

Ergodicity of Reversible Reaction Diffusion Processes

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Summary. Reaction–diffusion processes were introduced by Nicolis and Prigogine, and Haken. Existence theorems have been established for most models, but not much is known about ergodic properties. In this paper we study a class of models which have a reversible measure. We show that the stationary distribution is unique and is the limit starting from any initial distribution.

1. Introduction

We need some notation to describe our model. Let \mathbb{Z}^d be the d dimensional integer lattice, \mathbb{Z}_+ = the nonnegative integers, and $X = \{\eta: \mathbb{Z}^d \rightarrow \mathbb{Z}_+\}$. The reaction diffusion processes considered in this paper are continuous time Markov processes with state space X that evolve in the following way:

(i) at rate $\beta(\eta(x))$ a particle is born at x ,

(ii) at rate $\delta(\eta(x))$ a particle at x dies,

(iii) at rate $\eta(x)p(x, y)$ a particle jumps from x to y . Here $p(x, y)$ is the transition probability of an irreducible symmetric random walk on \mathbb{Z}^d with $p(x, x) = 0$.

The formal generator is:

$$\begin{aligned} \Omega f(\eta) = & \sum_x \left(\beta(\eta(x)) [f(\eta + e_x) - f(\eta)] + \delta(\eta(x)) [f(\eta - e_x) - f(\eta)] \right. \\ & \left. + \sum_y \eta(y)p(y, x) [f(\eta + e_x - e_y) - f(\eta)] \right). \end{aligned} \quad (1.1)$$

Here the sums are over all x and y in \mathbb{Z}^d , and $e_x \in X$ has $e_x(x) = 1$ and $e_x(y) = 0$ for $y \neq x$.

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In the cases we will consider

$$\beta(x) = \sum_{j=0}^k b_j x^{(j)} \quad \delta(x) = \sum_{j=1}^{k+1} c_j x^{(j)}$$

where $x^{(j)} = x(x-1)\dots(x-j+1)$, the coefficients b_j and c_j are nonnegative, $k \geq 1$, and $c_{k+1} > 0$. Three concrete examples are:

1. Schlögl's first model. $k = 1$.
2. Schlögl's second model. $k = 2$, $b_1 = 0$, $c_2 = 0$.
3. Autocatalytic reaction. $k = 1$, $b_0 = 0$, $c_1 = 0$.

Here the coefficients not mentioned are assumed to be > 0 .

Reaction-diffusion systems have been studied extensively in the physics literature, but only recently have been studied by probabilists. The first step in their analysis was to construct the process. Since we have supposed that the transition probability is symmetric, it follows from results in Liggett (1973) that there is a positive function $\rho(x)$ on \mathbb{Z}^d and a finite M so that

$$\sum_y p(x, y) \rho(y) \leq M \rho(x) \quad \text{for all } x \in \mathbb{Z}^d, \quad \text{and} \quad (1.1)$$

$$\sum_x \rho(x) < \infty. \quad (1.2)$$

In what follows we fix a ρ with these properties and let

$$X_0 = \{ \eta \in X : \|\eta\| = \sum_x \eta(x) \rho(x) < \infty \}.$$

The construction of the process on X_0 , which we will denote by η_t , can be found in Chen (1985, or 1986b, or 1987). If we let E_η denote expected value for the process starting from $\eta_0 = \eta$, then Theorem 1.1 in Chen (1985) implies that the semigroup $P_t f(\eta) = E_\eta f(\eta_t)$ defined on functions f with $|f(\eta) - f(\zeta)| \leq c(f) \|\eta - \zeta\|$ has generator Ω and the following Feller property:

$$|P_t f(\eta) - P_t f(\zeta)| \leq c(f) \|\eta - \zeta\| \exp(\gamma t), \quad (1.3)$$

where γ is a constant which is independent of f . Precise statements and further properties can be found in the paper cited.

With the existence of the process established, it becomes natural to ask about its stationary distributions and asymptotic behavior as $t \rightarrow \infty$. It has been known for a long time (see e.g. Janssen (1974)) that if

$$(B) \quad \text{there is a } \lambda > 0 \text{ so that } b_j = \lambda c_{j+1} \text{ for } 0 \leq j \leq k,$$

then ν , the product measure in which each coordinate has a Poisson distribution with mean λ , is a stationary (and reversible) distribution for the process. The main result of this paper is:

Theorem. *Assume (A), (B), and $b_0 > 0$. Then ν is the only stationary distribution and is the limit starting from any initial distribution.*

If $b_0 = 0$ then δ_0 , the pointmass on the configuration $\eta(x) \equiv 0$, is also a stationary distribution. Shiga (1988) (see pp. 350–351) has conjectured that if (B) holds for the autocatalytic reaction (Example 3 above), then $\eta_t \Rightarrow \nu$ whenever $P(\eta_0 \equiv 0) = 0$. ($c_1 = b_0/\lambda = 0$ so an isolated particle cannot die.) Our techniques allow us to conclude that in this case the only translation invariant stationary distributions are convex combinations of δ_0 and ν (see the remark at the end of Sect. 3), but we have not been able to prove Shiga's conjecture. (After the original version of this paper was written, Mountford (1989) proved this conjecture under a first moment assumption on $p(x, y)$.)

