Probab. Th. Rel. Fields 84, 67–82 (1990)

Probability Theory and Related Fields

Large Deviations for Independent Random Walks*

J. Theodore Cox[1] and Richard Durrett[2]

[1] Department of Mathematics, Syracuse University, Syracuse, NY 13210, USA
[2] Department of Mathematics, White Hall, Cornell University, Ithaca, NY 14853, USA

Summary. We consider a system of independent random walks on $\mathbb{Z}$. Let $\xi_n(x)$ be the number of particles at x at time n, and let $L_n(x) = \xi_0(x) + \ldots + \xi_n(x)$ be the total occupation time of x by time n. In this paper we study the large deviations of $L_n(0) - L_n(1)$. The behavior we find is much different from that of $L_n(0)$. We investigate the limiting behavior when the initial configurations has asymptotic density 1 and when $\xi_0(x)$ are i.i.d Poisson mean 1, finding that the asymptotics are different in these two cases.

1. Introduction

Consider a system of independent particles performing symmetric simple random walks on $\mathbb{Z}$. In what follows we will mostly be concerned with discrete time walks, but in order to discuss results from the literature, we will also need to consider continuous time systems which stay at a site for an exponential amount of time with mean one before they jump. Let $\xi_t(x)$ denote the number of particles at x at time t in the continuous time system in which $\xi_0(x)$, $x \in \mathbb{Z}$ are i.i.d. Poisson with mean 1. Cox and Griffeath (1984) showed that if $A \subset \mathbb{Z}$ is finite and we let

$$D_t = (t|A|)^{-1} \int_0^t \sum_{x \in A} \xi_s(x)\, ds$$

be the "mean particle density on A up to time t," then

$$\lim_{t \to \infty} t^{-1/2} \log P(D_t \in (a, b)) = - \inf_{x \in (a, b)} J(x) \tag{1}$$

where J is an explicitly given function which is independent of A.

Lee (1988) extended their result and clarified the answer by considering (in discrete time)

$$D_n(x) = n^{-1} \sum_{m=0}^{n} \xi_m(x)$$

* This work was done while the first author was on sabbatical at Cornell University. Both authors were partially supported by the National Science Foundation and the Army Research Office through the Mathematical Sciences Institute at Cornell

as an element of $\{\lambda: \mathbb{Z} \to [0, \infty)\}$ (with the product topology). He proved that if the $\xi_0(x) x \in \mathbb{Z}$ are i.i.d. Poisson mean 1 then

$$n^{-1/2} \log P(D_n \in S) \approx -\inf_{\lambda \in S} \mathrm{I}(\lambda) \tag{2}$$

where $\approx$ means

$$\limsup \mathrm{LHS} \leqq \mathrm{RHS} \text{ for closed sets } S$$

$$\liminf \mathrm{LHS} \geqq \mathrm{RHS} \text{ for open sets } S$$

and

$$I(\lambda) = \begin{cases} J(c) & \text{if } \lambda(x) \equiv c \\ \infty & \text{otherwise} . \end{cases}$$

To recover Cox and Griffeath's result, let

$$S = \left\{ \lambda : |A|^{-1} \sum_{x \in A} \lambda(x) \in (a, b) \right\} .$$

The function J is the same since Lee showed that the I function is the same in discrete or continuous time.

Lee's result implies that the large deviations behavior of weighted occupation times

$$L_n = \sum_{m=0}^{n} \sum_{x} V(x) \xi_m(x) \tag{3}$$

will be the same for all V with $\sum_x V(x) = c > 0$ (and $\{x : V(x) \neq 0\}$ finite). The last observation suggests the question: What happens when $\sum_x V(x) = 0$? In this paper we will study the special case $V(0) = 1$, $V(1) = -1$, $V(x) = 0$ otherwise, and show that if ξ_0 is a nonrandom initial configuration with

$$(2n)^{-1} \sum_{m=-n}^{n} \xi_0(m) \to 1 \quad \text{as} \quad n \to \infty , \tag{$*$}$$

then

$$\lim_{n \to \infty} n^{-\theta} \log P(D_n(0) - D_n(1) > a) = -K(a) . \tag{4}$$

Since one of the surprises is the value of θ, we invite the reader to guess the answer before we reveal it below. Hint: $\theta \in \{2/3, 3/4, 4/5, 5/6\}$.

The key to the proof of (4), like most large deviations results, is an examination of the Laplace transform of the weighted occupation time defined in (3). Since the particles are independent,

$$\log E \exp(\lambda L_n) = \sum_x \xi_0(x) \log E_x \exp\left(\lambda \sum_{m=0}^{n} V(S_m) \right) , \tag{5}$$

where E_x denotes the expected value for a symmetric simple random walk S_m with $S_0 = x$. The last computation and many others below do not require

$$V(0) = 1, \quad V(1) = -1 , \quad \text{and} \quad V(x) = 0 \quad \text{otherwise}$$

but for simplicity we will always restrict our attention to this special case.

(5) reduces the problem to questions about the behavior of a single particle. The first step is to understand the central limit behavior. Suppose $S_0 = 0$ and let

$$W_n = V(S_0) + \ldots + V(S_n) .$$

A result of Dobrushin (1955) (see Kesten (1962)) implies

$$W_n/n^{1/4} \Rightarrow \aleph , \tag{6}$$

where $\aleph$ is a mixture of normal distributions. Although the 1/4th power may be surprising at first, the result is easy to understand and not hard to prove. Let $R_0 = 0$ and for $k \geqq 1$ let

$$R_k = \inf\{m > R_{k-1} : S_m = 0\}$$

$$X_k = \sum_{R(k-1) \leqq m < R(k)} V(S_m)$$

$$N_n = \sup\{k : R_k \leqq n\}$$

so that

$$W_n = X_1 + \ldots + X_{N_n} + Y_n$$

where

$$Y_n = \sum_{m=R(N_n)}^{n} V(S_m) .$$

It is well known that for all $t \geqq 0$

$$P(N_n \leqq tn^{1/2}) \to G(t) = \int_0^t \pi^{-1/2} \exp(-x^2/4)dx . \tag{7}$$

