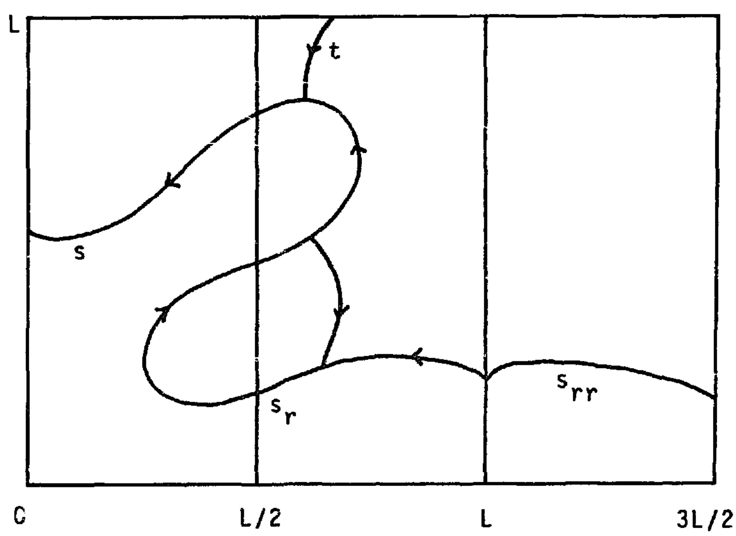

Fig. 1.

Having shown that u, t', and s can be combined to give the desired path the rest of the proof is the same as in Russo (1981). Let G be the union of $E_s \cap F'_s$ over all paths s for which the last point of s_r has y-coordinate $\leq L/2$. Let H be the event that there is an open right-left crossing of $(L/2, 3L/2) \times (0, L)$ which ends at a point with y-coordinate $\geq L/2$. Since the occurrence of G and H guarantees the desired crossing, it suffices to show

$$P(G\cap H)\geq(1-(1-R_{L,L})^{1/2})^3.$$

The first step in doing this is to observe that Harris' inequality implies $P(G\cap H)\geq P(G)P(H)$. Using the square root trick with $A=\{$there is a right-left crossing of $(L/2,3L/2)\times(0,L)\}$ and $A_1=H$ gives

$$P(H)\geq(1-(1-R_{L,L})^{1/2}).$$

To estimate $P(G)$ we write

$$P(G)=\sum_s P(E_s\cap F'_s)$$

$$=\sum_s P(E_s)P(F'_s),$$

and use the square root trick with $A_1=F'_s$ and $A=\{$there is a path from $(L/2,3L/2)\times\{L\}$ down to $s_r\cup s_{rr}$ in $\mathscr{A}(s_r\cup s_{rr})$ in the "new" system$\}$ to conclude

$$P(F'_s)\geq 1-(1-P(A))^{1/2}$$

$$\geq 1-(1-R_{L,L})^{1/2}.$$

Here we are using the fact that E_s depends only on the old variables in $\mathscr{B}(s)$, and hence is independent of F'_s, which depends only on the new variables.

With the last inequality in hand the proof is complete because it implies

$$P(G)\geq(1-(1-R_{L,L})^{1/2})\sum_s P(E_s).$$

Another use of the square root trick gives

$$P(G)\geq(1-(1-R_{L,L})^{1/2})^2,$$

the last piece of the puzzle. This completes the proof of (2.2).

With (2.2) in hand, the rest of the developments are almost exactly as in the ordinary case. The next step is to prove

$$1-R_{kL,L}\leq 4(1-R_{(k+1)L/2,L})\quad\text{for } k\geq 1. \tag{2.4}$$

Proof. To prove this we draw a picture (Fig. 2) and observe that if all 4 paths exist then there is a crossing. The inequality above results from

$$P\left(\bigcup_{i=1}^4 A_i^c\right)\leq\sum_{i=1}^4 P(A_i^c).$$

Using (2.2) and (2.4), we obtain

$$R_{3L/2,L}\geq(1-(1-R_{L,L})^{1/2})^3,$$

$$R_{2L,L}\geq 1-4(1-R_{3L/2,L}),$$

$$R_{3L,L}\geq 1-4(1-R_{2L,L}),$$

and so on. The point is that once $R_{L,L}$ is close to 1 all the $R_{kL,L}$ are.

