A Simple Proof of the Stability Criterion of Gray and Griffeath

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Summary. Gray and Griffeath studied attractive nearest neighbor spin systems on the integers having "all 0's" and "all 1's" as traps. Using the contour method, they established a necessary and sufficient condition for the stability of the "all 1's" equilibrium under small perturbations. In this paper we use a renormalized site construction to give a much simpler proof. Our new approach can be used in many situations as a substitute for the contour method.

Introduction

Consider a one dimensional nearest neighbor spin system on $\mathbb{Z}$ with birth rates β_{ij} and death rates δ_{ij}. That is, ξ_t is a Markov process with state space $\{0, 1\}^{\mathbb{Z}}$ which has

$$P(\xi_{t+s}(x)=1 \mid \xi_t(x-1)=i\, \xi_t(x)=0\, \xi_t(x+1)=j) = \beta_{ij} s + o(s),$$
$$P(\xi_{t+s}(x)=0 \mid \xi_t(x-1)=i\, \xi_t(x)=1\, \xi_t(x+1)=j) = \delta_{ij} s + o(s)$$

as $s \to 0$. Gray and Griffeath (1982) investigated the class of models which have

$$\beta_{00}=0 \quad \beta_{11} \geqq \beta_{01}, \beta_{10} \quad \delta_{11}=0 \quad \delta_{00} \geqq \delta_{01}, \delta_{10}.$$

The inequalities say that the system is "attractive," and imply that if $\xi_0(x) \leqq \bar{\xi}_0(x)$, then copies of the process starting from these initial configurations can be constructed so that $\xi_t(x) \leqq \bar{\xi}_t(x)$ for all x and t. The equalities $\beta_{00}=0$ and $\delta_{11}=0$ imply that the states $\mathbf{0}=$ "all 0's" and $\mathbf{1}=$ "all 1's" are absorbing states.

Gray and Griffeath investigated the stability of the fixed point $\mathbf{1}$ under the perturbation $\beta_{ij}^{\varepsilon}=\beta_{ij}, \delta_{ij}^{\varepsilon}=\delta_{ij}+\varepsilon$. The perturbed flip rates are attractive, so if we start the perturbed system from $\xi_0^{\varepsilon}(x) \equiv 1$ then $\xi_t^{\varepsilon} \Rightarrow \nu_1^{\varepsilon}$ where ν_1^{ε} is a transla-

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tion invariant stationary distribution. (See Liggett (1985) p. 135.) If $v_1^\varepsilon\{\xi: \xi(x) = 1\}\uparrow 1$ as $\varepsilon\downarrow 0$ (the last probability does not depend on x), then we say $\mathbf{1}$ is stable. If $v_1^\varepsilon\{\xi: \xi(x)=1\}=0$ for all $\varepsilon>0$, then we say $\mathbf{1}$ is unstable. Although it is not immediately clear from the definition that something which is not unstable is stable, the next result, due to Gray and Griffeath (1982), shows this is true and identifies when the two cases occur.

Theorem. *If $\beta_{01}+\beta_{10}\leqq\delta_{01}+\delta_{10}$, then $\mathbf{1}$ is unstable.*
If $\beta_{01}+\beta_{10}>\delta_{01}+\delta_{10}$, then $\mathbf{1}$ is stable.

To see the motivation for and content of this result consider two examples.

Example 1. Pure Birth Process. $\delta_{ij}=0$. In this example the result says that if $\beta_{01}+\beta_{10}>0$, then $\mathbf{1}$ is stable. An important special case is $\beta_{ij}=i+j$. In this special case if we change variables $\varepsilon=1/\lambda$, the result is the contact process (run at $1/\lambda$ times the usual speed). For the contact process, the theorem says that "$\lambda_c<\infty$", that is, if λ is large there is a nontrivial stationary distribution.

Example 2. Biased voter model. $\beta_{ij}=\lambda(i+j)$, $\delta_{ij}=i+j$. In this example the result implies $\mathbf{1}$ is stable if $\lambda>1$ and unstable if $\lambda\leqq 1$. In particular, in the voter model ($\lambda=1$) the equilibrium $\mathbf{1}$ is not stable under perturbation.