In our special case the X_k's are i.i.d. with

$$P(X_k = 1 - j) = (1/2)^{j+1} \quad \text{for} \quad j \geqq 0 .$$

These variables have mean 0 and variance 2, so

$$n^{-1/4}(X_1 + \ldots + X_{[\sqrt{n}t]}) \Rightarrow \sqrt{2} B_t \tag{8}$$

where B_t is a standard Brownian motion. A little thought reveals that the variables in (7) and (8) are asymptotically independent, and with a little work it follows that

$$E_0 \exp(\lambda W_n/n^{1/4}) \to \varphi(\lambda) = \int \exp(-\lambda^2 t/4) dG(t) . \tag{9}$$

The key to the proof of (4) is

Lemma 1. *If $\beta \in (0, 1/4)$ and $w = 0$ or 1 then as $n \to \infty$*

$$n^{4\beta-1} \log E_w \exp(\lambda W_n/n^\beta) \to \lambda^4/2 . \tag{10}$$

In the last result we have restricted our attention to points in the support of V, but that is good enough since if $x > 1$

$$E_x \exp(\lambda W_n/n^\beta) = P_x(T_1 > n) + \sum_{m=1}^{n} P_x(T_1 = m) E_1 \exp(W_{n-m}/n^\beta) . \tag{11}$$

where $T_1 = \inf\{m : S_m = 1\}$. A similar formula holds for $x < 0$. In fact $P_x(W_n \in \cdot) = P_{1-x}(-W_n \in \cdot)$, so throughout the paper it is enough to prove things for $x \geqq 1$ or $x \leqq 0$. Using some facts about simple random walk, (11) and Lemma 1 lead easily to

Lemma 2. *If* $\beta \in (0, 1/4)$ *and* $x_n/n^{1-2\beta} \to y$

$$n^{4\beta-1} \log E_{x_n} \exp(\lambda W_n/n^\beta) \to (\lambda^4/2 - \lambda^2|y|)^+ . \tag{12}$$

Using (5) and Lemma 2 now gives

Lemma 3. *If* $(*)$ *holds and* $\beta \in (0, 1/4)$ *then*

$$n^{6\beta-2} \log E \exp(\lambda L_n/n^\beta) \to \int (\lambda^4/2 - \lambda^2|y|)^+ \, dy = \lambda^6/4 . \tag{13}$$

When Lemma 3 is established standard large deviations arguments take over (see Lemma 1 in Cox and Griffeath (1984)) to prove:

Theorem 1. *If* $(*)$ *holds,* $3/4 < \alpha < 2$, *and* $a > 0$, *then as* $n \to \infty$

$$n^{(2-6\alpha)/5} \log P(L_n > an^\alpha) \to -ca^{6/5} \text{ where } c = (5/4)(2/3)^{6/5} .$$

To explain the power of n:

$$P(L_n > an^\alpha) = P(L_n/n^\beta > an^{\alpha-\beta}) \leqq \exp(-\lambda a n^{\alpha-\beta}) E \exp(\lambda L_n/n^\beta) . \tag{14}$$

Taking $\alpha - \beta = 2 - 6\beta$, i.e. $\beta = (2-\alpha)/5$, and optimizing over λ gives the upper bound.

Taking $\alpha = 1$ we see that the answer to the question in (4) is $\theta = 4/5$. Tracing back through the proof we see that $\beta = 1/5$ in this case, and the major contribution to the Laplace transform comes from $|x| \leqq (\lambda^2/2)n^{3/5}$. It is not hard to show that with probability at least $C\exp(-\varepsilon n^{4/5})$, all the particles at $|x| \leqq n^{3/5}$ hit 0 by time $n/2$ and have $W_n \geqq n^{2/5}$. The event which produces $L_n \geqq an$ must be something like this.

The upper limit $\alpha < 2$ is natural since the largest possible value of L_n is about n^2. To see the reason for restriction $\alpha > 3/4$ observe that under ordinary circumstances only about $n^{1/2}$ particles will hit the support of V by time n and their weighted occupation times have standard deviation $n^{1/4}$. Dividing the individual contributions by $n^{1/4}$, we see $P(L_n > an^{3/4})$ is the probability the sum of $n^{1/2}$ random variables with mean 0 and standard deviation $O(1)$ is $> an^{1/2}$. The last observation shows that $\alpha = 3/4$ corresponds to the "usual" large deviations setting while the deviations studied in Theorem 1 are enormous.

Using methods similar to the proof of Theorem 1 we can show

Theorem 2. *If* $(*)$ *holds then*

$$n^{-1/2} \log P(L_n > an^{3/4}) \to -I(a)$$

where $I(a) = \sup_\lambda \lambda a - \psi(\lambda)$ *and*

$$\psi(\lambda) = \int dx \log\left(1 + \int_0^1 P_x(\tau_0 \in ds)\{\varphi((1-s)^{1/4}\lambda) - 1\}\right) .$$

Here φ is the limit in (9), and $P_x(\tau_0 \in ds)$ is the distribution of the time to hit 0 for a Brownian motion started at x.

Since $(*)$ is satisfied for almost every initial configuration when we assume

$$\xi_0(x),\ x \in \mathbb{Z} \text{ are i.i.d Poisson with mean one}, \qquad (**)$$

it is easy to jump to the conclusion that the large deviations behavior will be the same under $(*)$ and $(**)$, but this is *wrong*.

Theorem 3. *If* $(**)$ *holds then the conclusions of Theorem* 2 *hold but*

$$\psi(\lambda) = \int dx \int_0^1 P_x(\tau_0 \in ds)\{\varphi((1-s)^{1/4}\lambda) - 1\}.$$

Since $\log(1+u) \leqq u$ for $u > -1$ with strict inequality for $u \neq 0$ the new ψ is strictly larger than the one in Theorem 2, and the Poisson process will more easily achieve large weighted occupation times.

Differences between the large deviations behavior under $(*)$ and $(**)$ become even more severe when $\alpha > 3/4$.