It would be interesting to know whether uniqueness of the stationary distribution always holds when (A) holds and $b_0 > 0$. Neuhauser (1988, 1990) has shown that if the particles jump from x to y at rate $\varepsilon\eta(x)p(x, y)$ then uniqueness holds for $\varepsilon < \varepsilon_0$ (depending on β and δ , but not on p). By rescaling time one can, of course, reformulate her theorem for $\varepsilon = 1$. In general the condition on β and δ that results is somewhat complicated, but in the case of Schlögl's first model it is very simple: $b_1 < c_1$. Intuitively, when $b_1 < c_1$ the process with $b_0 = 0$ is subcritical, so the influence of the initial configuration disappears.

The rest of the paper is devoted to the proof of our theorem. In Sect. 2 we will show that if (A) holds it is possible to define the process starting from $\bar{\eta}_0 \equiv \infty$. Since the process is attractive (see (2.2)), using ideas of Holley (1972) we see that the limiting distribution of $\bar{\eta}_t$ exists. Call it $\bar{\mu}$. A similar and easier argument shows that if we let η_t denote the process starting from $\eta_0 \equiv 0$ then the limiting distribution of η_t exists. Call it $\underline{\mu}$. Attractiveness implies that for any initial distribution η_0 we can construct η_t , $\bar{\eta}_t$, and $\tilde{\eta}_t$ on the same space with $\eta_t(x) \leq \tilde{\eta}_t(x) \leq \bar{\eta}_t(x)$. To prove our theorem then, it suffices to show that $\bar{\mu} = \underline{\mu}$. Let I be the set of stationary distributions (invariant measures), and let S be the set of translation invariant measures. Since $\bar{\mu}, \underline{\mu} \in S$, it suffices to show $|I \cap S| = 1$. This is done in Sects 3 and 4 using the free energy technique developed in Holley (1971), and Holley and Stroock (1977). See Sect. IV.5 of Liggett (1985).

2. Implosion, Construction of $\bar{\mu}$ and $\underline{\mu}$

The first word in the title of the section refers to the fact that here we will show that it is possible to start the process from $\eta_0(x) \equiv \infty$, and when we do this, $E(\eta_t(x)^k) < \infty$ for all $t > 0$. At the end of the section we will use the last conclusion with $k = 1$ to construct the stationary distributions $\bar{\mu}$ and $\underline{\mu}$ mentioned in the introduction. Let η_t^n denote the process starting from $\eta_0(x) \equiv n$.

(2.1) **Lemma.** *For $m \leq n < \infty$, we can construct η_t^m and η_t^n on the same space in such a way that $\eta_t^m \leq \eta_t^n$ for all $t \geq 0$.*

Proof. We use the obvious coupling. At sites with $\eta_t^m(x) < \eta_t^n(x)$ we allow the births and deaths to occur independently. When $\eta_t^m(x) = \eta_t^n(x)$ the births and deaths occur simultaneously. Finally, we move a particle from x to y in both processes at rate $\eta_t^m(x)p(x, y)$ and in the second process only at rate $(\eta_t^n(x) - \eta_t^m(x))p(x, y)$. Further details are left to the reader.

(2.2) *Remark.* From the proof just given it should be clear that if $\eta(x) \leq \zeta(x)$ then we can construct η_t and ζ_t with these initial configurations so that $\eta_t(x) \leq \zeta_t(x)$ for all x and t , i.e. the process is *attractive*.

Lemma (2.1) implies that η_t^n increases to a limit as $n \uparrow \infty$. To prove that the limit is not $\equiv \infty$ we will prove:

(2.3) **Lemma.** *Let E^n indicate expected value for the process with $\eta(x) \equiv n$. There is a decreasing function $\varphi(t)$ on $[0, \infty)$ which is independent of n and finite for all $t > 0$ so that*

$$E^n \eta_t(x) \leq \varphi(t) \quad \text{for all } t \geq 0.$$

Proof. We begin by computing formally, i.e. we will assume that all moments are finite etc., and at the end we will give the additional arguments needed to make our computations rigorous. To simplify the expressions we drop the superscript n from the expected value.

$$\frac{d}{dt} E(\eta_t(x)) = E \left\{ \beta(\eta_t(x)) - \delta(\eta_t(x)) - \eta_t(x) + \sum_y \eta_t(y) p(y, x) \right\}. \quad (2.4)$$

Since $\sum_y p(y, x) = 1$, the last two terms cancel by translation invariance. Now,

$$\beta(x) = \sum_{j=0}^k b_j x^{(j)} \quad \text{and} \quad \delta(x) = \sum_{j=1}^{k+1} c_j x^{(j)},$$

so

$$\beta(x) - \delta(x) = \sum_{j=0}^{k+1} a_j x^j \leq a_0 + \left(\sum_{j=1}^k |a_j| \right) x^k - c_{k+1} x^{k+1} \quad \text{for } x \in \mathbb{Z}.$$

