Stochastic Processes and their Applications 5 (1977) 213–220.

WEAK CONVERGENCE WITH RANDOM INDICES

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Received 15 April 1976
Revised 13 December 1976

Suppose $\{X_n, n \geq 0\}$ are random variables such that for normalizing constants $a_n > 0$, b_n, $n \geq 0$ we have $Y_n(\cdot) = (X_{[n\cdot]} - b_n)/a_n \Rightarrow Y(\cdot)$ in $D(0,\infty)$. Then a_n and b_n must vary in specific ways and the process Y possesses a scaling property. If $\{N_n\}$ are positive integer valued random variables we discuss when $Y_{N_n} \Rightarrow Y$ and $Y'_n = (X_{[N_n\cdot]} - b_n)/a_n \Rightarrow Y'$. Results given subsume random index limit theorems for convergence to Brownian motion, stable processes and extremal processes.

weak convergence	extremal process
random indices	regular variation
stable process	mixing
Brownian motion	

1. Introduction

The main question we consider is the following: If there is a functional limit theorem for a sequence of random variables (rv's) $\{X_n, n \geq 0\}$ then how is the convergence affected when the indices are rv's.

To state the question precisely we introduce the following:

Basic Assumption. For some $a_n > 0$, b_n, $n \geq 1$, $Y_n := (X_{[n\cdot]} - b_n)/a_n$ converges weakly as a sequence of random elements of $D = D(0,\infty)$ — the space of right continuous functions with finite left limits at each $t > 0$ — to a process $Y \in D$ which has a nondegenerate distribution at each $t > 0$.

For what follows, $Y_n \Rightarrow Y$ denotes weak convergence in the metric space $D(0,\infty)$ which means $Y_n \Rightarrow Y$ in $D[r,s]$ for each $0 < r < s < \infty$ which are not

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** S.I. Resnick was supported initially by NSF Grant MPS74–21416 and then by NSF Grant OIP75–14513 while on leave from Stanford University. The hospitality of CSIRO, Division of Mathematics and Statistics, Canberra and the Department of Statistics, SGS, Australian National University is gratefully acknowledged. Present address: Statistics Department, Colorado State University, Fort Collins, CO, U.S.A.

fixed discontinuities of the limit Y. (See [1, 7].) Observe that the convergence we have postulated is the usual one encountered in the convergence of random walks to stable processes (see [1, 23, 24]) and partial maxima to extremal processes (see [28, 29, or 31]).

In Section 2 we show that under the basic assumption the norming constants a_n and b_n must vary in specific ways and Y must be stochastically continuous. These results owe much to Lamperti (1962) and Weissman (1975). Our contribution is that stochastic continuity of the limit is now a conclusion rather than an assumption.

In Section 3 we discuss some conditions for the convergence of Y_{N_n} and $Y'_n = (X_{[N_n \cdot]} - b_n)/a_n$ which apply when the sequences $\{X_n\}$ and $\{N_n\}$ are independent or dependent. Of interest here is that no special properties of Y are needed to prove the results beyond those obtained from the basic assumption so that many random index limit theorems can be obtained as special cases. Section 4 deals with the convergence of $Y_{N_n}(1)$. Such results have been of interest in estimation theory using random sample sizes. We also give a counterexample to a conjecture of Mogyoródi and Guiasu.

2. Consequences of the basic assumption

Theorem 1. *If the basic assumption holds, then for each $s > 0$ the following two limits exist:*

$$\lim_{n\to\infty} a_{[ns]}/a_n = \alpha(s) > 0, \tag{a}$$

$$\lim_{n\to\infty} (b_{[ns]} - b_n)/a_n = \beta(s) \tag{b}$$

and satisfy

$$\{Y(st); t > 0\} \stackrel{d}{=} \{\alpha(s)\, Y(t) + \beta(s); t > 0\}. \tag{c}$$

Furthermore, for some constant h one of the following possibilities holds:

(i) $\alpha(s) = s^\rho$, $\rho > 0$, $\beta(s) = h(s^\rho - 1)$;
(ii) $\alpha(s) = 1$, $\beta(s) = h \log s$;
(iii) $\alpha(s) = s^\rho$, $\rho < 0$, $\beta(s) = h(1 - s^\rho)$.