Theorem 4. *Let* $\alpha > 3/4$ *and* $\gamma = \alpha - 1/4$. *If* n *is large then*

$$\hat{P}(L_n > n^\alpha) \geqq \exp(-2\gamma n^\gamma \log n).$$

Here and in what follows $\hat{P}$ indicates that we are assuming $(**)$. Since $(6\alpha - 2)/5 > \alpha - 1/4$ when $\alpha > 3/4$, the lower bound in Theorem 4 shows that enormous deviations from a Poisson initial configuration are much more likely than from a fixed configuration. The extra boost comes from large deviations in the initial configuration.

To prove Theorem 4 we observe that

$$\hat{P}(\xi_0(0) = k) = e^{-1}/k! \geqq \exp(-1 - k\log k).$$

Now if $k = n^\gamma$ and all the particles starting at 0 have $W_n > 2n^{1/4}$, an event of probability at least $\exp(-Cn^\gamma)$ by (6), we will have $L_n^0 \geqq 2n^\alpha$, where the superscript 0 indicates we are looking at only the contribution from particles starting at 0. By computing second moments it is not hard to show $\hat{P}(|L_n^+| > n^\alpha) \to 0$, where L_n^+ refers to the contribution of particles from $x \neq 0$ and the result follows. Since we have not been able to improve the lower bound in Theorem 4, we think it might be the right order of magnitude.

The paper is organized as follows. Lemmas 1–3, which make up the proof of Theorem 1, are proved in Sects. 2–4. In Sect. 5 we prove Theorems 2 and 3. Finally Theorem 4 is proved in Sect. 6. We would like to thank Bruno Remillard for his help with the proof of Lemma 1. He has proved Lemma 1 for a general V with $\Sigma V(x) = 0$ and $\{x: V(x) \neq 0\}$ finite, and solved the analogous problems for Brownian motion.

2. Proof of Lemma 1

We begin by computing solutions of

$$\{f(x+1) + f(x-1)\}/2 = (\cosh\theta)e^{-aV(x)}f(x), \qquad (1)$$

where $V(0) = 1$, $V(1) = -1$, and $V(x) = 0$ otherwise. The reason for interest in functions that satisfy (1) is that

$$(\cosh\theta)^{-n}\exp\left(\sum_{m=0}^{n-1} aV(S_m)\right)f(S_n) \text{ is a martingale .}$$

If we let

$$f(x) = \begin{cases} e^{\theta x} & \text{for} \quad x \leqq 0 \\ Ae^{-\theta(x-1)} & \text{for} \quad x \geqq 1 \end{cases} \tag{2}$$

then (1) holds for $x \leqq -1$ and $x \geqq 2$. When $x = 0$, (1) becomes

$$f(1) = 2(\cosh\theta)e^{-a}f(0) - f(-1)$$

so

$$A = (e^{\theta} + e^{-\theta})e^{-a} - e^{-\theta} . \tag{3}$$

Applying similar reasoning to the equation with $x = 1$ gives

$$Ae^{-\theta} = (e^{\theta} + e^{-\theta})e^{a}A - 1 . \tag{4}$$

Plugging in the value of A and simplifying

$$(1 + e^{-2\theta})e^{-a} - e^{-2\theta} = (e^{\theta} + e^{-\theta})^2 - (1 + e^{-2\theta})e^{a} - 1$$

$$(1 + e^{-2\theta})(e^{-a} - 2 + e^{a}) = (e^{2\theta} - 1)$$

$$(e^{2\theta} - 1)/(1 + e^{-2\theta}) = e^{a} - 2 + e^{-a} = (2\sinh a/2)^2 . \tag{5}$$

The right hand side is > 0. The left hand side is 0 at $\theta = 0$ and increases to ∞, so there is a unique positive slution $\theta(a)$. The left hand side $\sim \theta$ as $\theta \to 0$ so

$$\theta(a) \sim (e^{a} - 2 + e^{-a}) \sim a^2 \quad \text{as} \quad a \to 0 . \tag{6}$$

Let $a = \lambda n^{-\beta}$, let $\theta(a)$ be given by (5), and let $f_n(x)$ be the function defined by (2)–(5). The martingale property implies that

$$E_x\left[f_n(S_n)\exp\left(\lambda n^{-\beta}\sum_{m=0}^{n-1} V(S_m)\right)\right] = \{\cosh\theta(\lambda n^{-\beta})\}^n f_n(x) \tag{7}$$

so

$$E_x\left[\exp\left(\lambda n^{-\beta}\sum_{m=0}^{n-1} V(S_m)\right)\right] \geqq (\sup f_n(y))^{-1}\{\cosh\theta(\lambda n^{-\beta})\}^n f_n(x) . \tag{8}$$

From the definition of f_n, it is easy to see that as $n \to \infty$

$$\sup_y f_n(y) = \max(f_n(0), f_n(1)) \to 1 . \tag{9}$$

Using (6) and calculus gives $\theta(\lambda n^{-\beta}) \sim \lambda^2 n^{-2\beta}$ as $n \to \infty$, $\cosh\theta - 1 \sim \theta^2/2$ as $\theta \to 0$, and

$$\log(\cosh\theta(\lambda n^{-\beta})) \sim \lambda^4 n^{-4\beta}/2 \quad \text{as } n \to \infty . \tag{10}$$

Putting it all together we have

$$\liminf_{n\to\infty} n^{4\beta-1}\log E_x\exp\left(\lambda n^{-\beta}\sum_{m=0}^{n-1} V(S_m)\right) \geqq \lambda^4/2 . \tag{11}$$

To get the corresponding upper bound, let $p(x, y) = 1/2$ if $|x - y| = 1$ and 0 otherwise, let f be a positive solution of (1), and define a transition probability by

$$q(x, y) = \frac{e^{aV(x)}}{\cosh\theta} p(x, y) \frac{f(y)}{f(x)} . \tag{12}$$

It is easy to check that if X_n is a Markov chain with transition probability q and $h \geqq 0$ then

$$E_x\left[(\cosh\theta)^{-n}\exp\left(a\sum_{m=0}^{n-1} V(S_m)\right)\frac{f(S_n)}{f(x)} h(S_n)\right] = E_x h(X_n) .$$