Setting $a = a_0$, $b = \sum_{1 \leq j \leq k} |a_j|$, and $c = c_{k+1}$ gives a differential inequality:

$$\frac{d}{dt} E \eta_t(x) \leq a + b E(\eta_t(x)^k) - c E(\eta_t(x)^{k+1}) \quad (2.5)$$

where $a, b, c > 0$. If $k > 1$ we write $k = 1/k + (k^2 - 1)/k$ and use Hölder's inequality with $p = k$ and $q = k/(k - 1)$ to get

$$E(\eta_t(x)^k) \leq (E \eta_t(x))^{1/k} (E \eta_t(x)^{k+1})^{(k-1)/k}. \quad (2.6)$$

Now let $f(u, v) = a + bu^{1/k}v^{(k-1)/k} - cv$, and compute

$$\frac{\partial f}{\partial v} = \frac{k-1}{k} bu^{1/k}v^{-1/k} - c \leq 0 \quad \text{if } u \leq \left(\frac{ck}{b(k-1)} \right)^k v. \quad (2.7)$$

Taking $u = E \eta_t(x)$, we see that $f(u, v)$ is decreasing in v for $v \geq (E \eta_t(x))^{k+1}$, provided that $u \geq b(k-1)/ck$. Since

$$E(\eta_t(x)^{k+1}) \geq (E \eta_t(x))^{k+1} \quad (2.8)$$

it follows that as long as $E\eta_t(x) \geq b(k-1)/ck$

$$\begin{aligned} \frac{d}{dt} E\eta_t(x) &\leq a + b(E\eta_t(x))^{1/k} (E\eta_t(x)^{k+1})^{(k-1)/k} - cE(\eta_t(x)^{k+1}) \\ &= f(E\eta_t(x), E\eta_t(x)^{k+1}) \leq f(E\eta_t(x), (E\eta_t(x))^{k+1}) \\ &= a + bE\eta_t(x) - cE(\eta_t(x)^{k+1}). \end{aligned} \quad (2.9)$$

Writing $u(t) = E\eta_t(x)$ gives

$$u'(t) \leq a + bu(t)^k - cu(t)^{k+1} \quad \text{whenever } u(t) \geq b(k-1)/ck. \quad (2.10)$$

All of the analysis since (2.5) has been for the case $k > 1$. When $k = 1$, (2.5) and (2.10) are the same so (2.10) is true for all k . Let u_0 be the largest root of $0 = a + bx^k - cx^{k+1}$, and to prepare for a later proof let $\alpha = k + 1$. If $u(t) \geq u_0$ (which is $\geq (b/c) \geq b(k-1)/ck$) for all $t \in [0, T]$ then

$$\int_0^T (a + bu(t)^k - cu(t)^\alpha)^{-1} u'(t) dt \geq T. \quad (2.11)$$

Let $\Gamma(x) = \int_{u_0+1}^x (a + bv^k - cv^\alpha)^{-1} dv \leq 0$ for $x \geq u_0 + 1$.

If $\alpha > 1$ then $\Gamma(x)$ decreases to a finite limit $\Gamma(\infty)$ as $x \uparrow \infty$. Substituting the definition of Γ into (2.10) now gives

$$\Gamma(u(T)) - \Gamma(u(0)) \geq T \quad \text{if } u(t) \geq u_0 + 1 \quad \text{for all } t \in [0, T]. \quad (2.12)$$

Since the left-hand side can be at most $-\Gamma(\infty)$ we conclude that $u(t) \leq u_0 + 1$ for some $t \leq -\Gamma(\infty)$. Since $a + bv^k - cv^\alpha < 0$ when $v = u_0 + 1$, (2.10) implies that $u(t)$ cannot rise above $u_0 + 1$ once it falls below it.

The last observation and (2.12) combine to give the desired conclusion, so it remains to show that all the computations are justified. To avoid the problems that come from unboundedness, we consider a system in which there can be at most m particles per site and a particle which jumps onto a site with m particles disappears. If we call this process $\bar{\eta}_t(x)$ then results in Chapter 1 of Liggett (1985) imply

$$\frac{d}{dt} E(\bar{\eta}_t(x)) = E \left\{ \left(\beta(\bar{\eta}_t(x)) + \sum_y \bar{\eta}_t(y) p(y, x) \right) 1_{(\bar{\eta}_t(x) < m)} - \delta(\bar{\eta}_t(x)) - \bar{\eta}_t(x) \right\}. \quad (2.4')$$

Since $\bar{\eta}_t(x) \leq m$ there is no question about the existence of moments and repeating the proof above leads to an upper bound on $\bar{u}(t) = E\bar{\eta}_t(x)$ that is independent of the truncation level. Observing that Chen's construction implies $P(\bar{\eta}_t(x) \neq \eta_t(x)) \rightarrow 0$ as $m \rightarrow \infty$ and using the monotone convergence theorem gives the desired result.

For computations in Sects 3 and 4, it is important to obtain a similar estimate on the higher moments.

