Journal of Statistical Physics, Vol. 55, Nos. 5/6, 1989

Correlation Lengths for Oriented Percolation

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Received December 9, 1988

Oriented percolation has two correlation lengths, one in the "space" and one in the "time" direction. In this paper we define these quantities for the two-dimensional model in terms of the exponential decay of suitably chosen quantities, and study the relationship between the various definitions. The definitions are used in a companion paper to prove inequalities between critical exponents.

KEY WORDS: Correlation lengths; oriented percolation.

1. INTRODUCTION

We begin by describing the model under consideration. Let $\mathscr{L} = \{(m, n) \in \mathbb{Z}^2$: $m + n$ is even$\}$ and make $\mathscr{L}$ into a graph by drawing an oriented arc from each $z \in \mathscr{L}$ to $z + (1, 1)$ and to $z + (-1, 1)$. Each arc, also called a bond, is independently open with probability p and closed with probability $1 - p$. An open bond indicates that flow is allowed in the direction of the orientation. With this in mind, we write $x \to y$ (and say y can be reached from x) if there is an open path from x to y, i.e., there is a sequence of sites in $\mathscr{L}$, $x = x_0, x_1,..., x_n = y$, so that for each $m \leqslant n$ the arc from x_{m-1} to x_m is open.

Thinking of the vertical direction as time, we set

$$\xi_n^A = \{j\colon \quad \text{for some } i \in A,\ (i, 0) \to (j, n)\}$$

ξ_n^A is a set-valued Markov process often referred to as the "discrete time contact process." The superscript A denotes the initial state, i.e., $\xi_0^A = A$. Let

$$\tau^A = \inf\{n\colon \xi_n^A = \varnothing\}$$

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0022-4715/89/0600-0965$06.00/0

be the time at which the process dies out, i.e., reaches the absorbing state $\varnothing$. Let

$$p_c = \inf\{p: P(\tau^0 = \infty) > 0\}$$

where τ^0 is short for $\tau^{\{0\}}$. Let

$$C_0 = \{(x, n): (0, 0) \to (x, n)\}$$

be the cluster of the origin. It is easy to see that $\{|C_0| = \infty\} = \{\tau^0 = \infty\}$, where $|C_0|$ = the number of sites in C_0. So the definition of p_c given above coincides with the usual one for oriented bond percolation. See ref. 4 for more details.

We will now begin to define our correlation lengths. We need one for the time and one for the space direction. Following the practice in the physics literature, we will call these parallel ($||$) and perpendicular ($\perp$). For each correlation length we need a definition for the subcritical ($p < p_c$) and supercritical ($p > p_c$) regimes. To formulate our definitions, we need an argument which appears many times in the literature and is commonly known as "*supermultiplicativity*." Suppose A_n are events with

$$P(A_{n+m}) \geqslant P(A_n)\, P(A_m)$$

If we let $a_n = -\log P(A_n)$, then

$$a_{n+m} \leqslant a_n + a_m$$

An easy argument (see ref. 4, p. 1017) shows

$$a_n/n \to \inf_{m \geqslant 1} a_m/m$$

and if we use γ to denote the right-hand side then

$$P(A_n) \leqslant e^{-\gamma n} \qquad \text{for all } n$$

Definition 1. Since $|\xi_n^0| \geqslant 1$ on $\xi_n^0 \neq \varnothing$, it follows easily that

$$P(\tau^0 \geqslant n + m \mid \tau^0 \geqslant m) \geqslant P(\tau^0 \geqslant n)$$

so "supermultiplicativity" implies that

$$\gamma_{||}(p) = \lim_{n \to \infty} \left[-\frac{1}{n} \log P(\tau^0 \geqslant n) \right] \tag{1.1}$$

exists for all p. For more details see p. 1017 of ref. 4, where it is shown that $\gamma_{||}(p) > 0$ for $p < p_c$. Let $L_{||}(p) = 1/\gamma_{||}(p)$ for $p < p_c$. We use L instead of the traditional ξ for correlation length, since we use that letter for the contact process.