Taking $h = 1/f$ and rearranging gives

$$E_x\left[\exp\left(a\sum_{m=0}^{n-1} V(S_m)\right)\right] = f(x)(\cosh\theta)^n E_x(1/f(X_n)) . \tag{13}$$

When $x \geqq 2$, $V(x) = 0$ and $f(x) = Ce^{-\theta x}$, so

$$q(x, x+1) = e^{-\theta}/(e^{\theta} + e^{-\theta}) \quad q(x, x-1) = e^{\theta}/(e^{\theta} + {}^{-\theta}) . \tag{14}$$

A similar formula holds for $x \leqq -1$ with the probabilities reversed. This suggests letting $\psi(x) = -x$ for $x \leqq 0$ and $x - 1$ for $x \geqq 1$ and comparing with a chain Y_n on $\{0, 1, \ldots\}$ with transition probability given by $r(0, 1) = 1$ and for $x \geqq 1$

$$r(x, x+1) = e^{-\theta}/(e^{\theta} + e^{-\theta}) \quad r(x, x-1) = e^{\theta}/(e^{\theta} + e^{-\theta})$$

(15) **Lemma.** *If $\psi(X_0) = 0$ then we can construct X_n and Y_n on the same space in such a way that $\psi(X_n) \leqq 1 + Y_n$ for all n.*

Proof. There are several cases to consider. First if $Y_n = 0$ we have no choices to make in defining the coupled process since $Y_{n+1} = 1$ with probability one. If $Y_n > 0$ and $\psi(X_n) \in \{Y_n, Y_n + 1\}$ then $X_n \notin \{0, 1\}$, so the transition probabilities are equal and we move the two in parallel, i.e. $\psi(X_{n+1}) - Y_{n+1} = \psi(X_n) - Y_n$. Finally if $\psi(x_n) \leqq Y_n - 1$ we allow the chains to move independently. In all three cases the inequality is preserved and the proof is complete.

Remark. The last case in the proof is the "bad" case. If $\psi(X_n) = 0$ and $Y_n = 1$ the probability of $\psi(X_{n+1}) = 1$ may be $> r(1, 2)$ so we cannot guarantee $\psi(X_n) \leqq Y_n$.

Y_n is obviously positive recurrent. Being a birth and death process, its stationary distribution π satisfies

$$\pi(n)e^{-\theta}/(e^{\theta} + e^{-\theta}) = \pi(n+1)e^{\theta}/(e^{\theta} + e^{-\theta}) \quad \text{for } n \geqq 1,$$

i.e. $\pi(n+1) = e^{-2\theta}\pi(n)$. The equation for $n = 0$ is

$$\pi(0) = \pi(1)e^{\theta}/(e^{\theta} + e^{-\theta}) ,$$

so the exact formula for $\pi(n)$ is a little messy. It is easy to see that

$$\pi(n) = C(\theta)e^{-2\theta n} \quad \text{for } n \geqq 1, \text{ and} \tag{16}$$

$$C(\theta) \leqq \left(\sum_{n=1}^{\infty} e^{-2\theta n}\right)^{-1} = e^{2\theta}(1 - e^{-2\theta}) . \tag{17}$$

To get our upper bound now from (13), we start Y_n from its stationary distribution π and observe that (15) and the definition of f in (2)–(5) imply that if $w = 0$ or 1

$$E_w(1/f(X_n)) \leqq (f(1) \wedge 1)^{-1} E_\pi \exp(\theta(Y_n + 1)) . \tag{18}$$

Using (16) and (17) now gives

$$E_\pi \exp(\theta(Y_n + 1)) \leqq e^\theta \pi(0) + C(\theta) \sum_{k=1}^{\infty} e^{-2\theta k} e^{\theta(k+1)} \leqq e^\theta(1 + C(\theta)/(1 - e^{-\theta})) \tag{19}$$

$$\leqq e^{3\theta}\{1 + (1 - e^{-\theta} + e^{-\theta} - e^{-2\theta})/(1 - e^{-\theta})\} \leqq 3e^{3\theta} .$$

(13), (18), and (19) imply

$$E_w\left[\exp\left(a \sum_{m=0}^{n-1} V(S_m)\right)\right] \leqq f(x)(\cosh\theta(a))^n 3\exp(3\theta(a))/(f(1) \wedge 1) . \tag{20}$$

Letting $a = \lambda n^{-\beta}$ and observing $\theta(a) \to 0$, we get an upper bound which differs by a constant factor from the lower bound in (8), and it follows that for $w = 0$ or 1

$$\limsup_{n\to\infty} n^{4\beta-1} \log E_w\left[\exp\left(\lambda n^{-\beta} \sum_{m=0}^{n-1} V(S_m)\right)\right] \leqq \lambda^4/2 . \tag{21}$$

3. Proof of Lemma 2

Lemma 2. *If $\beta \in (0, 1/4)$ and $x_n/n^{1-2\beta} \to y$ then*

$$n^{4\beta-1} \log E_{x_n} \exp(\lambda W_n/n^\beta) \to (\lambda^4/2 - \lambda^2|y|)^+ .$$