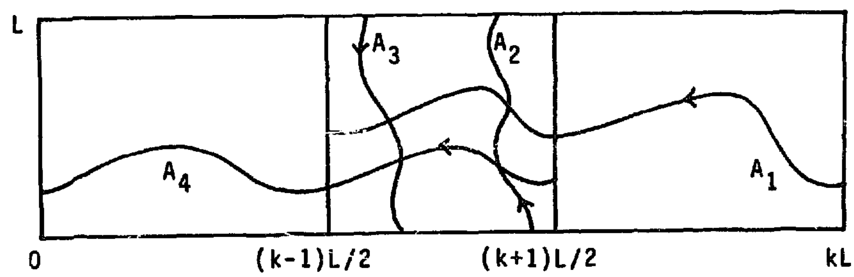

Fig. 2.

The next two inequalities (due to Aizenman, Chayes, Chayes, Frohlich, and Russo (1983)) will allow us to conclude that if $R_{L,L}$ is close enough to one for some L, then $R_{L,L} \to 1$ as $L \to \infty$.

$$1 - R_{4L,L} \leq 7(1 - R_{2L,L}), \tag{2.5a}$$

$$R_{4L,2L} \geq 1 - (1 - R_{4L,L})^2. \tag{2.5b}$$

Proof. For (2.5a) we draw another picture (Fig. 3), observe that if all seven paths exist then there is a crossing, and then argue as in the proof of (2.4). To prove (2.5b) we observe that the existence of open crossings in $(0, 4L) \times (0, L)$ and $(0, 4L) \times (L, 2L)$ are independent—this is the reason for using *open* rectangles in the definition of $R_{K,L}$.

Combining (2.5a) and (2.5b) gives

$$R_{4L,2L} \geq 1 - 49(1 - R_{2L,L})^2. \tag{2.6}$$

If we iterate (2.6) assuming that $R_{2L,L} = 1 - \lambda/49$ for some $\lambda < 1$ we get

$$R_{4L,2L} \geq 1 - \lambda^2/49,$$

$$R_{8L,4L} \geq 1 - \lambda^4/49$$

and by induction

$$R(2^k L, 2^{k-1} L) \geq 1 - \tfrac{1}{49} \exp(2^{k-1} \log \lambda).$$

Combining this with (2.5a) we also obtain

$$R(2^{k+1} L, 2^{k-1} L) \geq 1 - \tfrac{1}{7} \exp(2^{k-1} \log \lambda). \tag{2.7}$$

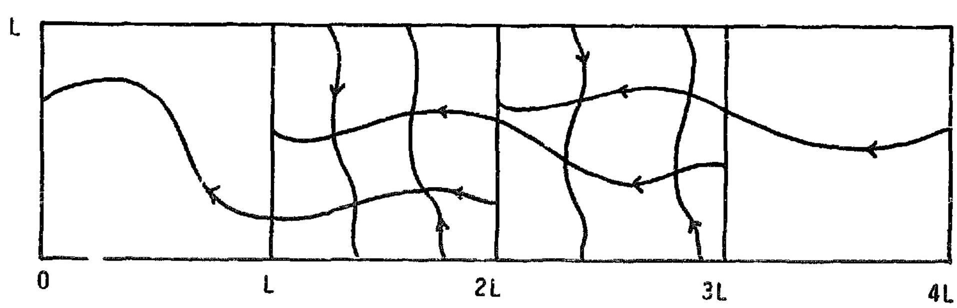

Fig. 3.

From this result it follows that if R_{2L_0,L_0} is close enough to one then $R(2^{k+1}L_0, 2^{k-1}L_0) \to 1$ as $k \to \infty$, and $R_{2L,L} \to 1$ as $L \to \infty$.

The development above motivates defining

$$L_0(\alpha) = \inf\{L: R_{2L,L}(\alpha) \geq 0.99\},$$

which must be finite for $\alpha > \alpha_s$. We will introduce a final critical value, α_c^i defined by

$$\alpha_c^i = \inf\{\alpha: P_\alpha(C_i \text{ is infinite}) > 0\}$$

where $C_i = \{x: 0$ can be reached from x by a path of open bonds$\}$. The next result shows that all three critical values are the same, and that the epidemic dies out at the critical value.