The first conclusion in the theorem is easy to prove. Suppose $\varepsilon=0$ and we start with $\xi_0(0)=0$ and $\xi_0(x)=1$ for $x\neq 0$. Since $\beta_{00}=0$ and $\delta_{11}=0$, the state at any time will be $\mathbf{1}$ or have the form $\xi_t(x)=0$ if $l_t\leqq x\leqq r_t$ and $=1$ otherwise. It is easy to see that when $r_t-l_t\geqq 1$

$$r_t \to \begin{cases} r_t+1 & \text{at rate } \delta_{01} \\ r_t-1 & \text{at rate } \beta_{01} \end{cases} \text{ and } l_t \to \begin{cases} l_t+1 & \text{at rate } \beta_{10} \\ l_t-1 & \text{at rate } \delta_{10}, \end{cases}$$

so the interval grows by 1 at rate $\delta_{01}+\delta_{10}$, and shrinks by 1 at rate $\beta_{01}+\beta_{10}$. If we add ε to the death rates then the interval grows at rate at least $\delta_{01}+\delta_{10}+2\varepsilon$ (0's outside $[l_t, r_t]$ may help the growth), and shrinks at rate $\beta_{01}+\beta_{10}$. So if $\delta_{01}+\delta_{10}+2\varepsilon>\beta_{01}+\beta_{10}$ there is positive probability that a single 0 leads to an interval of 0's which grows at a positive linear rate. Since 0's appear at a positive rate, an easy argument shows that starting from $\mathbf{1}$ the system converges to $\mathbf{0}$.

The intuition behind the second conclusion in the theorem is similar (intervals of 0's tend to shrink), but this result is much more difficult to prove because 0's outside an interval may help its growth. Gray and Griffeath used the "contour method" to prove their result. In essence, they developed a power series for $v_1^\varepsilon\{\xi: \xi(x)=1\}$ and proved that the series had a positive radius of convergence. When $\beta_{11}\geqq\beta_{01}+\beta_{10}$ the proof was a little tedious (8 journal pages). When $\beta_{11}<\beta_{01}+\beta_{10}$ they had to "resort to unpleasant surgical operations on the contour which obfuscate the essential idea". In the next section we will give a proof of Gray and Griffeath's result which does not depend on the size of β_{11}, and is much simpler than their proof. As the first paragraph of the proof should indicate, the method of proof can be applied to prove results about other systems.

Proof of Stability

If we leave out a few details the main idea of the proof can be described in three sentences. We show that when the process with $\varepsilon = 0$ is viewed on suitable length and time scales it dominates 1-dependent oriented percolation with $p = 1 - 3^{-37}$. Since the boxes involved in the construction have finite area the probability of a spontaneous deaths in a box is $< 3^{-37}$ if $\varepsilon < \varepsilon_0$. The last observation implies that the processes with $\varepsilon < \varepsilon_0$ dominate 1-dependent oriented percolation and the stability result follows from facts in Section 10 of Durrett (1984). It will take a number of words (mostly definitions) to turn the sketch into a proof. We will have to replace 3^{-37} by any $\gamma > 0$ to get the desired result. We hope the sketch has made the following *philosophy* clear: the hard work is done for the process with $\varepsilon = 0$ and then the continuity of probabilities of events in boxes of finite area extends the result to small $\varepsilon > 0$.

Turning to the details, we begin by considering what the process with $\varepsilon = 0$ looks like starting from an interval of 1's. In what follows it will be convenient to regard ξ_t as a set-valued process ($= \{x : \xi_t(x) = 1\}$) and think of the points in ξ_t as occupied by particles. Suppose $\varepsilon = 0$ and start with $\xi_0 = [l_0, r_0]$. Since $\beta_{00} = 0$ and $\delta_{11} = 0$, the state at any time will be ϕ or an interval $[l_t, r_t]$. It is easy to see that when $r_t - l_t \geqq 1$

$$r_t \to \begin{cases} r_t + 1 & \text{at rate } \beta_{10} \\ r_t - 1 & \text{at rate } \delta_{10} \end{cases} \quad \text{and} \quad l_t \to \begin{cases} l_t + 1 & \text{at rate } \delta_{10} \\ l_t - 1 & \text{at rate } \beta_{01}. \end{cases}$$

The right edge r_t has drift $c_2 = \beta_{10} - \delta_{10}$, and the left edge l_t has drift $c_1 = \delta_{01} - \beta_{01}$. (To help remember the notation, recall $2 = 10$ in binary.) Our assumption is that $c_2 - c_1 > 0$, so:

(1) If we pick $\delta > 0$ and L large, then the system starting with $[-\delta L, \delta L]$ occupied at time 0 will at time L contain $[c_1 L, c_2 L]$ and be contained in $[(c_1 - 2\delta) L, (c_2 + 2\delta) L]$ with high probability.