Proof. As remarked in the introduction $P_x(W_n \in \cdot) = P_{1-x}(-W_n \in \cdot)$, so we can suppose that $x_n \geqq 1$ for all n. From (2.8)

$$E_{x_n} \exp(\lambda W_n/n^\beta) \geqq \left(\sup_z f_n(z)\right)^{-1} \{\cosh\theta(\lambda n^{-\beta})\}^n f_n(x_n)$$

where

$$f_n(x_n) = f_n(1)\exp(-\theta(\lambda n^{-\beta})(x_n - 1)) .$$

(2.9), (2.6), and (2.10) imply that as $n \to \infty$ $\sup_z f_n(z) \to 1$, $\theta(\lambda n^{-\beta}) \sim \lambda^2 n^{-2\beta}$, and

$$\log\cosh(\theta(\lambda n^{-\beta})) \sim \lambda^4 n^{-4\beta}/2 .$$

So if $x_n/n^{1-2\beta} \to y$ (necessarily $\geqq 0$) then $n^{4\beta-1}\log f_n(x_n) \to \lambda^2 y$ and it follows that

$$\liminf_{n\to\infty} n^{4\beta-1} \log E_{x_n} \exp(\lambda W_n/n^\beta) \geqq (\lambda^4/2 - \lambda^2 y)^+ ,$$

the positive part coming from the fact that (1.6) and $\beta \in (0, 1/4)$ imply

$$E_{x_n} \exp(\lambda W_n/n^\beta) \to \infty .$$

For the upper bound we use the approach mentioned in the introduction. To simplify the formulas we will drop the subscript from the x. We write for $x \geqq 1$

$$E_x \exp(\lambda W_n/n^\beta) = P_x(T_1 > n) + \sum_{m=1}^{n} P_x(T_1 = m) E_1 \exp(\lambda W_{n-m}/n^\beta) , \tag{2}$$

where $T_1 = \inf\{m : S_m = 1\}$. The first term on the right is $\leqq 1$. We will bound the second by n times the largest term. To identify and bound that term we observe

$$P_x(T_1 = m) \leqq P_x(S_m = 1) \leqq P_1(S_m \geqq x) , \tag{3}$$

and for $\theta > 0$

$$P_1(S_m \geqq x) \leqq e^{-\theta x}(\cosh\theta)^m .$$

Setting $\theta = x/m$

$$P_1(S_m \geqq x) \leqq \exp(-x^2/m)(\cosh(x/m))^m . \tag{4}$$

Since $\cosh\theta - 1 \sim \theta^2/2$ as $\theta \to 0$, it follows that if $x/m^{1-2\beta} \to z$ then

$$\limsup_{m\to\infty} m^{4\beta-1} \log P_1(S_m \geqq x) \leqq -z^2/2 .$$

So if $m/n \to t \in (0, 1]$ and $x/n^{1-2\beta} \to y$ then $z = y/t^{1-2\beta}$ and

$$\limsup_{n\to\infty} n^{4\beta-1} \log P_1(S_m \geqq x) \leqq -t^{1-4\beta}(x/t^{1-2\beta})^2/2 = -y^2/2t . \tag{5}$$

From (2.20) we get

$$\limsup_{n\to\infty} n^{4\beta-1} \log E_1 \exp(\lambda W_{n-m}/n^\beta) = (1-t)\lambda^4/2 . \tag{6}$$

Intuitively this comes from multiplying and dividing by $(n-m)^\beta$ in the exponential and using Lemma 1. By using the formula quoted we do not have to exclude the case $t = 1$. Adding the right hand sides of (5) and (6) gives $-y^2/2t + (1-t)\lambda^4/2$, a quantity that we will call $g(t)$. For fixed y the maximum occurs when $g'(t) = y^2/2t^2 - \lambda^4/2 = 0$, i.e. $t = y/\lambda^2$. If $0 < y \leqq \lambda^2$ the maximum value is $\lambda^4/2 - \lambda^2 y$. If $y > \lambda^2$, $g'(t) < 0$ for $t \leqq 1$ and the maximum value is $g(1) = -y^2/2 < 0$. Considering the two cases and recalling that we have $P_x(T_1 \geqq n) \leqq 1$ on the right hand side of (2), it should be easy to believe that if $x/n^{1-2\beta} \to y$ then

$$\limsup_{n\to\infty} n^{4\beta-1} \log E_x \exp(\lambda W_n/n^\beta) \leqq (\lambda^4/2 - \lambda^2 y)^+ . \tag{7}$$

To turn the calculations above into a proof, we need to control the values for small m. To do this we observe

$$\sum_{m=1}^{n\varepsilon} P_x(T_1 = m) E_1 \exp(\lambda W_{n-m}/n^\beta) \leqq P_x(T_1 \leqq n\varepsilon) \sup_{m \leqq n\varepsilon} E_1 \exp(\lambda W_{n-m}/n^\beta) . \tag{8}$$

Symmetry and the reflection principle imply

$$P_x(T_1 \leqq n\varepsilon) = P_1(T_x \leqq n\varepsilon) \leqq 2P_1(S_{n\varepsilon} \geqq x) . \tag{9}$$

Using (5) and a strengthening of (6) which follows easily from (2.20),

$$\limsup_{n\to\infty} n^{4\beta-1} \sup_{m \leqq n\varepsilon} \log E_1 \exp(\lambda W_{n-m}/n^{\beta}) \leqq \lambda^4/2 .$$

So if $x/n^{1-2\beta} \to y$ then

$$\limsup_{n\to\infty} n^{4\beta-1} \log\left[\sum_{m=1}^{n\varepsilon}\right] \leqq -y^2/2\varepsilon + \lambda^4/2 \tag{10}$$

where the symbol inside the log is shorthand for the left hand side of (8).

To bound the rest of the sum we observe

$$\sum_{m=n\varepsilon}^{n} \leqq n \sup_{n\varepsilon \leqq m \leqq n} P_X(T_1 = m) E_1(\exp(\lambda W_{n-m}/n^{\beta})) \tag{11}$$

Using (3), (5), (6), and the calculation which led to (7), it follows that

$$\limsup_{n\to\infty} n^{4\beta-1} \log\left[\sum_{m=n\varepsilon}^{n}\right] \leqq (\lambda^4/2 - \lambda^2 y)^+ . \tag{12}$$

Combining (10) and (12), and using the trivial inequality

$$\log(x+y) \leqq \log 2 + \max(\log x, \log y) ,$$

it follows easily that

$$\limsup_{n\to\infty} n^{4\beta-1} \log\left[\sum_{m=1}^{n}\right] \leqq (\lambda^4/2 + \lambda^2 y)^+ .$$

Using (2) and (13) now completes the proof.