Definition 2. Let $r_n^0 = \sup \xi_n^0$ and $R^0 = \sup\{r_n^0 : n \geqslant 0\}$. By considering the state of the process when it first reaches $[n, \infty)$, we see

$$P(R^0 \geqslant n+m \mid R^0 \geqslant n) \geqslant P(R^0 \geqslant m)$$

so "supermultiplicativity" implies that

$$\gamma_{\perp}(p) = \lim_{n \to \infty} \left[-\frac{1}{n} \log P(R^0 \geqslant n) \right] \tag{1.2}$$

exists for all p. The limit is positive for $p < p_c$ because

$$P(R^0 \geqslant n) \leqslant P(\tau^0 \geqslant n)$$

{In order to reach $[n, \infty)$, the process must live for n units of time.} Let $L_{\perp}(p) = 1/\gamma_{\perp}(p)$ for $p < p_c$. To see the relationship between (1.1) and (1.2), let $H_n^{\|} = \{(i, j): j = n\}$ and $H_n^{\perp} = \{(i, j): i = n\}$, and observe that

$$\{\tau^0 \geqslant n\} = \{C_0 \cap H_n^{\|} \neq \varnothing\}$$
$$\{R^0 \geqslant n\} = \{C_0 \cap H_n^{\perp} \neq \varnothing\}$$

Definition 3A. The traditional way to extend Definition 1 to $p > p_c$ is to look at

$$\gamma_{\|}(p) = \lim_{n \to \infty} \left[-\frac{1}{n} \log P(n \leqslant \tau^0 < \infty) \right] \tag{1.3}$$

and let $L_{\|}(p) = 1/\gamma_{\|}(p)$. This time we cannot use "supermultiplicativity" and it is not so easy to prove that the limit exists. This was done in ref. 11 using ideas from ref. 2. In this paper we will prove the existence of the limit by relating it to a second definition in terms of the dual percolation system introduced in refs. 3, 9, and 10.

Following ref. 5, we define a dual graph by letting $\mathscr{L}^* = \{(m, n) \in \mathbb{Z}^2 : m+n \text{ is odd}\}$ and drawing oriented bonds from (m, n) to $(m-1, n+1)$ and to $(m-1, n-1)$. Define the new bonds to be open (resp. closed) if the bonds that they cross on the original graph are closed (resp. open). We complete the dual by drawing bonds which are always open from (m, n) to $(m+1, n+1)$ and to $(m+1, n-1)$. This corresponds to the fact that on the original graph the bonds from (m, n) to $(m+1, n-1)$ and to $(m-1, n-1)$ are open with probability 0.

To see that the dual is a natural object, suppose that C_0 is finite, let $D = \{(x, y): |x| + |y| \leqslant 1\}$ with the boundary oriented in a counterclockwise fashion, and let

$$W = \bigcup_{x \in C(0)} x + D$$

If we add up the boundaries of the squares in the union allowing oppositely directed segments to cancel, then the result is a family of open paths on the dual. The one which is the boundary of the unbounded component of W^c is usually called the contour associated with the finite cluster C_0. See Section 10 of ref. 4 for more details.

In what follows we will need several facts, such as the following: C_0 is finite if and only if there is a dual path from $(1, 0)$ to $(-1, 0)$ in $\mathbb{R} \times [0, \infty)$. This fact and the others we will use below are not hard to prove using ideas in the last paragraph. If C_0 is finite, the contour contains such a path. To prove the other direction, suppose that the dual path π is self-avoiding and observe that when a path from the origin first crosses π that bond is closed. A complete account can be found in Section 2 of ref. 5. Therefore, when such facts are needed below we will just say that they come from "planar graph duality."

Definition 3B. Returning to our main subject, we let

$$\gamma_{\|}^D(p) = \lim_{n \to \infty} \left[-\frac{1}{2n} \log P((m, 0) \overset{*}{\to} (m, 2n)) \right] \tag{1.4}$$

where $x \overset{*}{\to} y$ means there is a dual path from x to y, and let $L_{\|}^D(p) = 1/\gamma_{\|}^D(p)$ for $p > p_c$. "Supermultiplicativity" implies that the limit exists. In Section 2 we will show

$$\gamma_{\|}(p) = 2\gamma_{\|}^D(p) > 0 \tag{1.5}$$

Remark. In the companion paper,[7] we introduce a third definition $L_{\|}^{\varepsilon}(p)$, which is the analogue of the definition in terms of sponge crossings for oriented percolation.

We turn now to definitions for the perpendicular correlation length.

Definition 4A. By analogy with Definitions 1, 2, and 3A, we can set

$$\gamma_{\perp}(p) = \lim_{n \to \infty} \left[-\frac{1}{n} \log P(R^0 \geqslant n, \tau^0 < \infty) \right] \tag{1.6}$$

and let $L_\perp(p)=1/\gamma_\perp(p)$. Using the "duplication trick" in ref. 2, one can show that the limit exists, but the proof is tedious, so we omit it. The absence of a superscript here (and in Definition 3A) indicates that we consider this to be the natural definition.