Theorem 2. *$\alpha_s = \alpha_c^i = \alpha_c^\circ$ and*

$$P_{\alpha_s}(|C_0| = \infty) = P_{\alpha_s}(|C_i| = \infty) = 0.$$

Proof. First let $\alpha > \alpha_s$, let $n = L_0(\alpha)$, and for $j \geq 1$ let

$$B_{2j-1} = (2^{2j-2}n, 2^{2j-1}n) \times (0, 2^{2j}n),$$

$$B_{2j} = (0, 2^{2j+1}n) \times (2^{2j-1}n, 2^{2j}n),$$

$$A_{2j-1} = \{\text{there are top-bottom and bottom-top crossings of } B_{2j-1}\},$$

$$A_{2j} = \{\text{there are left-right and right-left crossings of } B_{2j}\}$$

(see Fig. 4). By (2.7) and Harris' inequality

$$P\left(\bigcap_{k=1}^{\infty} A_k\right) \geq \prod_{k=1}^{\infty} R(2^{k+1}n, 2^{k-1}n)^2 > 0. \tag{2.8}$$

Since there is positive probability that all the bonds on the segment from $(0,0)$ to $(2n, 0)$ are open it follows that both $P_\alpha(|C_0| = \infty)$ and $P_\alpha(|C_i| = \infty)$ are strictly positive.

The construction above shows $\alpha_s \geq \max(\alpha_c^i, \alpha_c^\circ)$. To prove the other inequality observe that if $\alpha < \alpha_s$ then $R_{2L,L} \leq 48/49$ for all L, or else (2.7) would imply $R(2^{k+1}L, 2^{k-1}L) \to 1$ and hence $\alpha > \alpha_s$. So it follows from (2.2) and (2.4) that there is an $\varepsilon_0 > 0$ so that $R_{L,L}(\alpha) \leq 1 - \varepsilon_0$ for all L and $\alpha < \alpha_s$. By continuity, the last conclusion implies $R_{L,L}(\alpha_s) \leq 1 - \varepsilon_0$. With the probabilities of sponge crossings bounded away from 1, we can now use the original argument of Harris (1960) to show there is no percolation. Introduce the dual percolation process with sites $Y^2 = (\frac{1}{2}, \frac{1}{2}) + Z^2$, and call the bond (u, v) between neighboring points in Y^2 open (closed) if the bond on the original lattice obtained by rotating it 90° counterclockwise around its midpoint is closed (respectively open).

This duality is the natural generalization to oriented percolation of the duality used in the ordinary case (see Chapter 2 of Kesten (1982)) and has many of the same properties. In particular, we have:

(2.9) Either there is a right-left crossing of $(0, L) \times (0, L)$ or a top-bottom crossing of $(\frac{1}{2}, L - \frac{1}{2}) \times (\frac{1}{2}, L - \frac{1}{2})$ on the dual, but not both.