4. Proof of Lemma 3

Lemma 3. *If ξ_0 is a nonrandom initial configuration with*

$$\frac{1}{2n} \sum_{m=-n}^{n} \xi_0(m) \to 1 \quad \text{as } n \to \infty \tag{$*$}$$

then

$$n^{6\beta-2} \log E \exp(\lambda L_n/n^{\beta}) \to \lambda^6/4 .$$

Proof. From (1.5)

$$\log E \exp(\lambda L_n/n^{\beta}) = \sum_{x=-n}^{n} \xi_0(x) \log E_x \exp(\lambda W_n/n^{\beta}) . \tag{1}$$

First we dispense with the terms that are too far out to contribute.

$$\sum_{m=1}^{n} P_x(T_1 = m) E_1 \exp(\lambda W_{n-m}/n^{\beta}) \tag{2}$$

$$\leqq P_x(T_1 \leqq n) \sup_{m \leqq n} E_1 \exp(\lambda W_{n-m}/n^{\beta}) .$$

As we argued in the proof of (3.10)

$$\limsup_{n\to\infty} n^{4\beta-1}\log\left[\sup_{0\leqq m\leqq n} E_1\exp(\lambda W_{n-m}/n^{\beta})\right] = \lambda^4/2\,. \tag{3}$$

For the other factor on the right in (2), we observe that if $x \geqq cn^{1-2\beta}$

$$P_x(T_1 \leqq n) \leqq 2P_1(S_n \geqq x) \leqq 2P_1(S_n \geqq cn^{1-2\beta})\,. \tag{4}$$

Using (3.5)

$$\limsup_{n\to\infty} n^{4\beta-1}\log P(S_n \geqq cn^{1-2\beta}) \leqq -c^2/2\,. \tag{5}$$

If we let $c = 2\lambda^2$ then it follows from (3) and (5) that if n is large

$$P_1(S_n \geqq 2\lambda^2 n^{1-2\beta}) \sup_{0\leqq m\leqq n} E_1\exp(\lambda W_{n-m}/n^{\beta}) \leqq \exp(-\lambda^4 n^{1-4\beta})\,. \tag{6}$$

Combining (3.2), (2), (4), and (6) we have

$$E_x\exp(\lambda W_n/n^{\beta}) \leqq 1 + 2\exp(-\lambda^4 n^{1-4\beta}) \tag{7}$$

for all $x \geqq 2\lambda^2 n^{1-2\beta}$ if n is large. If we let $\sum_x^0$ denote the sum over $2\lambda^2 n^{1-2\beta} < |x| \leqq n$ (o for outside), and use (∗) it follows that

$$n^{6\beta-2}\sum\nolimits_x^0 \xi_0(x)\log E_x\exp(\lambda W_n/n^{\beta}) \to 0\,. \tag{8}$$

To deal with $\sum_x^i$ = the sum over $|x| \leqq 2\lambda^2 n^{1-2\beta}$, we start by supposing

$$\frac{1}{n}\sum_{m=0}^{n}\xi_0(m) \to 1 \quad\text{and}\quad \frac{1}{n}\sum_{m=-n}^{-1}\xi_0(m) \to 1\,. \tag{$*'$}$$

In this case if we define measures μ_n by $\mu([-2\lambda^2, 2\lambda^2]^c) = 0$ and

$$\mu_n(A) = n^{2\beta-1}\sum_{x\in n^{1-2\beta}A}\xi_0(x) \quad\text{for}\quad A \subset [-2\lambda^2, 2\lambda^2]\,, \tag{9}$$

then μ_n converges weakly to μ, Lebesgue measure on $[-2\lambda^2, 2\lambda^2]$. If we let

$$G_n(y) = n^{4\beta-1}\log E_{[yn^{1-2\beta}]}\exp(\lambda W_n/n^{\beta})$$

then Lemma 2 implies that $G_n(y_n) \to G(y) = (\lambda^4/2 - \lambda^2|y|)^+$ when $y_n \to y$. To get

$$\int G_n(y)\mu_n(dy) \to \int G(y)\mu(dy) \tag{10}$$

we use:

(11) **Lemma.** *If measures* $\nu_n \Rightarrow \nu$ *a finite measure,* $|f_n(y_n)| \leqq M$ *for all* y_n *in the support of* ν_n, *and* $f_n(y_n) \to f(y)$ *whenever* $y_n \to y$ *in the support of* ν, *then*

$$\int f_n(y)\nu_n(dy) \to \int f(y)\nu(dy)\,.$$

Proof. By dividing ν_n and multiplying f_n by $\nu_n(\mathbb{R})$ we can suppose without loss of generality that the ν_n's are probability measures. Let X_n have distribution ν_n and converge a.s. to X with distribution ν. Using the bounded convergence theorem, we conclude $Ef_n(X_n) \to Ef(X)$.

Applying (11) proves (10) and we have

$$n^{6\beta-2} \sum_x^i \xi_0(x) \log E_x \exp(\lambda W_n/n^\beta) \to \int_{-2\lambda^2}^{2\lambda^2} G(y)\,dy \,. \tag{12}$$

(12) and (8) give the desired result. To prove the last conclusion under the weaker assumption (∗) we observe that the sequence μ_n defined in (9) is tight. If $\mu_{n(k)} \Rightarrow \mu'$ then (11) implies

$$\int G_{n(k)}(y)\mu_{n(k)}(dy) \to \int G(y)\mu'(dy) = \int G(y)\mu(dy)$$

since $G(-y) = G(y)$ and (∗) implies $\mu_n([-a, a]) \to 2a$ for $a \leqq 2\lambda^2$.

5. Proofs of Theorems 2 and 3

We start with a little notation:

$$F_n(x, \lambda) = E_x\{\exp(\lambda W_n/n^{1/4}) - 1\}$$

$$H_n(y_n, \lambda) = F_n(y_n n^{1/2}, \lambda)$$

$$H(y, \lambda) = \int_0^1 P_y(\tau^0 \in ds)\{\varphi((1-s)^{1/4}\lambda) - 1\}$$

As with Theorem 1, it suffices to prove

$$\lim_{n\to\infty} n^{-1/2} \log E \exp(\lambda L_n/n^{1/4}) = \int dy \log(1 + H(y, \lambda)) \tag{1a}$$

$$\lim_{n\to\infty} n^{-1/2} \log \hat{E} \exp(\lambda L_n/n^{1/4}) = \int dy H(y, \lambda) \tag{1b}$$

where E and $\hat{E}$ indicate expected value starting from a nonrandom initial configuration satisfying (∗) and starting from $\xi_0(x)$, $x \in \mathbb{Z}$ i.i.d. Poisson mean 1, respectively.