Definition 4B. As in the case of Definition 3A, there is a second definition in terms of the dual process which is easier to work with. Let

$$\gamma_\perp^D(p)=\lim_{n\to\infty}\left[-\frac{1}{2m}\log P((1,0)\xrightarrow{*}(-2m+1,0))\right] \tag{1.7}$$

and let $L_\perp^D(p)=1/\gamma_\perp(p)$ for $p>p_c$. As in Definition 3B, "supermultiplicativity" implies that $\gamma_\perp^D(p)$ exists. If we notice that $(-2m+1,0)\xrightarrow{*}(1,2m)$ has probability 1, it follows that $\gamma_\perp^D(p)\geqslant\gamma_\parallel^D(p)>0$. In Section 3 we will show that, for any $\delta>0$, there are $c, C\in(0,\infty)$ so that

$$c\exp[-(2+\delta)\gamma_\perp^D n]\leqslant P(R^0\geqslant n, \tau^0<\infty)\leqslant C(n+1)\exp(-\gamma_\perp^D n) \tag{1.8}$$

Using the fact that the limit in (1.7) exists, this implies

$$\gamma_\perp^D(p)\leqslant\gamma_\perp(p)\leqslant 2\gamma_\perp^D(p) \qquad \text{for} \quad p>p_c \tag{1.9}$$

Readers (or authors) of ref. 2 might ask if the last two inequalities can be replaced by a single equality. This seems difficult to prove and might even be false.

Definition 4C. It is known (see ref. 4, Section 8) that if we start the contact process with all sites occupied, then, as $n\to\infty$, $\xi_{2n}^{2\mathbb{Z}}$ converges in distribution to a limit which has the same distribution as $\eta=\{x\in 2\mathbb{Z}: \tau^x=\infty\}$, τ^x being short for $\tau^{\{x\}}$. Let $\eta(y)=1$ if $y\in\eta$ and 0 otherwise, and

$$\operatorname{Cov}(\eta(0),\eta(x))=P(0,x\in\eta)-P(0\in\eta)\,P(x\in\eta)$$

which is $\geqslant 0$ by Harris' inequality. In Section 4 we will show that, for any $\delta>0$, there is a constant $C>0$ so that

$$C\exp[-(2+\delta)\gamma_\perp^D x]\leqslant\operatorname{Cov}(\eta(0),\eta(x))\leqslant\exp(-\gamma_\perp^D x) \tag{1.10}$$

Tightening up the bounds in (1.8) and (1.10) to remove the factor of 2 difference between the exponents seems a difficult problem. Indeed, we do not have a good feeling for which (if any) of the inequalities is sharp.

In the above discussion we have been careful to point out that the correlation lengths are all known to be finite when $p\neq p_c$. In ref. 6 it is shown that they all diverge as p approaches p_c and bounds on the associated critical exponents are given. In the case of Definition 2 we can show that if $\nu_\perp$ exists, then it is at least 4/7, which is greater than the mean field value 1/2.

2. THE PROOF OF $\gamma_{||}(p) = 2\gamma_{||}^D(p)$

We begin by observing that "supermultiplicativity" implies that

$$\gamma_{||,a}^D = \lim_{n \to \infty} \left[-\frac{1}{2n} \log P\{(1, 0) \xrightarrow{*} (1, -2n) \text{ in } [-a, a] \times \mathbb{R}\} \right] \quad (2.1)$$

exists. A simple argument of Chayes and Chayes (see ref. 5, Lemma 2) implies

$$\gamma_{||}^D = \lim_{a \to \infty} \gamma_{||,a}^D \quad (2.2)$$

We give the proof, since it is short and we will need two related results below.

Proof of (2.2). First observe that $a \to \gamma_{||,a}^D$ is decreasing. Let $A_{n,a} = \{(1, 0) \xrightarrow{*} (1, -2n) \text{ in } [-a, a] \times \mathbb{R}\}$, let $\delta > 0$, and pick n large enough so that

$$\exp[-(\gamma_{||}^D + \delta)\, 2n] \leqslant P(A_{n,\infty})$$

With n fixed,

$$P(A_{n,\infty}) = \lim_{a \to \infty} P(A_{n,a})$$

"Supermultiplicativity" implies

$$P(A_{n,a}) \leqslant \exp(-\gamma_{||,a}^D 2n)$$

Combining the last three equations gives

$$\gamma_{||}^D + \delta \geqslant \lim_{a \to \infty} \gamma_{||,a}^D$$

which proves the desired result.