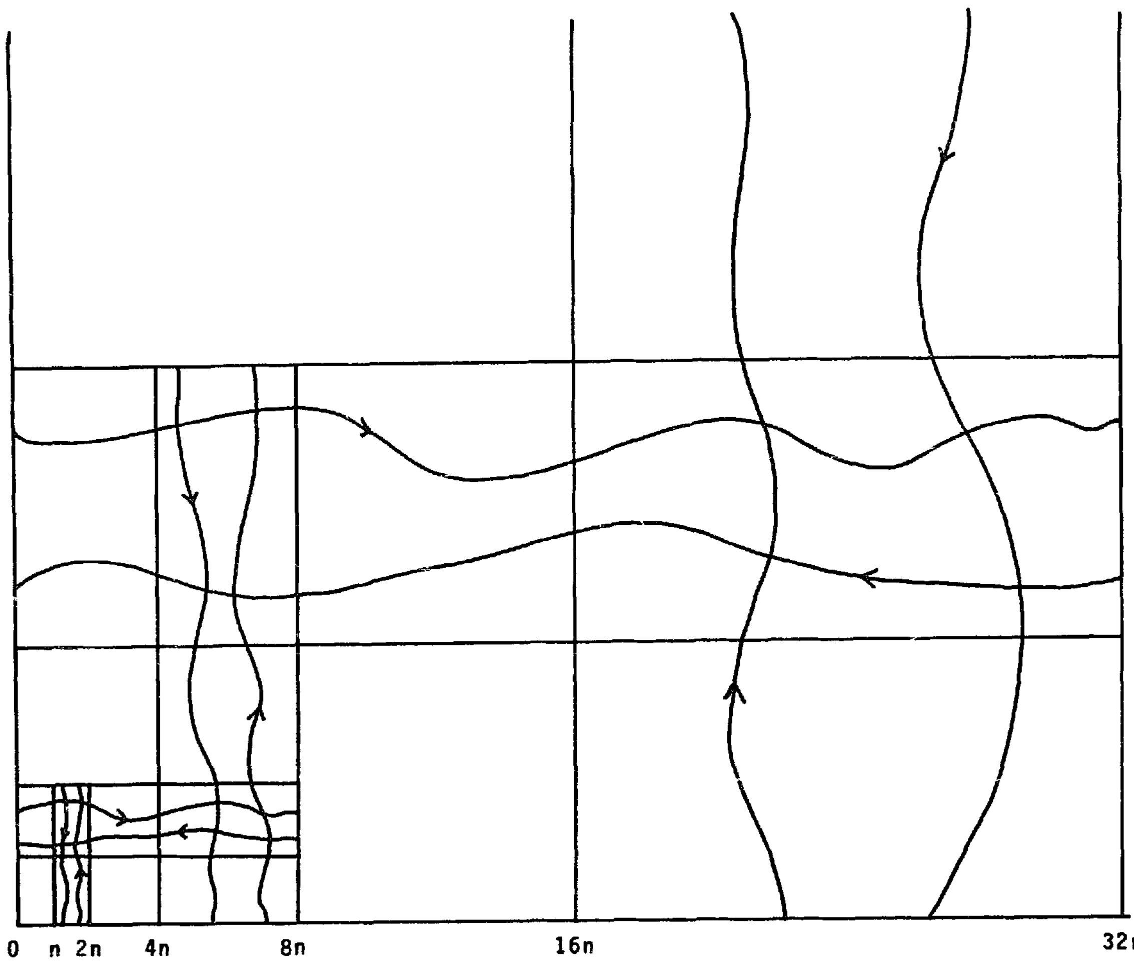

Fig. 4.

The proof of (2.9) is almost the same as in the ordinary case. A detailed proof is given in Section 2 of Durrett and Schonmann (1987). For the convenience of the reader we will sketch the argument here. First, if there is a top-bottom crossing on the dual then there is a self-avoiding one (i.e. each site appears at most once in the path). If we call the path σ, an application of the Jordan curve theorem shows that σ divides the interior of the square into 2 parts—one we call T_1 which lies to the right of σ, and one we call T_2 which lies to the left of σ. If we move along σ in the direction of the orientation then T_1 is always on our left and T_2 is always on our right. From this, we see that if there is a path of open bonds from right to left then any time it crosses from T_1 to T_2 it does so along a bond which is a 90° clockwise rotation of a bond on σ. But such bonds are closed, so no open path exists.

To prove the other direction, we will suppose there is no right-left crossing and construct a top-bottom one. Let C be the set of points which can be reached from the right edge by a path of open bounds. Let $D = \{(a, b) \in R^2 : |a|, |b| \le \frac{1}{2}\}$, and orient the boundary of D in a counterclockwise fashion. Finally let $W = \bigcup_{z \in C} (z + D)$. If we combine the boundaries of the $z + D$ with $z \in C$, and let oppositely directed segments cancel, then the boundaries which remain are closed paths on the dual.

One of them, Γ = the boundary of the component of $(\frac{1}{2}, L-\frac{1}{2}) \times (\frac{1}{2}, L-\frac{1}{2}) \setminus W$ which contains the left side of the box, is the path we want. For more details see Durrett and Schonmann (1987). The reader should note that a similar construction can be used to prove that there is a lowest right-left crossing.