Proof. Let $T = \{t > 0 : \mathrm{P}[Y(t) \neq Y(t-)] = 0\}$. From [1], p. 124, $(0, \infty) - T$ is countable. Pick $t, s > 0$ so that $t, st \in T$. Using Theorem 5.5 of [1] gives that if $s_n \to s$,

$$(X_{[ns_nt]} - b_{[ns_n]})/a_{[ns_n]} \Rightarrow Y(t),$$

$$(X_{[ns_nt]} - b_n)/a_n \Rightarrow Y(st).$$

Now by the convergence to types theorem there exist $\alpha(s) > 0$ and $\beta(s)$ which satisfy (a), (b), and

$$Y(st) \stackrel{d}{=} \alpha(s)Y(t) + \beta(s)$$

so that

$$\alpha(st)Y(1) + \beta(st) \stackrel{d}{=} \alpha(s)\alpha(t)Y(1) + \alpha(s)\beta(t) + \beta(s)$$

(where we have assumed for convenience $1 \in T$). Therefore for $t, st \in T$,

$$\alpha(st) = \alpha(s)\alpha(t),$$

$$\beta(st) = \alpha(s)\beta(t) + \beta(s)$$

for which (i), (ii), (iii) are the only measurable solutions with $\alpha(1) = 1$.

To show equality of the finite dimensional distributions in (c) it suffices to repeat the above procedure using a multivariate analogue of the convergence to types theorem. See for example [3], p. 148.

Examples of the convergence described in (i) are weak convergence to stable laws of index $1/\rho$ $(0 < 1/\rho \leq 2,\ \rho \neq 1)$ or to the extremal process generated by $\Phi_{1/\rho}(x) = \exp\{-x^{-1/\rho}\}$, $x > 0$. The variation in (iii) arises in weak convergence to the extremal process generated by $\Psi_{|1/\rho|}(x) = \exp\{-(-x)^{|1/\rho|}\}$, $x \leq 0$, $= 1$, $x > 0$. Finally situation (ii) arises when the limit Y is an extremal process generated by $\Lambda(x) = \exp\{-e^{-x}\}$ and also (as pointed out to us by L. de Haan) in connection with weak convergence to a stable law of index 1 as follows: If ξ_n, $n \geq 1$ are iid rv's in the domain of attraction of a stable law of index 1, set $\bar{X}_n = \sum_{i=1}^{n} \xi_i / n$. Then there exist a_n, b_n varying as described in (ii) such that $(\bar{X}_{[n\cdot]} - b_n)/a_n \Longrightarrow Y$.

Theorem 2. *If $[X_n, n \geq 0]$ satisfies the basic assumption, then Y is stochastically continuous.*

Proof. If $u \in T$, then for each $\varepsilon > 0$

$$\lim_{s \nearrow u} \mathbf{P}[|Y(s) - Y(u)| > \varepsilon] = 0.$$

From Theorem 1, $\{Y(s\tau), \tau > 0\} \stackrel{d}{=} \{\alpha(s)Y(\tau) + \beta(s), \tau > 0\}$. If $t > 0$ is chosen arbitrarily and $s = t/u$ then letting $v' = v/s$ gives

$$\lim_{v \nearrow t} \mathbf{P}[|Y(v) - Y(t)| > \varepsilon] = \lim_{v' \nearrow u} \mathbf{P}[|Y(sv') - Y(su)| > \varepsilon].$$

Since $(Y(sv'), Y(su)) \stackrel{d}{=} (\alpha(s)Y(v') + \beta(s), \alpha(s)Y(u) + \beta(s))$, the above equals

$$\lim_{v' \nearrow u} \mathbf{P}[|Y(v') - Y(u)| > \varepsilon/\alpha(s)] = 0$$

(recall $u \in T$). Therefore Y is left stochastically continous at t and since right stochastic continuity follows from Y being a.s. right continuous the proof is complete.

3. Functional limit theorems with random indices

Throughout this section we assume the basic assumption holds. We will be concerned with conditions that guarantee

$$Y_{N_n} \Rightarrow Y, \tag{1}$$

$$Y'_n = (X_{[N_n \cdot]} - b_n)/a_n \Rightarrow Y' \tag{2}$$

where $\{N_n\}$ are integer valued random variables. The first step is the following:

Theorem 3. *If* $(Y_n, N_n/n) \Rightarrow (Y, N)$ *with* $P(0<N<\infty)=1$, *then* $Y'_n \Rightarrow Y(N\cdot)$ *and* $Y_{N_n} \Rightarrow [Y(N\cdot)-\beta(N)]/\alpha(N)$.