For the next result recall that if $r_n = \sup \xi_n^{\{0, -2, -4, \ldots\}}$ then there is a constant $\alpha(p)$ so that $r_n/n \to \alpha(p)$ a.s. as $n \to \infty$, and $p_c = \inf\{p \colon \alpha(p) > 0\}$. See ref. 4, Section 3 for a proof. In Section 11 of that paper it was shown that if $p > p_c$, then

$$\lim_{n \to \infty} \left[-\frac{1}{n} \log P\{r_n \leqslant 0\} \right] = \gamma_{||}^r > 0 \quad (2.3)$$

We did not mention this definition in the introduction because

$$\gamma_{\|}^{D} = \gamma_{\|}^{r} \tag{2.4}$$

Proof. To prove that the limits in (1.4) and (2.3) are equal, it suffices to consider what happens when $n = 2m$. Planar graph duality implies

$$\{r_{2m} \leqslant k\} = \{(1, 0) \overset{*}{\to} (k+1, 2m) \text{ in } \mathbb{R} \times [0, 2m]\} \tag{2.5}$$

from which it follows immediately that $\gamma_{\|}^{r} \geqslant \gamma_{\|}^{D}$. To prove the opposite inequality, let a be an odd positive integer and let

$$A_m = \{(a, 0) \overset{*}{\to} (a, 2m) \text{ in } [0, 2a] \times \mathbb{R}\}$$
$$B_m = \{(2a-1, 2m) \overset{*}{\to} (2a-2, 2m-1) \overset{*}{\to} (2a-3, 2m) \cdots \overset{*}{\to} (1, 2m)\}$$

Since we can always go $(1, 0) \overset{*}{\to} (2, 1) \overset{*}{\to} (3, 0) \cdots \overset{*}{\to} (2a-1, 0)$, it follows from (2.5) that

$$\{r_{2m} \leqslant 0\} \supset A_m \cap B_m$$

For large m

$$P(A_m) \geqslant \exp[-(\gamma_{\|,a}^{D} + \delta)\, 2m]$$

and we always have

$$P(B_m) = (1-p)^{2a-2}$$

Using Harris' inequality now gives

$$P(r_{2m} \leqslant 0) \geqslant P(A_m)\, P(B_m)$$

so

$$\limsup_{m \to \infty} \left(-\frac{1}{2m} \log P\{r_{2m} \leqslant 0\} \right) \leqslant \gamma_{\|,a}^{D} + \delta$$

Since $\delta > 0$ and $a < \infty$ are arbitrary, the desired result follows from (2.2) and the proof is complete.

Lemma. For $p > p_c$,

$$\lim_{n \to \infty} \left[-\frac{1}{2n} \log P(0 \in \xi_{2n}^{0}, \tau^{0} = 2n) \right] = 2\gamma_{\|}^{D} \tag{2.6}$$

Proof. Planar graph duality implies that on $\{0 \in \xi^0_{2n}, \tau^0 = 2n\}$ there are dual paths from $(1, 0)$ to $(0, 2n+1)$ and from $(0, 2n+1)$ to $(-1, 0)$ which only have $(0, 2n+1)$ in common. Noting that $(-1, 0) \overset{*}{\to} (0, -1)$ and $(0, -1) \overset{*}{\to} (1, 0)$ have probability 1 and using the van den Berg–Kesten inequality gives

$$P(0 \in \xi^0_{2n}, \tau^0 = 2n) \leqslant P\{(0, -1) \overset{*}{\to} (0, 2n+1)\}\ P\{(0, 2n+1) \overset{*}{\to} (0, -1)\}$$

The two events on the right-hand side have the same probability by symmetry, so it follows from "supermultiplicativity" that

$$P(0 \in \xi^0_{2n}, \tau^0 = 2n) \leqslant \exp[-2\gamma^D_{||}(2n+2)]$$

and we have shown

$$\liminf_{n \to \infty} \left(-\frac{1}{2n} \log P\{0 \in \xi^0_{2n}, \tau^0 = 2n\} \right) \geqslant 2\gamma^D_{||}$$

To prove the opposite inequality, let a be an odd positive integer and let

$$A_n = \{(2a+1, 0) \overset{*}{\to} (2a+1, 2n) \text{ in } [a, 3a+2] \times \mathbb{R}\}$$
$$B_n = \{(-2a-1, 2n) \overset{*}{\to} (-2a-1, 0) \text{ in } [-3a-2, -a] \times \mathbb{R}\}$$
$$C_n = \{(3a+2, 2n) \overset{*}{\to} (3a+1, 2n+1) \overset{*}{\to} (3a, 2n) \cdots \overset{*}{\to} (-3a-2, 2n)\}$$
$$D_n = \{(0, 0) \to (0, 2n) \text{ in } (-a, a) \times [0, 2n]\}$$
$$E_n = A_n \cap B_n \cap C_n \cap D_n$$