With (2.9) established, we can conclude that the probability of a top-bottom crossing of $(\frac{1}{2}, L-\frac{1}{2}) \times (\frac{1}{2}, L-\frac{1}{2})$ is bounded away from 0 when $\alpha = \alpha_s$. If we let $\bar{R}_{L,L}$ denote the last probability then applying (2.3) generalized to the model under consideration we have

$$\bar{R}_{3L/2,L} \geqslant (1-(1-\bar{R}_{L,L})^{1/2})^3. \tag{2.10}$$

To see that this is legitimate, recall the note after the statement of (2.2). Although the dual bonds $(x, y) \to (x+1, y), \to (x+1, y+1), \to (x, y+1), \to (x, y)$ are dependent, bonds which go counterclockwise around different squares are independent. From the last observation we see that if there is a right-left crossing σ then all the bonds above σ are independent of it and the previous argument works.

With (2.10) the rest is easy and follows Harris (1960). Using the construction used to prove (2.4) but making a different estimate shows

$$\bar{R}_{kL,L} \geqslant (\bar{R}_{(K-1)L/2,L})^4, \tag{2.11}$$

so we have

$$\bar{R}_{2L,L} \geqslant (\bar{R}_{3L/2,L})^4,$$

and

$$\bar{R}_{3L,L} \geqslant (\bar{R}_{2L,L})^4.$$

By combining crossings of $3L \times L$ rectangles, we get a circuit in an annulus (see Fig. 5). Since $\bar{R}_{L,L}$ is bounded away from 0, we have a ridiculously small but, nonetheless, positive lower bound on the occurrence of a dual circuit in an annulus. By considering an infinite disjoint sequence of annuli we see there is no percolation out or in when $\alpha = \alpha_s$. This completes the proof of Theorem 2.

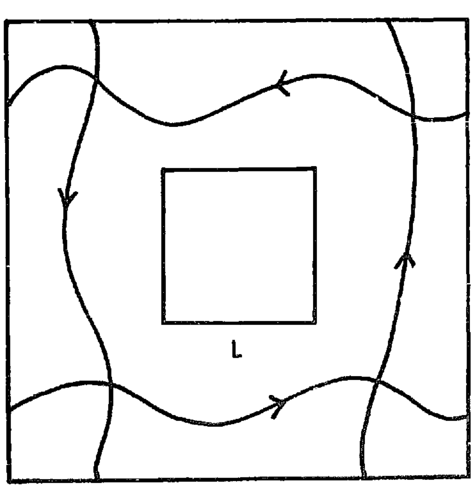

Fig. 5.

We will need one more estimate for the next section. Let $\Gamma(k)$ be the event that there is an open circuit around the square $[-k, k]^2$ that is contained in the square $[-2k, 2k]^2$ and that there are infinite open paths *starting* and *ending* on $[-k, k]^2$. If $n = L_0(\alpha)$ then there is a finite positive constant γ' such that

$$P(\Gamma(2^k n)) \geq 1 - \gamma' \exp(2^k \log \lambda). \tag{2.12}$$

The proof is now standard. The circuit in the annulus can be constructed by constructing four paths each at cost $R(2^{k+2}n, 2^k n)$ and paths to and from infinity can be constructed as in the proof of Theorem 2. The relevant probability estimates are (2.7) and (2.8).

3. Radial limits

In this section we will prove Theorem 3, the existence of radial limits. Since the result is trivial if $\alpha \leq \alpha_c^\circ$, we now assume in the following that $\alpha > \alpha_c^\circ$. With the percolation estimates of the last section established we can prove Theorem 3 using the approach of Section 2 of CD. There are two differences: (i) bonds (and paths) are oriented, and (ii) if $\tau(x, y) < \infty$ then $\tau(x, y)$ is bounded above by an exponential random variable with mean $4/\alpha$.

The first difference makes things a little harder, as we must take care in tying paths together. The second difference makes things a little easier since truncation is no longer required. In what follows, we will use the notation of Section 2 of CD as much as possible to bring out similarities in the proof, and to make it possible to use some results from that paper without reproving them. We start with some notation.