With (1) established, it is easy to show that the system with $\varepsilon = 0$ dominates 1-dependent oriented percolation. See Fig. 1 for help with the definitions. The first coordinate corresponds to space and the second to time. We set:

$$0 < \delta < (c_2 - c_1)/10 \quad b = (c_2 - c_1 - 2\delta)/2 \geqq 4\delta > 0 \quad a = (c_2 + c_1)/2$$

$$v_{m,n} = ((na + mb) L, nL) \in \mathbb{Z}^2 \quad \mathscr{L} = \{(m, n) \in \mathbb{Z}^2 : m + n \text{ is even}\}$$

A = the parallelogram with vertices $(-2bL, 0)$, $(2bL, 0)$,
$((a + 2b) L, L)$, $((a - 2b) L, L)$

$$A_{m,n} = v_{m,n} + A \quad I = [-\delta L, \delta L] \quad I_{m,n} = (na + mb) L + I$$

Here $x + S = \{x + y : y \in S\}$. In what follows we will often use z to denote points of $\mathscr{L}$. In this case for obvious typographical reasons we let
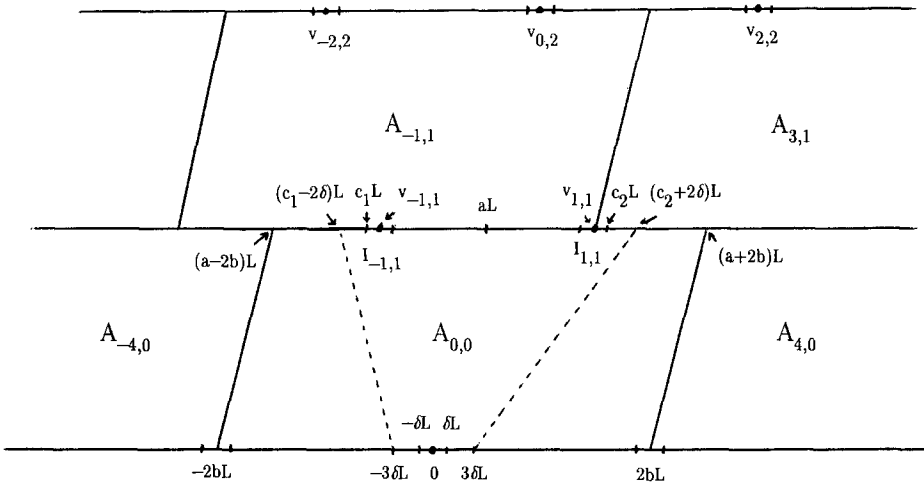
$$I(z) = I_{z_1, z_2}.$$

Fig. 1.

To see the reason for the above definitions, notice that $a+b=c_2-\delta$ and $a-b=c_1+\delta$, so

$$(2) \qquad [c_1 L, c_2 L] \supset (I_{-1,1} \cup I_{1,1}).$$

A second important observation is that

$$a+2b \geq a+b+4\delta > c_2+2\delta,$$

and the parallelogram is convex so the dotted lines in Fig. 1 lie in A. Combining the last two observation with (1) shows:

(3)   With high probability, the right edge of the process starting with $[-\delta L, \delta L]$ occupied will not escape from the "box" A by time L. An analogous statement holds for the left edge.

We will say that (m, n) is open and set $\eta_{m,n}=1$ if the process starting with $I_{m,n}$ occupied at time nL, which we will call $\zeta_t^{m,n}$, contains $I_{m-1,n+1}$ and $I_{m+1,n+1}$ at time $(n+1)L$ and has $\zeta_t^{m,n} \times \{t\} \subset A_{m,n}$ for $nL \leq t \leq (n+1)L$. For the last statement to be meaningful, we have to have all the processes referred to constructed on the same probability space. To do this, we use an approach similar to the one in Sect. 2 of Gray and Griffeath (1982)

First of all, suppose without loss of generality that $\max(\delta_{00}, \beta_{11})=1$. For each $x \in \mathbb{Z}$ let $\{B_n^x, n \geq 1\}$, $\{D_n^x, n \geq 1\}$ be the arrival times of two rate-one Poisson processes; let $\{U_n^x, n \geq 1\}$, $\{V_n^x, n \geq 1\}$ be i.i.d. sequences of random variables which have uniform distributions on $(0, 1)$; and suppose that all these processes are independent. The B_n^x and D_n^x are possible birth and death times of particles at x. At these times we look at the state of $x-1$, x, and $x-1$ and use U_n^x or V_n^x to see if a birth or death should occur. For example, if at time $t=B_n^x$, $\xi_t(x)=0$, $\xi_t(x-1)=i$, and $\xi_t(x+1)=j$, then x will become occupied if and only if $U_n^x < \beta_{ij}$; if $\xi_t(x)=1$, then nothing happens. A similar definition applies at times $t=D_n^x$. Since thinning a rate 1 Poisson process by flipping a coin with probability

p of heads results in a rate p Poisson process, it is easy to see that the above recipe allows us to construct, on one probability space, processes $\xi_s^{A,t} s \geqq t$ which start with A occupied at time t. For more details see Sect. 2 of Gray and Griffeath (1982).