(1.5) implies

$$n^{-1/2} \log E \exp(\lambda L_n/n^{1/4}) = n^{-1/2} \sum_x \xi_0(x) \log(1 + F_n(x, \lambda)) \,. \tag{2a}$$

To compute the corresponding quantity for a Poisson initial configuration, we observe that if $\theta(x) \neq 0$ for only finitely many x

$$\hat{E} \exp\left(\sum_x \theta(x)\xi_0(x)\right) = \prod_x \exp(e^{\theta(x)} - 1) \,.$$

So $\hat{E} \exp(aL_n) = \prod_x \exp\{E_x \exp(aW_n) - 1\}$ and

$$n^{-1/2} \log \hat{E} \exp(\lambda L_n/n^{1/4}) = n^{-1/2} \sum_x F_n(x, \lambda) \,. \tag{2b}$$

We turn our attention now to computing the limit of $F_n(x, \lambda)$. (1.11) implies

$$E_x\{\exp(\lambda W_n/n^{1/4}) - 1\} = \sum_{m=1}^{n} P_x(T_1 = m) E_1\{\exp(\lambda W_{n-m}/n^{1/4}) - 1\} \tag{3}$$

(1.9) and scaling implies that if $m/n \to t \in [0, 1]$

$$E_1 \exp(\lambda W_{n-m}/n^{1/4}) \to \varphi((1-t)^{1/4}\lambda) . \tag{4}$$

If $n \to \infty$ and $x_n/n^{1/2} \to y$ then it follows from Donsker's theorem that

$$P_{x_n}(T_1/n^{1/2} \in ds) \Rightarrow P_y(\tau_0 \in ds) . \tag{5}$$

Combining (3)–(5) and using (4.11) it follows that if $y_n \to y$ then

$$H_n(y_n, \lambda) \to H(y, \lambda) . \tag{6}$$

($*$) guarantees $n^{-1/2} \sum_{|x| \leqq cn^{1/2}} \xi_0(x) \to 2c$ and if $y_n \to y$ we have

$$\log(1 + H_n(y_n, \lambda)) \to \log(1 + H(y, \lambda)) .$$

The limit is symmetric, so the argument at the end of Section 4 implies

$$n^{-1/2} \sum_{|x| \leqq An^{1/2}} \xi_0(x) \log(1 + F_n(x, \lambda)) \to \int_{-A}^{A} dy \log(1 + H(y, \lambda)) . \tag{7a}$$

A similar but easier argument shows

$$n^{-1/2} \sum_{|x| \leqq An^{1/2}} F_n(x, \lambda) \to \int_{-A}^{A} dy\, H(y, \lambda) . \tag{7b}$$

To control the contribution from outside we use:

(8) **Lemma.** *Fix $\lambda_0 > 0$. There is a constant $K(\lambda_0) < \infty$ so that*

$$|F_n(x, \lambda)| \leqq K(\lambda_0) P_x(T_1 \leqq n) \quad \text{for} \quad \lambda \leqq \lambda_0 .$$

Proof. For $|\lambda| \leqq \lambda_0$, $E_1 \exp(\lambda W_n/n^{1/4})$ is smaller than

$$E_1 \exp(\lambda_0 W_n/n^{1/4}) + E_1 \exp(-\lambda_0 W_n/n^{1/4}) \to \varphi(\lambda_0) + \varphi(-\lambda_0) ,$$

so $K(\lambda_0) = \sup_n \sup_{|\lambda| \leqq \lambda_0} E_1 \exp(\lambda W_n/n^{1/4}) < \infty$. From (3) we get

$$|F_n(x, \lambda)| \leqq \sum_{m=1}^{n} |P_x(T_1 = m) E_1\{\exp(\lambda W_{n-m}/n^{1/4}) - 1\}|$$

and the proof is complete.

Using (8) now gives

$$\begin{aligned} n^{-1/2} \sum_{x > An^{1/2}} \xi_0(x) |F_n(x, \lambda)| &\leqq K(\lambda) n^{-1/2} \sum_{x > An^{1/2}} \xi_0(x) P_x(T_1 \leqq n) \\ &\leqq 2K(\lambda) n^{-1/2} \sum_{x > An^{1/2}} \xi_0(x) P_1(S_n \geqq x) \\ &\leqq 2K(\lambda) n^{-1/2} \sum_{m=A}^{\infty} \eta_m P_1(S_n \geqq mn^{1/2}) \end{aligned} \tag{9}$$

where $\eta_m = \xi_0(mn^{1/2} + 1) + \ldots + \xi_0((m+1)n^{1/2})$. $(*)$ implies that there is a $C < \infty$ with $\xi_0(1) + \ldots + \xi_0(l) \leqq Cl$, so $\eta_m \leqq C(m+1)n^{1/2}$, and it follows that

$$n^{-1/2} \sum_{x > An^{1/2}} \xi_0(x)|F_n(x, \lambda)| \leqq 2K(\lambda) \sum_{m=A}^{\infty} C(m+1)P_1(S_n \geqq mn^{1/2}) . \tag{10}$$

For $n \geqq 1$,

$$P_1(S_n \geqq mn^{1/2}) \leqq P_0(S_n \geqq (m-1)n^{1/2}) \leqq B/(m-1)^4$$

where $B = \sup_n E(S_n/n^{1/2})^4 < \infty$. Using this in (10) gives

$$\lim_{A \to \infty} \sup_{n \geqq 1} n^{-1/2} \sum_{|x| > An^{1/2}} \xi_0(x)|F_n(x, \lambda)| = 0 . \tag{11}$$

Since $\log(1+u) \leqq u$, and $\xi_0(x) \equiv 1$ satisfies $(*)$, the last result implies

$$\lim_{A \to \infty} \sup_{n \geqq 1} n^{-1/2} \sum_{|x| > An^{1/2}} \xi_0(x) \log(1 + |F_n(x, \lambda)|) = 0 , \tag{12a}$$

$$\lim_{A \to \infty} \sup_{n \geqq 1} n^{-1/2} \sum_{|x| > An^{1/2}} |F_n(x, \lambda)| = 0 , \tag{12b}$$

and the proofs are complete.