For each $z \in \mathbb{Z}^2$ let $\kappa(z)$ be the smallest $k \geq 1$ such that: (i) there are infinite open paths *to* and *from* the square $z + [-k, k]^2$, and (ii) there is an open circuit around $z + [-k, k]^2$ contained in $z + [-2k, 2k]^2$. For each $k \geq 1$ determine an arbitrary ordering of circuits around $z + [-k, k]^2$ contained in $z = [-2k, 2k]^2$. Now let $\Delta(z)$ be the "minimal" open circuit around z, where "minimal" means if $\kappa(z) = k$ then $\Delta(z)$ is the first open circuit in our ordering.

Having defined $\Delta(z)$, let $\tilde{\Delta}(z)$ be the union of $\Delta(z)$ and all open bonds inside $\Delta(z)$, and let $\hat{t}(x, y)$ be the minimum passage time from a site of $\tilde{\Delta}(x)$ to a site of $\tilde{\Delta}(y)$. Finally, let $u(z)$ be the sum of the passage times of all the open bonds of $\tilde{\Delta}(z)$. Observe that

$$\text{If } t(x, y) < \infty, \quad \text{then} \quad \hat{t}(x, y) \leq t(x, y) \leq \hat{t}(x, y) + u(x) + u(y). \tag{3.1}$$

$$\text{If } \xi(x, y) = \hat{t}(x, y) + u(y), \quad \text{then} \quad \xi(x, z) \leq \xi(x, y) + \xi(y, z). \tag{3.2}$$

Following the development in Section 2 of CD, here are some facts needed to prove Theorem 3.

$$P(\kappa(z) \geq n) \to 0 \quad \text{exponentially fast as } n \to \infty. \tag{3.3}$$

This is an immediate consequence of (2.12).

$$E(|\tilde{\Delta}(z)|^m)<\infty \quad \text{for } m=1,2,\ldots. \tag{3.4}$$

This is an immediate consequence of (3.3) and the crude estimate

$$P(|\tilde{\Delta}(z)|>4\cdot 2k(2k+1))\leq P(\kappa(z)>k).$$

$$E(u(z)^m)<\infty \quad \text{for } m=1,2,\ldots. \tag{3.5}$$

This fact uses (3.4) and the observation that if $\tau(x,y)<\infty$, then $\tau(x,y)$ is bounded by an exponential random variable with mean $4/\alpha$ (in Section 2 of CD truncation was used).

$$E(\hat{t}(x,y)^m)<\infty \quad \text{for } m=1,2,\ldots. \tag{3.6}$$

If $z_0, z_1, \ldots, z_k$ is a path from $z_0=x$ to $z_k=y$ then there is a path from $\tilde{\Delta}(x)$ to $\tilde{\Delta}(y)$ contained in $\bigcup_{i=0}^k \tilde{\Delta}(z_i)$. (This tedious detail can be proved by following the proof of Lemma 4.1. To see that the orientation poses no problem notice that if $x, y\in\Delta(z)$, then we can get from x to y, and the algorithm in Lemma 4.1 guarantees that the Δ's intersect.) Consequently, $\hat{t}(x,y)\leq\sum_{i=0}^k u(z_i)$. Now apply (3.5) to obtain (3.6).

Proof of Theorem 3. For fixed $\theta\in\mathbb{Z}^2$ let $\xi_{m,n}=\xi(m\theta, n\theta)$, $0\leq m<n<\infty$. Then, by (3.2), (3.5), and (3.6), $\xi_{m,n}$ is subadditive in the sense of Kingman, and as in the proof of Theorem 1 of CD, the limit

$$\mu(\theta):=\lim_{n\to\infty} n^{-1}\xi_{0,n} \quad \text{a.s. and in } L_1$$

exists and is constant. To finish the proof, observe that

$$\left|\frac{\hat{t}(0,n\theta)}{n}-\frac{t(0,n\theta)}{n}\right|\cdot 1_{\{n\theta\in C_0\}}\leq\frac{u(0)+u(n\theta)}{n}.$$

Since $Eu^2(n\theta)=Eu^2(0)<\infty$ by (3.5), Chebyshev and Borel-Cantelli arguments imply that $u(n\theta)/n\to 0$ a.s.

4. The shape theorem

In this section we will prove the shape theorem for the epidemic model. The program is to prove a shape result for $\hat{A}_t=\{z: \hat{t}(0,z)\leq t\}$, and use this to get the shape result for the epidemic model. We will follow Sections 3 and 4 of CD as closely as possible. Recall $\alpha>\alpha_c^\circ$.