(2.13) **Lemma.** *Let E^n indicate expected value for the process with $\eta(x) \equiv n$. There is a decreasing function $\varphi_m(t)$ on $[0, \infty)$ which is independent of n and finite for all $t > 0$ so that*

$$E^n(\eta_t(x)^m) \leq \varphi_m(t) \quad \text{for all } t \geq 0.$$

Proof. The computations we are about to do can be justified by an argument similar to the one given at the end of the last proof, so we will leave those details to the reader. As before we will drop the superscript n on the E . Let $\Delta_m(x) = (x + 1)^m - x^m$

$$\begin{aligned} \frac{d}{dt} E(\eta_t(x)^m) &= E \left\{ \left(\beta(\eta_t(x)) + \sum_y \eta_t(y) p(y, x) \right) \Delta_m(\eta_t(x)) \right. \\ &\quad \left. - (\delta(\eta_t(x)) + \eta_t(x)) \Delta_m(\eta_t(x) - 1) \right\} \end{aligned} \quad (2.14)$$

We use the fact that $-\eta_t(x) \Delta_m(\eta_t(x) - 1) \leq 0$ to drop that term. To deal with the sum over y we observe that using Hölder's inequality with $p = j + 1$ and $q = (j + 1)/j$ shows

$$E(\eta_t(y) \eta_t(x)^j) \leq E(\eta_t(y)^{j+1})^{1/(j+1)} E(\eta_t(x)^{j+1})^{j/(j+1)} = E(\eta_t(0)^{j+1})$$

by translation invariance. Using the last inequality in (2.14) and applying the reasoning used to convert (2.4) into (2.5) gives

$$\frac{d}{dt} E(\eta_t(x)^m) \leq a + bE(\eta_t(x)^{m+k-1}) - cE(\eta_t(x)^{m+k}), \quad (2.15)$$

where $a, b, c > 0$. Writing $m + k - 1 = m/k + (k^2 + km - m - k)/k$ and using Hölder's inequality with $p = k$ and $q = k/(k - 1)$ gives

$$E(\eta_t(x)^{m+k-1}) \leq E(\eta_t(x)^m)^{1/k} E(\eta_t(x)^{m+k})^{(k-1)/k} \quad (2.16)$$

Combining (2.15)–(2.16) with (2.7), using $E(\eta_t(x)^{m+k}) \geq E(\eta_t(x)^m)^{(m+k)/m}$ and writing $u(t) = E(\eta_t(x)^m)$ gives

$$u'(t) \leq a + bu(t)^{(m+k-1)/m} - cu(t)^{(m+k)/m} \quad \text{when } u(t) \geq c_{m,k} \quad (2.17)$$

where $c_{m,k}$ is a constant that depends on m and k . $\alpha = (m + k)/m > 1$ so the rest of the argument is the same as before.

Turning to the question of existence of stationary distributions, we observe that Lemma (2.1) implies that η_t^n increases to a limit η_t^∞ as $n \rightarrow \infty$, and from Lemma (2.3) it follows that $E \eta_1^\infty(x) \leq \varphi(1) < \infty$, so Fubini's theorem implies that $\eta_1^\infty \in X_0$ with probability 1. Let $\zeta \in X_0$ and let ζ_t^n denote the process starting from $\zeta_0^n(x) = \zeta(x) \wedge n$. By (2.2) we can construct ζ_t^n and η_t^n on the same space in such a way that $\zeta_t^n(x) \leq \eta_t^n(x)$ for all t . (1.3) implies that as $n \rightarrow \infty$, $\zeta_t^n \Rightarrow \zeta_t$, the process starting from $\zeta_0 = \zeta$, so we have

$$\eta_1^\infty \text{ and } \zeta_1 \text{ can be constructed on the same space with } \eta_1^\infty(x) \geq \zeta_1(x). \quad (2.18)$$

Applying the last conclusion to the process $\bar{\eta}_t$ starting from $\bar{\eta}_0 = \eta_1^\infty$, we see that $\bar{\eta}_0$ and a copy of $\bar{\eta}_1$ can be constructed on the same space with $\bar{\eta}_0(x) \geq \bar{\eta}_1(x)$. Iterating shows that if f is a bounded monotone function then

$$\lim_{n \rightarrow \infty} Ef(\bar{\eta}_n) \text{ exists,}$$

and from this it follows easily that $\bar{\eta}_n \Rightarrow$ a limit we call $\bar{\mu}$. (Here \Rightarrow denotes weak convergence, which in this setting is convergence of finite dimensional distributions.) Since P_t is Feller, a standard result (see Liggett (1985), Prop. 1.8) implies that $\bar{\mu}$ is a stationary distribution. A similar and easier argument shows that if we let η_t denote the process starting from $\underline{\eta} \equiv 0$ then $\underline{\eta} \Rightarrow$ a limit $\underline{\mu}$ that is a stationary distribution.