Since $(1, 0) \overset{*}{\to} (2, 1) \overset{*}{\to} (3, 0) \cdots \overset{*}{\to} (3a+2, 0)$ and $(-3a-2, 0) \overset{*}{\to} (-3a-1, 1) \overset{*}{\to} (-3a, 0) \overset{*}{\to} \cdots \overset{*}{\to} (-1, 0)$ have probability 1, planar graph duality implies

$$\{0 \in \xi^0_{2n}, \tau^0 = 2n\} \supset E_n$$

(See Fig. 1.) Independence of events in disjoint regions of the plane and Harris' inequality give

$$P(E) = P(A_n \cap B_n \cap C_n)\ P(D_n) \geqslant P(A_n)\ P(B_n)\ P(C_n)\ P(D_n)$$

If $\delta > 0$ and n is large,

$$P(A_n),\ P(B_n) \geqslant \exp[-(\gamma^D_{||,a} + \delta)\ 2n]$$

For all n

$$P(C_n) = (1-p)^{6a+4}$$

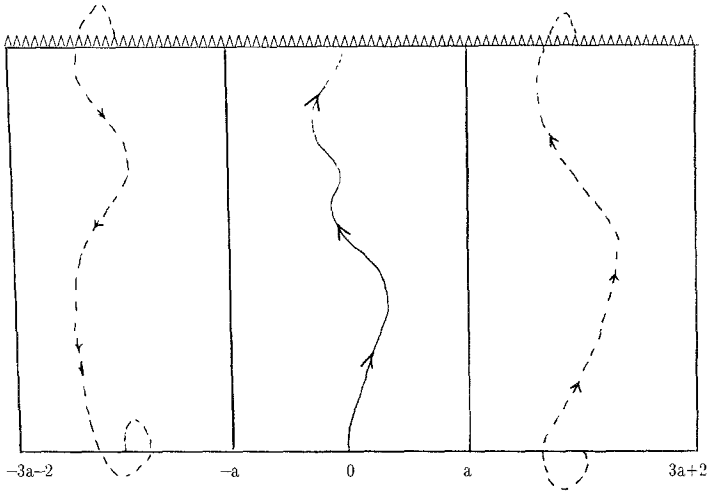

Fig. 1

To handle D_n, we observe that "supermultiplicativity" implies

$$\gamma_k = \lim_{n \to \infty} \left[-\frac{1}{2n} \log P\{(0,0) \to (0,2n) \text{ in } (-k,k) \times \mathbb{R}\} \right] \tag{2.7}$$

exists for $1 \leqslant k \leqslant \infty$, and the argument which leads to (2.2) implies

$$\gamma_\infty = \lim_{k \to \infty} \gamma_k \tag{2.8}$$

To show $\gamma_\infty = 0$, we observe that Harris' inequality and symmetry imply

$$\begin{aligned} P\{(0,0) \to (0,2n)\} &\geqslant P\{(0,0) \to [0,\infty) \times \{2n\}\}\, P([0,\infty) \times \{0\} \to (0,2n)\} \\ &\geqslant (P\{\xi_n^0 \neq \varnothing\}/2)^2 \geqslant [P(\Omega_\infty)/2]^2 \end{aligned}$$

where $\Omega_\infty = \{\xi_m^0 \neq \varnothing \text{ for all } m\}$.

Combining the observations in the last paragraph, we see that if n is large,

$$P(D_n) \geqslant \exp[-(\gamma_a + \delta)\, 2n]$$

so

$$\limsup_{n \to \infty} -\frac{1}{2n} \log P(E_n) \leqslant 2\gamma^D_{\|,a} + \gamma_a + \delta$$

Since $\delta > 0$ and $a < \infty$ are arbitrary, it follows from (2.8) that we have proved the desired result.

Lemma. For $p > p_c$

$$\lim_{n\to\infty}\left[-\frac{1}{n}\log P(\tau^0 = n)\right] = 2\gamma_{||}^D \tag{2.9}$$

Proof. Note that $P(\tau^0 = n) \geqslant P(0 \in \xi_n^0, \tau^0 = n)$. Equation (2.6) implies

$$\limsup_{n\to\infty}\left[-\frac{1}{2n}\log P(\tau^0 = 2n)\right] \leqslant 2\gamma_{||}^D$$