Let $g(x)=E(\xi(0,x))$ for $x\in\mathbb{Z}^2$, and extend the domain of g to all of $\mathbb{R}^2$ by making it linear on triangles of the form (x,y), $(x,y+1)$, $(x+1,y)$ and $(x,y+1)$, $(x+1,y)$, and $(x+1,y+1)$. The proof of Lemma 3.2 of CD applies without change to prove

(4.1) There is a function φ on $\mathbb{R}^2$ such that $\lim_{n\to\infty} g(nx)/n=\varphi(x)$ uniformly on compact subsets of $\mathbb{R}^2$, $\varphi(x)=\mu(x)$ for $x\in\mathbb{Z}^2$.

Next we let $\hat{t}(0,x)=\hat{t}(0,z)$ for $x\in z+(-\frac{1}{2},\frac{1}{2}]^2$, $z\in\mathbb{Z}^2$ and prove

$$\lim_{k\to\infty}\hat{t}(0,kx)/k=\varphi(x)\quad\text{a.s.},\quad x\in\mathbb{Q}^2. \tag{4.2}$$

The argument on p. 592 of CD applies but some modifications are needed; here are the details. Fix $x\in\mathbb{Q}^2$ and let $n=\min\{n\geq 1: nx\in\mathbb{Z}^2\}$. Then $a_{m,n}=\xi(mNx,nNx)$ defines a subadditive process in the sense of Kingman, and so a.s. as $n\to\infty$,

$$a_{0,n}/nN\to\varphi(Nx)/n=\varphi(x).$$

In view of (3.5) we also have

$$\hat{t}(0,nNx)/nN\to\varphi(x)\quad\text{a.s.}$$

To obtain the limit along the full sequence of integers note that if $0\leq j\leq n-1$,

$$|\hat{t}(0,(j+nN)x)-\hat{t}(0,nNx)|\leq u(nNx)+\xi(nNx,(j+Nn)x).$$

This follows from the subadditivity of $\xi(x,y)$. Consequently, for any $\varepsilon>0$, and $0\leq j\leq N$,

$$\sum_{n=1}^{\infty}P(|\hat{t}(0,(j+nN)x)-\hat{t}(0,nNx)|>\varepsilon n)\leq\sum_{n=1}^{\infty}P(|u(0)+\xi(0,jx)|>n\varepsilon)<\infty,$$

and (4.2) must hold.

As in CD, we introduce new circuits $c(z)$ which do not depend on being connected to infinity. Let $c(z)$ be the "minimal" circuit as in the definition of $\Delta(z)$, except drop requirement (i). Let $\tilde{c}(z)$ be the union of $c(z)$ and all open bonds inside $c(z)$, clearly $\tilde{c}(z)\subset\Delta(z)$.

(4.3) If $z_0,z_1,\ldots,z_m$ is a path from $z_0=0$ to $z_m=z$, then there is a path from $\tilde{\Delta}(0)$ to $\tilde{\Delta}(z)$ contained in $\bigcup_{i=0}^{m}\tilde{c}(z_i)$.

This fact is proved exactly as is Lemma 4.1 of CD.

(4.4) For any positive $q<\infty$, and $x\neq y\in\mathbb{Z}^2$,

$$\operatorname{Cov}(|\tilde{c}(x)|,|\tilde{c}(y)|)\leq 2^{1-q}E(|\tilde{c}(0)|^{q+3})\|x-y\|^{-q}.$$

This fact is proved exactly as is Lemma 4.2 of CD.

Let $c=E|\tilde{c}(0)|$ and $\sigma^2=\sum_{x\in\mathbb{Z}^2}\operatorname{cov}(|\tilde{c}(0)|,|\tilde{c}(x)|)$ which must be finite by (4.4). Chebyshev's inequality gives us:

(4.5) If $z_0,\ldots,z_m$ is any path from $z_0=0$ to $z_m=z$, then

$$P\left(\sum_{i=0}^{m}|\tilde{c}(z_i)|>c(k+1)(m+1)\right)\leq\frac{\sigma^2}{k^2(m+1)}.$$

The importance of this estimate is that it implies, when combined with (4.3), that there is a $K<\infty$ such that

$$P(\hat{t}(0,z)\geq K|z|)=O(|z|^{-1})\quad\text{as }|z|\to\infty. \tag{4.6}$$

This prepares us for the main probability estimate:

(4.7) There is a $K<\infty$ such that

$$P(\hat{t}(0,z)\geq K|z|)=O(|z|^{-3})\quad\text{as } |z|\to\infty,$$

so that $\sum_z P(\hat{t}(0,z)>K|z|)<\infty$.