The last paragraph allows us to construct

$$\zeta_t^{m,n} \equiv \xi_t^{I_{mn},nL}, \ t \geqq nL.$$

This construction has the property that if $\xi_t^A \supset B$, then

(4) $$\xi_s^A \supset \xi_s^{B,t} \quad \text{for all } s \geqq t.$$

Moreover, one can check that the $\eta_{m,n}$ have the following three properties:

(i) The random variables $\eta(z), z \in \mathscr{L}$ are 1-dependent, that is, if we let $\|(m,n)\| = (|m| + |n|)/2$, and $z_1, \ldots z_k$ are points in $\mathscr{L}$ with $\|z_i - z_j\| > 1$ for $i \neq j$, then $\eta(z_1), \ldots \eta(z_k)$ are independent.

(ii) If there is a sequence $z_i = (m_i, i)$ so that $\eta(z_i) = 1$ and $m_{i+1} \in \{m_i - 1, m_i + 1\}$ for $0 \leqq i \leqq n - 1$, then

$$\xi_{nL}^{I(z_0)} \supset I(z_n).$$

(iii) If $\gamma > 0$, then we can pick L large enough so that $P(\eta(z) = 1) \geqq 1 - \gamma$.

(i) follows from the fact that if $z, w \in \mathscr{L}$ have $\|z - w\| > 1$ then the parallelograms A_z and A_w do not intersect. To check (ii) observe that (4) and the definition of η imply by induction that

$$\xi_{nL}^{I(z_0)} \supset I(z_m) \quad \text{for } m \geqq 0.$$

Finally, to check (iii) notice that if L is large then (1)-(3) imply

$$\xi_L^{[-\delta L, \delta L]} \supset [c_1 L, c_2 L] \supset (I_{-1,1} \cup I_{1,1}),$$

and

$$\xi_t^{[-\delta L, \delta L]} \times \{t\} \subset A_{0,0} \quad \text{for all } t \leqq L$$

have probabilities close to one.

Let $W_n = \{m : (m, n) \in \mathscr{L}$ can be reached, i.e., there is a path starting at some $(l, 0) \in \mathscr{L}$ and ending at (m, n) with the properties given in (ii)$\}$. (i) and (iii) imply that $\{W_n : n \geqq 0\}$ dominates a 1-dependent oriented percolation having open sites with probability $1 - \gamma$. In Durrett (1984) (see (1) on p. 1026), a result was proved which implies that

(5) $$P(m \in W_n) \geqq \rho(\gamma) \quad \text{with } \rho(\gamma) \to 1 \quad \text{as } \gamma \to 0.$$

(The argument there provides lower bounds on the probability of an infinite path of open sites starting at a given point, and hence by reversing time, lower bounds on the probability of an open path from (m, n) down to $2\mathbb{Z} \times \{0\}$.)

At this point we have worked ridiculously hard to prove a trivial result: if we start the system with $\varepsilon = 0$ from $\xi_0 = \mathbb{Z}$ then it does not converge to ϕ.

(It is $=\mathbb{Z}$ for all t!) However, as announced at the beginning of the proof, our reasoning allows us to easily conclude that the same result holds when $\varepsilon > 0$ is small. To construct the perturbed process ξ_t^ε, we add independent rate-ε Poisson processes $\{T_n^x, n \geq 1\}$ to the above construction. At times T_n^x we kill the particle at x (if one is present). To analyze ξ_t^ε, we define new random variables $\eta_{m,n}^\varepsilon$ by $\eta_{m,n}^\varepsilon = 1$ if (and only if) $\eta_{m,n} = 1$ and

$$H_{m,n} \equiv \{(x, T_n^x): n \geq 1, x \in \mathbb{Z}\} \cap A_{m,n}$$

is ϕ. Let $\gamma > 0$ and pick L so that (i)–(iii) hold for $\eta_{m,n}$. It is easy to see that (i) and (ii) are valid for $\eta_{m,n}^\varepsilon$. If ε is small, $P(H_{m,n} \neq \phi) < \gamma$, so $P(\eta_{m,n}^\varepsilon = 1) \geq 1 - 2\gamma$ and (iii) holds with γ replaced by 2γ.

Using (5) now gives

(6) $P(m \in W_n) \geq \rho(2\gamma)$ with $\rho(2\gamma) \to 1$ as $\gamma \to 0$.

To get from this to the desired result we observe that since the flip rates are "attractive" (see Liggett (1985), p. 135 again),

$$v_1^\varepsilon \{\xi: \xi(x) = 1\} = \lim_{n \to \infty} P(x \in \xi_{nL}^\varepsilon).$$

Translation invariance implies that the last probability does not depend on x. So by picking a point x_n which lies in one of the $I_{m,n}$, it follows from the construction and (6) that

$$P(0 \in \xi_{nL}^\varepsilon) = P(x_n \in \xi_{nL}^\varepsilon) \geq P(m \in W_n) \geq \rho(2\gamma).$$

$\gamma > 0$ is arbitrary so we have proved the stability half of our theorem.

References

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