To extend the result to odd times, observe that

$$P\{x \in \xi_n^0, \xi_{n+1}^0 = \{x+1\}\} \geqslant (p/1-p)\, P\{x \in \xi_n^0, \tau^0 = n\} \tag{2.10}$$

since if we take an outcome in $\{x \in \xi_n^0, \tau^0 = n\}$ and change the state of $(x, n) \to (x+1, n+1)$ from closed to open, then we have $x \in \xi_n^0$ and $\xi_{n+1} = \{x+1\}$. Taking $x = 0$ and n an even integer, we have

$$\begin{aligned} P(\tau^0 = n+1) &\geqslant (1-p)^2\, P(\xi_{n+1}^0 = \{1\}) \\ &\geqslant p(1-p)\, P(0 \in \xi_n^0, \tau^0 = n) \end{aligned}$$

so

$$\limsup_{n\to\infty}\left[-\frac{1}{n}\log P(\tau^0 = n)\right] \leqslant 2\gamma_{||}^D$$

To prove the opposite inequality, we need to consider

$$\xi_n^{(x,m)} = \{y\colon \quad (x, m) \to (y, n)\}$$

Translation invariance in time and space gives

$$\begin{aligned} P(\xi_{2n+2}^{(x+1,n+1)} = \{0\}) &= P(\xi_{n+1}^0 = \{-x-1\}) \\ &\geqslant (p/1-p)\, P(x \in \xi_n^0, \tau^0 = n) \end{aligned}$$

by (2.10) and reflection symmetry. Another use of (2.10) now give

$$P(\xi_{2n+2}^0 = \{0\}) \geqslant (p/1-p)^2\, P(x \in \xi_n^0, \tau^0 = n)^2$$

The fact that $\xi_n^0 \subset \{-n,..., n\}$ implies

$$\max_{|x| \leqslant n} P(x \in \xi_n^0, \tau^0 = n) \geqslant \frac{1}{2n+1}\, P(\tau^0 = n)$$

So we have

$$P(\xi_{2n+2}^0 = \{0\}, \tau^0 = 2n+2) \geqslant (p/2n+1)^2\, P(\tau^0 = n)^2$$

or

$$P(\tau^0 = n) \leqslant \frac{2n+1}{p} P(0 \in \xi^0_{2n+2}, \tau^0 = 2n+2)^{1/2}$$

and the proof of (2.9) is complete.

The last step is to show the following.

Lemma. For $p > p_c$

$$\lim_{n \to \infty} \left[-\frac{1}{n} \log P(n \leqslant \tau^0 < \infty) \right] = 2\gamma^D_{\|} \tag{2.11}$$

Proof. Since $P(n \leqslant \tau^0 < \infty) \geqslant P(\tau^0 = n)$, half of the result is an immediate consequence of (2.9). To prove the other half, observe that (2.9) implies that if $\delta > 0$, then for $m \geqslant m_0(\delta)$,

$$P(\tau^0 = m) \leqslant \exp[-(2\gamma^D_{\|} - \delta) m]$$

Summing the geometric series from $m = n$ to ∞ gives the desired result, since δ is arbitrary. [Sticklers for detail should note that (2.3) and (2.4) imply $\gamma^D_{\|} > 0$.]

3. THE PROOF OF $\gamma^D_\perp \leqslant \gamma_\perp \leqslant 2\gamma^D_\perp$

In this section we will prove (1.8). Half of the proof is easy because the work has already been done. An immediate consequence of Lemma 4 in ref. 5 is the following.

Suppose $p > p_c$. There is a constant C so that if $m \geqslant 0$, then

$$P((1, 0) \xrightarrow{*} \{-m+1\} \times \mathbb{R}) \leqslant C(m+1) \exp(-\gamma^D_\perp m) \tag{3.1}$$

Planar graph duality and symmetry imply

$$P(R^0 > n, \tau^0 < \infty) \leqslant P(\{n+1\} \times \mathbb{R} \xrightarrow{*} (-1, 0)) = P((1, 0) \xrightarrow{*} \{-n-1\} \times \mathbb{R})$$

from which it follows that $\gamma_\perp \geqslant \gamma^D_\perp$.

To prove the other inequality, we have to do a little work. As in Section 2, we begin by observing that "supermultiplicativity" implies that

$$\gamma^D_{\perp, a} = \lim_{n \to \infty} \left[-\frac{1}{2n} \log P\{(1, 0) \xrightarrow{*} (-2n+1, 0) \text{ in } \mathbb{R} \times [-a, a]\} \right] \tag{3.2}$$