3. Free Energy Computations

Let ν be the product measure in which each marginal has a Poisson distribution with mean λ . The aim of this section is to show

Theorem. *Let $\mu \in I \cap S$ then $\mu = \nu$.*

To begin the proof, let $[-n, n]^d = \{z \in \mathbb{Z}^d: -n \leq z_i \leq n\}$, let $X(\Lambda) = \mathbb{Z}_+^\Lambda$, and let $X_n = X([-n, n]^d)$. If $\zeta \in X(\Lambda)$, let $A_\zeta = \{\eta: \eta(x) = \zeta(x) \text{ for } x \in \Lambda\}$, let $\mu(\zeta) = \mu(A_\zeta)$, and let $f_\zeta(\eta) = 1$ on A_ζ , 0 on A_ζ^c . We begin our proof with what may be a puzzling observation: If $\mu \in I$ then

$$0 = \sum_{\zeta} \int \mu(d\eta) \Omega f_\zeta(\eta) [\log \mu(\zeta) - \log \nu(\zeta)], \quad (3.1)$$

where the sum is over all $\zeta \in X_n$. (3.1) is trivial to verify, since

$$0 = \int \mu(d\eta) \Omega f_\zeta(\eta) \quad \text{for all } \zeta.$$

To explain why we want to look at this quantity, and to show the reader that our proof will not suddenly self-destruct if a minus sign is wrong somewhere, we will now give an abstract description of the proof. Suppose we have a system in which $\eta \rightarrow \eta + v$ at rate $c(v, \eta)$ where v is a vector in which only finitely many components are nonzero. In the example under consideration we have

$$\begin{array}{ll} \frac{v}{e_x} & \frac{c(v, \eta)}{\beta(\eta(x))} \\ -e_x & \delta(\eta(x)) \\ e_x - e_y & \eta(y)p(y, x) \end{array}$$

Ignoring boundary terms (i.e. summing only over v for which the support of v is contained in $[-n, n]^d$), we have

$$\int \mu(d\eta) \Omega f(\eta) = \sum_v [c(v, \xi - v) \mu(\xi - v) - c(v, \xi) \mu(\xi)]$$

Multiplying by $\log \mu(\xi)$, summing over ξ , and changing variables $\zeta = \xi - v$ and $\zeta = \xi$ in the two parts of the sum gives

$$\sum_{\zeta} \int \mu(d\eta) \Omega f_\zeta(\eta) \log \mu(\zeta) = \sum_{\zeta} \sum_v c(v, \zeta) \mu(\zeta) \log \mu(\zeta + v) / \mu(\zeta).$$

Subtracting the corresponding expression with $\log v(\zeta)$ gives

$$\sum_{\zeta} \int \mu(d\eta) \Omega f_{\zeta}(\eta) [\log \mu(\zeta) - \log v(\zeta)] = \sum_{\zeta} \sum_v c(v, \zeta) \mu(\zeta) \log \frac{\mu(\zeta + v)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta + v)}$$

Symmetrizing, the right hand side becomes

$$\frac{1}{2} \sum_{\zeta} \sum_v \{c(v, \zeta) \mu(\zeta) - c(-v, \zeta + v) \mu(\zeta + v)\} \log \frac{\mu(\zeta + v)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta + v)}.$$

If v is a reversible measure, which it is in our application, $c(v, \eta)v(\eta) = c(-v, \eta + v)v(\eta + v)$ and using (3.1) we have

$$\frac{1}{2} \sum_{\zeta} \sum_v c(v, \zeta) v(\zeta) \left\{ \frac{\mu(\zeta)}{v(\zeta)} - \frac{\mu(\zeta + v)}{v(\zeta + v)} \right\} \log \frac{\mu(\zeta + v)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta + v)} = \text{boundary terms}.$$

The key feature of the left hand side is that it is a sum of terms which all have the same sign ($(s - t) \log(t/s) \leq 0$), so if we can conclude that the sum is 0, then all the terms in the sum are, and $\mu = v$. To show that the sum is 0 we prove that if it is not then the left hand side $\sim Cn^d$ but the right hand side $= o(n^d)$.

We will now compute the right hand side of (3.1). This involves a lot of algebra.

We will prove in Sect. 4 that all the sums are absolutely convergent. Let \sum_x^i denote a sum over $x \in [-n, n]^d$ and \sum_x^o a sum over $x \notin [-n, n]^d$. (i is for in, o for out.) To split the expression into a more manageable size we will treat separately the terms that correspond to births and deaths and the ones involving motion of particles.

$$\int \Omega f_{\zeta}(\eta) \mu(d\eta) = (3.2a) + (3.2b)$$

$$\sum_x^i [\beta(\zeta(x) - 1) \mu(\zeta - e_x) - \beta(\zeta(x)) \mu(\zeta) + \delta(\zeta(x) + 1) \mu(\zeta + e_x) - \delta(\zeta(x)) \mu(\zeta)] \quad (3.2a)$$

$$\begin{aligned} & \sum_x^i \sum_y^i p(x, y) \{(\zeta(x) + 1) \mu(\zeta + e_x - e_y) - \zeta(x) \mu(\zeta)\} \\ & + \sum_x^i \sum_y^o p(x, y) \{(\zeta(x) + 1) \mu(\zeta + e_x) - \zeta(x) \mu(\zeta)\} \\ & + \sum_x^o \sum_y^i p(x, y) \sum_{k=0}^{\infty} k \{ \mu(\zeta - e_y \times k_x) - \mu(\zeta \times k_x) \} \end{aligned} \quad (3.2b)$$