The proof of (4.7) uses (4.6) and the construction of 3 (almost) disjoint paths with independent travel times. The proof of pp. 598–599 of CD applies if we replace M there with $4/\alpha$.

With the main estimate one easily derives, exactly as in CD, that $\hat{A}_t$ contains (at least) a small ball with radius growing linearly in t. More precisely, from the Borel-Cantelli lemma it follows that, with $\delta=1/2K$,

$$P(C_0\cap\{x: |x|\leq\delta t\}\subset\hat{A}_t \text{ for all sufficiently large } t)=1. \tag{4.8}$$

With (4.8) and the existence of limits for $\hat{t}(0,nx)$ established the proof of the shape theorem can be completed using arguments which are now standard. By picking a finite set of x's so that balls of radii $\delta\varepsilon/2$ cover $\{x: \varphi(x)\leq 1-\varepsilon\}$ we prove that $t^{-1}\hat{A}_t\supset(1-\varepsilon)D$. To deal with the outside, reasoning which led to (4.8) also implies that if $\varphi(x)>1$ then there is a $\delta>0$ so that $t^{-1}\hat{A}_t\cap B_\delta(x)=\emptyset$ for all sufficiently large t. See CD page 595 for more details. We have now established that for any fixed $\varepsilon>0$

$$P((1-\varepsilon)D\subset t^{-1}\hat{A}_t\subset(1+\varepsilon)D \text{ for sufficiently large } t)=1. \tag{4.9}$$

Proof of the shape theorem. In addition to (4.9) we need

$$\text{If } \varepsilon>0 \quad\text{then}\quad P(u(z)>\varepsilon|z| \text{ i.o.})=P(T_z>\varepsilon|z| \text{ i.o.})=0. \tag{4.10}$$

This is a consequence of $Eu(z)^2<\infty$ and $ET_z^2<\infty$. We now wish to prove that the infected region does not sit far inside D. More precisely, we will prove

$$\text{if } \varepsilon>0 \quad\text{then}\quad P(\xi_t\cap(1-\varepsilon)tD=\emptyset \text{ for all sufficiently large } t)=1.$$

By (4.9) we know that a.s. for all large t, if $x\in(1-\varepsilon)tD$ then

$$\hat{t}(0,z)\leq\left(1-\frac{\varepsilon}{2}\right)t.$$

Add $u(0)+u(z)+T_z$ to both sides of this inequality and use (3.1) to obtain

$$t(0,z)+T_z\leq\left(1-\frac{\varepsilon}{2}\right)t+u(0)+T_z+u(z).$$

With $d=\sup_{x\in D}|x|$, we have from (4.10) that a.s., for all large t,

$$u(0)+u(z)+T_z\leq\frac{\varepsilon}{3d}|z|\leq\frac{\varepsilon}{3}(1-\varepsilon)t$$

(since $z \in (1-\varepsilon)tD$). Combining this with the previous inequality gives us

$$t(0,z) + T_z \leq \left(1 - \frac{\varepsilon}{6}\right)t,$$

and so z belongs to ζ_t, not ξ_t. This proves (4.11) and

$$P(t(1-\varepsilon)D \cap C_0 \subset \zeta_t \text{ for all sufficiently large } t) = 1.$$

On the other hand, if $z \in \xi_t$ or $z \in \zeta_t$, then $t(0,z) \leq t$ so certainly $\hat{t}(0,z) \leq t$, and by (4.9) $z \in (1+\varepsilon)tD$. That is,

$$P(\xi_t \subset (1+\varepsilon)tD \text{ for all sufficiently large } t) = 1$$

and

$$P(\zeta_t \subset (1+\varepsilon)tD \text{ for all sufficiently large } t) = 1.$$

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