exists, and the Chayes' argument implies

$$\gamma^D_\perp = \lim_{a \to \infty} \gamma^D_{\perp, a} \tag{3.3}$$

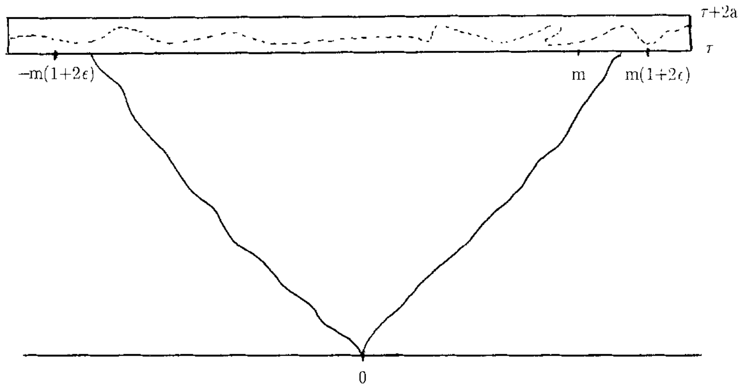

Fig. 2

Next define r_n^0 to be the rightmost site at time t in the process starting from $\{0\}$, i.e., $r_n^0 = \sup \xi_n^0$. Let $l_n^0 = \inf \xi_n^0$. From ref. 4, Section 3, we know that

$$r_n^0/n \to \alpha \quad \text{and} \quad l_n^0/n \to -\alpha \qquad \text{a.s.} \quad \text{on } \{\tau^0 = \infty\} \tag{3.4}$$

and $\alpha(p) > 0$ if $p > p_c$. Let $\varepsilon > 0$, $\tau = (m/\alpha)(1+\varepsilon)$, $\Gamma = \mathbb{R} \times [\tau, \tau + 2a]$, and let

$$A = \{m \leqslant r_\tau^0 \leqslant m(1+2\varepsilon)\}$$

$$B = \{l_\tau^0 \geqslant -m(1+2\varepsilon)\}$$

$$C = \{(m(1+2\varepsilon) + 2a, \tau + a) \xrightarrow{*} (-m(1+2\varepsilon) - 2a, \tau + a) \text{ in } \Gamma\}$$

A look at Fig. 2 shows

$$P(R^0 > n, \tau^0 < \infty) \geqslant P(A \cap B \cap C) = P(A \cap B)\, P(C)$$

by the independence of events based on disjoint regions of the plane. (3.4) implies that when m is large, $P(A \cap B) \geqslant P(\tau^0 = \infty)/2$. Relation (3.1) and translation invariance imply

$$P(C) \geqslant \exp[-2(1+\varepsilon)(\gamma_{\perp,a}^D + \delta)\, 2m]$$

for large m. Since $\delta, \varepsilon > 0$ and $a < \infty$ are arbitrary, the proof is complete.

4. BOUNDS ON Cov(η(0), η(*x*))

In this section we will prove (1.10). We will prove the second inequality first. It is based on ideas in ref. 2. Let $C(x)$ denote the set of sites which can be reached from $(x, 0)$. Then

$$\begin{aligned}\text{Cov}(\eta(0), \eta(x)) &= P(\tau^0 < \infty, \tau^x < \infty) - P(\tau^0 < \infty)\, P(\tau^x < \infty) \\ &= P(\tau^0 < \infty, \tau^x < \infty, C(0) \cap C(x) = \varnothing) \\ &\quad - P(\tau^0 < \infty)\, P(\tau^x < \infty) \\ &\quad + P(\tau^0 < \infty, \tau^x < \infty, C(0) \cap C(x) \neq \varnothing) \end{aligned} \tag{4.1}$$

On $\{\tau^0 < \infty, \tau^x < \infty, C(0) \cap C(x) = \varnothing\}$ there are two disjoint sets of bonds which determine $\{\tau^0 < \infty\}$ and $\{\tau^x < \infty\}$. [Readers who worry that the two dead clusters could share some boundary bonds should note that the closed bonds which block percolation from 0 must begin in $C(0)$.] Since the events in question are decreasing, an application of the van den Berg–Kesten inequality gives

$$P(\tau^0 < \infty, \tau^x < \infty, C(0) \cap C(x) = \varnothing) \leqslant P(\tau^0 < \infty)\, P(\tau^x < \infty)$$

and we have

$$\text{Cov}(\eta(0), \eta(x)) \leqslant P(\tau^0 < \infty, \tau^x < \infty, C(0) \cap C(x) \neq \varnothing)$$

Planar graph duality implies that the last probability is bounded above by

$$P((x+1, 0) \overset{*}{\rightarrow} (-1, 0)) \leqslant \exp(-\gamma_\perp^D x)$$

by "supermultiplicativity," so we have proved half of (4.1).