Proof. For $t>0$ and $f\in D$ let $\psi(f,t)=f(t\cdot)\in D$. ψ is continuous on $D\times R$ so by the continuous mapping theorem (5.1 in [1]), $Y'_n=\psi(Y_n,N_n/n)\Rightarrow\psi(Y,N)=Y(N\cdot)$.

To prove the second result, note that the mapping $\psi'(f,t)=(f(t\cdot),t)$ is continuous so that

$$(Y'_n, N_n/n)=\psi'(Y_n,N_n/n)\Rightarrow\psi'(Y,N)=(Y(N\cdot),N).$$

Applying the theorem for sequences of continuous mappings (Theorem 5.5 in [1]) with

$$\psi_n(f,x)=f(\cdot)\frac{a_n}{a_{[xn]}}+\left(\frac{b_n-b_{[xn]}}{a_n}\right)\frac{a_n}{a_{[nx]}}$$

and the fact that the convergences in (a) and (b) of Theorem 1 are uniform gives

$$Y_{N_n}=\frac{a_n}{a_{N_n}}Y'_n+\frac{b_n-b_{N_n}}{a_{N_n}}\Rightarrow\frac{Y(N\cdot)-\beta(N)}{\alpha(N)}.$$

It is easy to construct examples to show that in Theorem 3 if Y and N are dependent, it is not necessarily true that $(Y(N\cdot)-\beta(N))/\alpha(N)\overset{d}{=}Y$. However this is true in the independence case by Theorem 1. To get general sufficient conditions for independence of N and Y we need (cf. [1, 12, 13, 21]):

Definition. Suppose $\{V_n; n\geq 0\}$ are random elements of a metric space S defined on $(\Omega,\mathcal{F},P)$. The sequence V_n is *R-mixing* if for some V, $\mathbf{P}\{V_n\in\cdot\mid E\}\Rightarrow\mathbf{P}\{V\in\cdot\}$ for all $E\in\mathcal{F}$ such that $P(E)>0$.

The reason for using this property can be seen in the following characterization ([1, 21]):

Lemma 1. *If* $V_n\Rightarrow V$, *then* V_n *is R-mixing if and only if for any sequence of random elements* Z_n *of a metric space* S' *such that* $Z_n\xrightarrow{P}Z$ *we have* $(V_n,Z_n)\Rightarrow(V,Z)$ *where* V *and* Z *are independent.*

From this lemma we can immediately conclude:

Theorem 4. *If Y_n is R-mixing and $N_n/n \xrightarrow{P} N$ with $\mathbf{P}(0<N<\infty)=1$, then $Y_{N_n} \Rightarrow Y$ and $Y'_n \Rightarrow \alpha(N)Y+\beta(N)$.*

Proof. From Theorem 4 and Lemma 1, $Y_{N_n} \Rightarrow [Y(N\cdot)-\beta(N)]/\alpha(N)$ where Y and N are independent, so by Theorem 1 the limit has the same law as Y. The second statement follows from the calculations in Theorem 3.

It is easy to show that if $\{X_n; n\geq 0\}$ has a trivial tail σ-field then Y_n is R-mixing. (See [2], p. 45.) Using the Hewitt–Savage zero–one law now gives the conclusions of Theorem 4 for random walks and partial maxima of iid rv's. For examples of dependent sequences for which the mixing property can be verified and Theorem 4 applied, see [21].

Minor modifications of these methods allow one to quickly prove variants where either $\mathbf{P}[N=0]>0$ or $N_n/c_n \xrightarrow{P} N$ and c_n are constants not asymptotic to n. See [19, 20].

4. Convergence of $Y_{N_n}(1)$

To complete our study, we will derive conditions for the convergence of $Y_{N_n}(1)$. In general this will require additional hypotheses since the convergence of processes discussed above occurs in $D(0,\infty)$ and so we only have convergence of one-dimensional distributions for times at which the limit process $Y(N\cdot)$ is continuous in probability.

Theorem 5. *If $Y_n(1) \Rightarrow Y$ and*

(1) *$N_n/n \xrightarrow{P} N$ with $\mathbf{P}(0<N<\infty)=1$,*

(2) *$Y_n(1)$ is R-mixing with respect to $\sigma(N)$, that is, for each A such that $\mathbf{P}\{N\in A\}>0$*

$$\mathbf{P}\{Y_n(1)\in\cdot \mid N\in A\}\to\mathbf{P}\{Y\in\cdot\},$$

(3) *if $\Delta_{n,c}=\max_{|m-n|<nc}|Y_m(1)-Y_n(1)|$, then*

$$\lim_{c\to 0}\ \sup_{B:\mathbf{P}\{N\in B\}>0}\ \limsup_{n\to\infty} \mathbf{P}\{\Delta_{n,c}>\varepsilon \mid N\in B\}=0.$$

Then $Y_{N_n}(1) \Rightarrow Y(1)$.