6. Proof of Theorem 4

By remarks in the introduction the proof of Theorem 4 will be complete when we show that if $\alpha > 3/4$ then $\hat{P}(|L_n^+| > n^\alpha) \to 0$, where L_n^+ is the contribution of the particles starting from $x \neq 0$. To prove this it suffices to show $E(L_n^+)^2 \leqq A + Bn$. For then the result follows from Chebyshev's inequality

$$\hat{P}(|L_n^+| > n^\alpha) \leqq E|L_n^+|^2/n^{2\alpha} \leqq (A + Bn)/n^{3/2} \to 0 .$$

To compute $E|L_n^+|^2$, we begin with the first and second moments of $W_n = V(S_0) + \ldots + V(S_n)$.

$$\begin{aligned} E_0 W_{2n+1} &= \sum_{m=0}^{n} P_0(S_{2m} = 0) - P_0(S_{2m+1} = 1) \\ &= \sum_{m=0}^{n} 2^{-2m} \begin{bmatrix} 2m \\ m \end{bmatrix} - 2^{-(2m+1)} \begin{bmatrix} 2m+1 \\ m \end{bmatrix} \\ &= \sum_{m=0}^{n} 2^{-2m} \begin{bmatrix} 2m \\ m \end{bmatrix} \left(1 - \frac{1}{2} \cdot \frac{2m+1}{m+1}\right) \geqq 0 . \end{aligned}$$

Since $E_0 W_{2n+2} = E_0 W_{2n+1} + P_0(S_{2n+2} = 0)$, it follows that $E_0 W_k \geqq 0$ for all k. To estimate the size of this quantity observe that $E_1 W_k = -E_0 W_k$ and define a stopping time by $N = \inf\{m \geqq 1 : S_m = 1\}$. Then

$$E_0 W_n = E_0\left[\sum_{m < N \wedge n} V(S_m)\right] + \sum_{m=1}^{n} P(N = m) E_1 W_{n-m} \leqq 2 ,$$

since the second term is negative, and if we replace $N \wedge n$ by N in the first we get 2. To extend the bound to $x < 0$ we observe

$$0 \leqq E_x W_n = \sum_{m=1}^{n} P_x(T_0 = m) E_0 W_{n-m} \leqq 2P_x(T_0 \leqq n) \tag{1}$$

For the second moment we observe that (1.9) implies that for all λ

$$E_0(\exp(\lambda W_n/n^{1/4})) \to \varphi(\lambda) < \infty ,$$

so

$$\sup_n E_0\{\exp(W_n/n^{1/4}) + \exp(-W_n/n^{1/4})\} < \infty .$$

"Dominated convergence" gives $E_0(W_n/n^{1/4})^2 \to E(\aleph)^2$ and it follows that

$$E_0 W_n^2 \leqq Cn^{1/2} . \tag{2}$$

Our next step in computing the moments of L_n^+ is to look at L_n^x the contribution of the particles starting at x. For this we need

(3) **Lemma.** *Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2, and let $Y_n = X_1 + \ldots + X_n$. If N is independent of the sequence $EY_N = \mu EN$ and*

$$\mathrm{Var}(Y_N) = \sigma^2 EN + \mu^2 \mathrm{Var}(N) .$$

Proof. By conditioning on the value of N we see $E((Y_N - \mu N)(\mu N - \mu EN)) = 0$, so

$$E(Y_N - \mu EN)^2 = E(Y_N - \mu N)^2 + \mu^2 E(N - EN)^2 .$$

Using (3) gives $EL_n^x = E_x W_n$ and

$$\mathrm{Var}(L_n^x) = \mathrm{Var}_x(W_n) + (E_x W_n)^2 = E_x(W_n^2) .$$

To compute the last quantity we observe that if $x < 0$

$$E_x(W_n^2) = \sum_{m=1}^{n} P_x(T_0 = m) E_0(W_{n-m}^2) \leqq Cn^{1/2} P_x(T_0 \leqq n) \tag{4}$$

by (2). Putting things together

$$EL_n^+ = -E_0 W_n \in [-2, 0]$$

$$\mathrm{Var}(L_n^+) \leqq \sum_x E_x W_n^2 = 2 \sum_{x \leqq 0} E_x W_n^2 \leqq 2Cn^{1/2} \sum_{x \leqq 0} P_x(T_0 \leqq n)$$

$$\leqq 4Cn^{1/2} \sum_{x \geqq 0} P_0(S_n \geqq x) \leqq 4Cn^{1/2} E_0|S_n| \leqq C'n .$$

So $E(L_n^+)^2 \leqq 4 + C'n$ and the proof is complete.

References

Cox, J.T., Griffeath, D.: Large deviations for Poisson systems of independent random walks. Z. Wahrscheinlichkeitstheor. Verw. Geb. **66**, 543–558 (1984)

Darling, D.A., Kac, M.: On occupation times for Markov processes. Trans. Am. Math. Soc. **84**, 444–458 (1957)
Dobrushin, R.L.: Two limit theorems for the simplest random walk on the line. Usp. Mat. Nauk **10**, 139–146 (1955)
Donsker, M.D., Varadhan, S.R.S.: Large deviations for noninteracting particle systems. J. Stat. Phys. **46**, 1195–1232 (1987)
Kesten, H.: Occupation times for Markov chains and semi-Markov chains. Trans. Am. Math. Soc. **103**, 82–112 (1962)
Lee, T.Y.: Large deviations for noninteracting recurrent particles. Preprint 1988
Remillard, B.: Private communication, 1988

Received August 24, 1988; in revised form April 25, 1989