To prove the other inequality, notice that if $\mathscr{F}$ is the collection of finite sets of sites and $\mathscr{G} \subset \mathscr{F}$, then

$$\begin{aligned}\text{Cov}(\eta(0), \eta(x)) &= P(\tau^0 < \infty)\, P(\tau^x = \infty) - P(\tau^0 < \infty, \tau^x = \infty) \\ &= \sum_{S \in \mathscr{F}} P(C(0) = S)\{P(\tau^x = \infty) - P((x, 0) \text{ percolates in } S^c)\} \\ &= \sum_{S \in \mathscr{F}} P(C(0) = S)\, P((x, 0) \text{ only percolates through } S) \\ &\geqslant P(C(0) \in \mathscr{G}) \min_{S \in \mathscr{G}} P((x, 0) \text{ only percolates through } S) \end{aligned} \tag{4.2}$$

The next step is to choose a good $\mathscr{G}$. Let $\tau = (x/2\alpha)(1 + 2\varepsilon)$ and

$$\mathscr{G} = \{S\colon\ \ |S| < \infty, P(C(0) = S) > 0 \text{ and } S \cap [x/2, \infty) \times \{\tau\} \neq \varnothing\}$$

From the proof of (1.8) it follows that if $\delta > 0$,

$$P(C(0) \in \mathscr{G}) \geqslant c \exp[-(\gamma_\perp^D + \delta)x] \tag{4.3}$$

To force $(x, 0)$ to percolate through $S \in \mathscr{G}$, we let

$$\rho = x + (x/2)(1 + 3\varepsilon)$$
$$\lambda = x - (x/2)(1 - \varepsilon) = x/2 - \varepsilon x/2$$
$$A = \{r^x_\tau \leqslant \rho, l^x_\tau \leqslant \lambda\}$$
$$\Gamma = [x/2 - a, \rho + a] \times [\tau, \tau + 2a]$$
$$B = \{(\rho + a, \tau + a) \xrightarrow{*} (x/2 - a, \tau + a) \text{ in } \Gamma\}$$
$$C = \{\text{process starting with } l^x_\tau \text{ occupied at time } \tau \text{ percolates}\}$$

See Fig. 3 for help with the definitions. To make B and C independent, we want x large enough so that $x\varepsilon/2 \geqslant 3a$. From the definitions above it should be clear that if $S \in \mathscr{G}$, then

$$P((x, 0) \text{ only percolates through } S) \geqslant P(A)\, P(B)\, P(C) \tag{4.4}$$

Relation (3.4) implies that $P(A) \geqslant P(\tau^0 = \infty)/2$ for large x. One has $P(C) = P(\tau^0 = \infty)$. As for the other term, observe that if $\delta > 0$ and x is large,

$$P(B) \geqslant \exp\{-(\gamma^D_{\perp,a} + \delta)[x(1+\varepsilon) + 2a]\} \geqslant \exp[-(\gamma^D_{\perp,a} + \delta)(1 + 4\varepsilon/3)x]$$

since we chose x so that $x\varepsilon/2 \geqslant 3a$. Combining (4.2)–(4.4) with the bounds on $P(A)$, $P(B)$, and $P(C)$ just derived gives

$$\mathrm{Cov}(\eta(0), \eta(x)) \geqslant c' \exp[-(\gamma^D_{\perp,a} + \delta)(2 + 4\varepsilon/3)x]$$

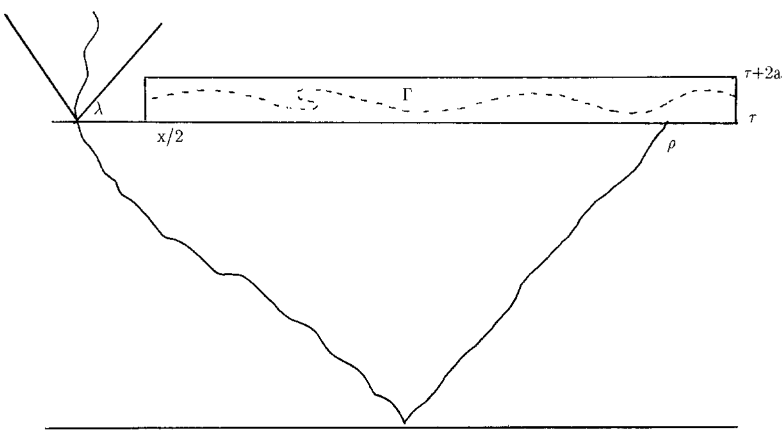

Fig. 3

Since the last result holds for large x whenever $\varepsilon, \delta > 0$ and $a < \infty$, the desired lower bound follows from (3.3).

ACKNOWLEDGMENTS

The work of R.D. and R.H.S. was partially supported by the NSF and the Army Research Office through the Mathematical Sciences Institute at Cornell. The work of N.I.T. was supported by CAPES (Brazil).

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