Note that we do not suppose the basic assumption holds so we cannot apply the continuous mapping theorem as we did in Theorems 3 and 4. We can prove

Theorem 5 however by using computations from the proof of the continuous mapping theorem as a substitute for applying the result.

Proof. Given $\varepsilon, \eta > 0$. Choose $s > 1$ so large that $\mathbf{P}[N \notin [1/s, s]] < \varepsilon$. Let m be a positive integer and let $\delta = (s - 1/s)/m$ (so that $\delta \to 0$ if $m \to \infty$) and $t_k = 1/s + k\delta$, $k = 0, \ldots, m$. If $Y_{N_n}(1) \leq x$, $N \in [1/s, s]$ and $|N_n/n - N| \leq \delta$, then $t_{k-1} \leq N < t_k$ for some k and $Y_{[nt_k]}(1) \leq x + \eta$ unless the oscillation of $Y_m(1)$ when $nt_{k-2} \leq m \leq nt_{k+1}$ is $> \eta$. Therefore

$$\mathbf{P}[Y_{N_n}(1) \leq x] \leq \varepsilon + \mathbf{P}[|N_n/n - N| > \delta] \tag{1'}$$

$$+ \sum_{k=1}^{m} \mathbf{P}[t_{k-1} \leq N < t_k,\ Y_{[nt_k]}(1) \leq x + \eta] \tag{2'}$$

$$+ \sum_{k=1}^{m} \mathbf{P}[t_{k-1} \leq N < t_k,\ \max_{nt_{k-2} \leq m \leq nt_{k+1}} |Y_m(1) - Y_{[nt_k]}(1)| > \eta]. \tag{3'}$$

Using the three hypotheses to evaluate the limits of the correspondingly numbered terms above we obtain that

$$\lim_{s \nearrow \infty} \limsup_{n \to \infty} (1') = 0,$$

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} (3') = 0,$$

$$\lim_{n \to \infty} (2') = \mathbf{P}[N \in [1/s, s]]\mathbf{P}[Y \leq x + \eta]$$

if $x + \eta$ is a continuity point. Therefore for all points x

$$\limsup_{n \to \infty} \mathbf{P}[Y_{N_n}(1) \leq x] \leq \mathbf{P}[Y \leq x].$$

A related lower bound is derived similarly and the proof is complete.

Mogyoródi (1967) and Guiasu (1971) have conjectured that for Theorem 5 it was sufficient that (3) hold with $B = (0, \infty)$ (this is Anscombe's condition). The example given below shows this is not true. [Note: There is also a counterexample due to Richter (1965).]

Example. Let $U, Z_1, Z_2, \ldots$ be independent r.v.'s such that U has a uniform distribution on $(0, 1)$ and for each $n \geq 1$, Z_n has a normal distribution with mean zero and variance one. Let $Y_n = n^{-1/2} \sum_{k=1}^{n} Z_k$, let $I(\omega) = \{[2^n U(\omega)] + 1, 1 \leq n < \infty\}$, and let $Y_n' = Y_n 1_{\{n \notin I\}}$.

It is easy to see that Y_n' is R-mixing and converges to a normal distribution. To do this, observe that if $P(E) > 0$ then for each $a < b$

$$|\mathbf{P}\{Y_n' \in [a, b] \mid E\} - \mathbf{P}\{Y_n \in [a, b] \mid E\}| \leq \mathbf{P}\{Y_n \neq Y_n'\}/P(E)$$

and by construction the right-hand side goes to 0. A similar argument shows that if $\Delta'_{n,c} = \max_{|m-n|<nc} |Y'_m - Y'_n|$, then for all $\varepsilon > 0$

$$\lim_{c \downarrow 0} \limsup_{n \to \infty} \mathrm{P}\{\Delta'_{n,c} > \varepsilon\} = 0$$

so that Anscombe's condition is satisfied. However it is easy to see that the conclusion of Theorem 5 is false: if $N_n = [2^n U + 1]$, then $N_n/2^n \xrightarrow{P} U$ but $Y'_{N_n} \not\Rightarrow 0$.

Acknowledgment

We would like to thank David Aldous for several helpful comments.

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