Here $\zeta - e_y \times k_x \in X([-n, n]^d \cup \{x\}) = \zeta - e_y$ on $[-n, n]^d$ and $= k$ at x . Multiplying (3.2a) by $\log \mu(\zeta)$, summing over ζ , using

$$\sum_{\zeta} \beta(\zeta(x) - 1) \mu(\zeta - e_x) \log \mu(\zeta) = \sum_{\zeta} \beta(\zeta(x)) \mu(\zeta) \log \mu(\zeta + e_x),$$

and making a similar change of variables in the fourth term converts (3.2a) into

$$\sum_x^i \sum_{\zeta} \{ \beta(\zeta(x)) \mu(\zeta) - \delta(\zeta(x) + 1) \mu(\zeta + e_x) \} \log(\mu(\zeta + e_x) / \mu(\zeta)). \quad (3.3a)$$

To deal with (3.2b) we notice that changing variables $\zeta = \xi + e_y$ and $\zeta = \xi + e_x$

$$\begin{aligned} & \{(\zeta(x) + 1)\mu(\zeta + e_x - e_y) - \zeta(x)\mu(\zeta)\} \log \mu(\zeta) \\ &= (\xi(x) + 1)\mu(\xi + e_x) \log \mu(\xi + e_y) - (\xi(x) + 1)\mu(\xi + e_x) \log \mu(\xi + e_x) \end{aligned}$$

With similar changes of variables in the second two sums we can rewrite (3.2b) as

$$\begin{aligned} & \sum_x^i \sum_y^i \sum_\zeta^i p(x, y) (\zeta(x) + 1) \mu(\zeta + e_x) \log \mu(\zeta + e_y) / \mu(\zeta + e_x) \\ &+ \sum_x^i \sum_y^o \sum_\zeta^o p(x, y) \zeta(x) \mu(\zeta) \log \mu(\zeta - e_x) / \mu(\zeta) \\ &+ \sum_x^o \sum_y^i \sum_\zeta^i p(x, y) \sum_{k=0}^\infty k \mu(\zeta \times k_x) \log \mu(\zeta + e_y) / \mu(\zeta) \end{aligned} \quad (3.3b)$$

Here and below $0 \log 0 = 0$, e.g., in the second term the summand is 0 when $\zeta(x) = 0$.

To compute the second term in (3.1) we observe that by almost the same computation which led to (3.3a) and (3.3b)

$$\begin{aligned} & \sum_\zeta \int \Omega f_\zeta(\eta) \mu(d\eta) \log v(\zeta) = \\ &= \sum_x^i \sum_\zeta^i \{ \beta(\zeta(x)) \mu(\zeta) - \delta(\zeta(x) + 1) \mu(\zeta + e_x) \} \log(v(\zeta + e_x) / v(\zeta)) \\ &+ \sum_x^i \sum_y^i \sum_\zeta^i p(x, y) (\zeta(x) + 1) \mu(\zeta + e_x) \log v(\zeta + e_y) / v(\zeta + e_x) \\ &+ \sum_x^i \sum_y^o \sum_\zeta^o p(x, y) \zeta(x) \mu(\zeta) \log v(\zeta - e_x) / v(\zeta) \\ &+ \sum_x^o \sum_y^i \sum_\zeta^i p(x, y) \sum_{k=0}^\infty k \mu(\zeta \times k_x) \log v(\zeta + e_y) / v(\zeta) \end{aligned} \quad (3.4)$$

By (3.1), we have (3.3a) + (3.3b) - (3.4) = 0, so

$$\begin{aligned} 0 &= \sum_x^i \sum_\zeta^i \{ \beta(\zeta(x)) \mu(\zeta) - \delta(\zeta(x) + 1) \mu(\zeta + e_x) \} \log \left(\frac{\mu(\zeta + e_x)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta + e_x)} \right) \\ &+ \sum_x^i \sum_y^i \sum_\zeta^i p(x, y) (\zeta(x) + 1) \mu(\zeta + e_x) \log \left(\frac{\mu(\zeta + e_y)}{\mu(\zeta + e_x)} \cdot \frac{v(\zeta + e_x)}{v(\zeta + e_y)} \right) \\ &+ \sum_x^i \sum_y^o \sum_\zeta^o p(x, y) \zeta(x) \mu(\zeta) \log \left(\frac{\mu(\zeta - e_x)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta - e_x)} \right) \\ &+ \sum_x^o \sum_y^i \sum_\zeta^i p(x, y) \sum_{k=0}^\infty k \mu(\zeta \times k_x) \log \left(\frac{\mu(\zeta + e_y)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta + e_y)} \right) \end{aligned} \quad (3.5)$$

Using reversibility $\beta(\zeta(x))v(\zeta) = \delta(\zeta(x) + 1)v(\zeta + e_x)$ in the first sum and using

$$v(\zeta) = \prod_x \{ e^{-\lambda} \lambda^{\zeta(x)} / \zeta(x)! \}$$

in the last three gives

$$\begin{aligned}
0 &= \sum_x^i \sum_\zeta \beta(\zeta(x))v(\zeta) \left\{ \frac{\mu(\zeta)}{v(\zeta)} - \frac{\mu(\zeta + e_x)}{v(\zeta + e_x)} \right\} \log \left(\frac{\mu(\zeta + e_x)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta + e_x)} \right) \\
&+ \sum_x^i \sum_y^i \sum_\zeta p(x, y)(\zeta(x) + 1)\mu(\zeta + e_x) \log \left(\frac{\mu(\zeta + e_y)}{\mu(\zeta + e_x)} \cdot \frac{\zeta(y) + 1}{\zeta(x) + 1} \right) \\
&+ \sum_x^i \sum_y^o \sum_\zeta p(x, y)\zeta(x)\mu(\zeta) \log \left(\frac{\mu(\zeta - e_x)}{\mu(\zeta)} \cdot \frac{\lambda}{\zeta(x)} \right) \\
&+ \sum_x^o \sum_y^i \sum_\zeta p(x, y) \sum_{k=0}^\infty k \mu(\zeta \times k_x) \log \left(\frac{\mu(\zeta + e_y)}{\mu(\zeta)} \cdot \frac{\zeta(y) + 1}{\lambda} \right) \quad (3.6)
\end{aligned}$$

Rearranging now and using symmetry $p(x, y) = p(y, x)$ in the second sum gives

$$\sum_x^i D_n(x) + \sum_x^i \sum_y^i D'_n(x, y) = R_n \quad (3.7)$$

where

$$\begin{aligned}
D_n(x) &= \sum_\zeta \beta(\zeta(x)) v(\zeta) \left(\frac{\mu(\zeta)}{v(\zeta)} - \frac{\mu(\zeta + e_x)}{v(\zeta + e_x)} \right) \log \left(\frac{\mu(\zeta + e_x)}{\mu(\zeta)} \cdot \frac{v(\zeta)}{v(\zeta + e_x)} \right) \\
D'_n(x, y) &= \frac{1}{2} p(x, y) \sum_\zeta \{ (\zeta(x) + 1)\mu(\zeta + e_x) \\
&\quad - (\zeta(y) + 1)\mu(\zeta + e_y) \} \log \left(\frac{\mu(\zeta + e_x)(\zeta(x) + 1)}{\mu(\zeta + e_y)(\zeta(y) + 1)} \right) \\
R_n &= \sum_x^i \sum_y^o p(x, y) \sum_\zeta \zeta(x)\mu(\zeta) \log \left(\frac{\mu(\zeta - e_x)}{\mu(\zeta)} \cdot \frac{\lambda}{\zeta(x)} \right) \\
&\quad + \sum_x^o \sum_y^i p(x, y) \sum_\zeta \sum_{k=0}^\infty k \mu(\zeta \times k_x) \log \left(\frac{\mu(\zeta + e_y)}{\mu(\zeta)} \cdot \frac{\zeta(y) + 1}{\lambda} \right)
\end{aligned}$$

The summands in D and D' have the form $(s - t) \log(s/t) \geq 0$, so they are nonnegative. A second crucial property is

(3.8) **Lemma.** If $m < n$ and $x, y \in [-m, m]^d$ then $0 \leq D_m(x) \leq D_n(x)$ and

$$0 \leq D'_m(x, y) \leq D'_n(x, y).$$

Proof. $\varphi(s, t) = (s - t) \log(s/t)$ is convex and homogeneous of degree one so it is subadditive. The result now follows from the fact that if $x \in [-m, m]^d$ and $\zeta \in X_m$ then

$$(\zeta(x) + 1)\mu(\zeta + e_x) = \sum_{\zeta \in X_m, \zeta = \zeta \text{ on } [-m, m]^d} (\zeta(x) + 1)\mu(\zeta + e_x)$$

and two similar identities.

If $D_n(x) \neq 0$ for some x and n , (3.8) and translation invariance imply that the left hand side of (3.7) is $\geq An^d$ for large n . To rule out this possibility we will show that R_n , which only involves terms across the boundary, is $= o(n^d)$. There are four

terms to estimate. Since $-x \log x \leq e^{-1}$ for $x > 0$.

$$-\sum_{\zeta} \zeta(x) \mu(\zeta) \log \mu(\zeta) / \mu(\zeta - e_x) \leq e^{-1} \sum_{\zeta} \zeta(x) \mu(\zeta - e_x). \quad (3.9)$$

Throwing away a negative term

$$\sum_{\zeta} \zeta(x) \mu(\zeta) \log (\lambda / \zeta(x)) \leq (\log \lambda) \sum_{\zeta} \zeta(x) \mu(\zeta). \quad (3.10)$$

Using $\log a = 2a^{1/2} (-a^{-1/2} \log a^{-1/2}) \leq a^{1/2}$,

$$\begin{aligned} \sum_{\zeta} \sum_{k=0}^{\infty} k \mu(\zeta \times k_y) \log \mu(\zeta + e_x) / \mu(\zeta) &\leq \\ &\leq \sum_{\zeta} \sum_{k=0}^{\infty} k \mu(\zeta \times k_y)^{1/2} \{ \mu(\zeta \times k_y) \mu(\zeta + e_x) / \mu(\zeta) \}^{1/2} \\ &\leq \left(\sum_{\zeta} \sum_{k=0}^{\infty} k^2 \mu(\zeta \times k_y) \right)^{1/2} \left(\sum_{\zeta} \sum_{k=0}^{\infty} \mu(\zeta \times k_y) \mu(\zeta + e_x) / \mu(\zeta) \right)^{1/2} \\ &\leq (E^{\mu} \zeta(y)^2)^{1/2} \end{aligned} \quad (3.11)$$

Again throwing away a negative term

$$\sum_{\zeta} \sum_{k=0}^{\infty} k \mu(\zeta \times k_y) \log (\zeta(x) + 1) / \lambda \leq E^{\mu} (\zeta(y) \log (\zeta(x) + 1)). \quad (3.12)$$

The results in Sect. 4 will show that the right-hand sides of (3.9)–(3.12) are finite so using symmetry again

$$R_n \leq C \sum_x^i \sum_y^o p(x, y).$$

To estimate the double sum we let $\varepsilon > 0$, pick K so that $\sum_{|x| > K} p(0, x) \leq \varepsilon$, and observe

$$\sum_x^i \sum_y^o p(x, y) \leq (2n+1)^d \varepsilon + |[-n, n]^d - [-(n-K), (n-K)]^d|.$$

Since the second term is $o(n^d)$ and ε is arbitrary, we have shown that $R_n = o(n^d)$ and it follows from the last observation that $D_n(x) = 0$ for all x and n . When $\beta(y) > 0$ for all $y \geq 0$ the last conclusion implies that for all n

$$\mu(\zeta) / \nu(\zeta) = \mu(\zeta + e_x) / \nu(\zeta + e_x) \quad \text{for all } \zeta \in X([-n, n]^d),$$

and it follows that $\mu = \nu$.

Remark. If $\beta(0) = 0$ but $\beta(y) > 0$ for $y > 0$, then we only get

$$\mu(\zeta) / \nu(\zeta) = \mu(\zeta + e_x) / \nu(\zeta + e_x) \quad \text{for all } \zeta \in X([-n, n]^d) \text{ with } \zeta(x) > 0.$$

Now $R_n = o(n^d)$ also implies $D'_n(x, y) = 0$ for all x, y . Using this and recalling

$$\nu(\zeta + e_x) / \nu(\zeta + e_y) = (\zeta(y) + 1) / (\zeta(x) + 1)$$

gives the result we claimed in the introduction: $\mu = \theta \delta_0 + (1 - \theta) \nu$.

4. Finiteness Lemmas

In this section we complete the proof by proving some results that show that all the sums which appear in the last section are absolutely convergent. Inspecting the terms that appear in the proof, we see that the worst one is

$$\sum_{\zeta} \delta(\zeta(x) + 1) \mu(\zeta + e_x) \log \mu(\zeta),$$

which appears when we go from (3.2a) to (3.3a). The first step in bounding this is to write

$$\log \mu(\zeta) = \log \mu(\zeta + e_x) + \log \mu(\zeta) / \mu(\zeta + e_x).$$

To bound the second term we imitate (3.9)

$$- \sum_{\zeta} \delta(\zeta(x) + 1) \mu(\zeta + e_x) \log \mu(\zeta + e_x) / \mu(\zeta) \leq e^{-1} \sum_{\zeta} \delta(\zeta(x) + 1) \mu(\zeta) < \infty$$

by results in Sect. 2. To bound the first term we need to show

$$\sum_{\eta} \eta(x)^{k+1} \mu(\eta) \log \mu(\eta) < \infty.$$

To do this we begin by-proving

(4.1) **Lemma.** *Let p_j be a probability density on \mathbb{Z}_+ . For any $\delta > 0$ and $\alpha \geq 0$,*

$$\sum_j j^{\alpha+\delta} p_j < \infty \quad \text{implies} \quad - \sum_j j^{\alpha} p_j \log p_j < \infty.$$

Proof. Let $\varepsilon = \delta / (\alpha + 1)$. Let $r = (1 + \varepsilon) / \varepsilon$ and $s = 1 + \varepsilon$, so $r\varepsilon > 1$ and $1/r + 1/s = 1$. We begin by observing that $-p_j \log p_j = -p_j \log(j^\varepsilon p_j) + \varepsilon p_j \log j$. Since

$$\log j = \int_1^j x^{-1} dx \leq \frac{1}{\delta} \int_1^j \delta x^{\delta-1} dx \leq \frac{1}{\delta} j^\delta,$$

the second term is trivial to deal with. To bound the first we observe

$$\begin{aligned} -p_j \log(j^\varepsilon p_j) &= (j^\varepsilon p_j)^{1/s} \{ -r(j^\varepsilon p_j)^{1/r} \log((j^\varepsilon p_j)^{1/r}) \} / j^\varepsilon \\ &\leq r(j^\varepsilon p_j)^{1/s} e^{-1/j^\varepsilon}, \end{aligned}$$

since $-x \log x \leq 1/e$ for $0 < x < 1$. Using the last result and Hölder's inequality we get

$$- \sum_j j^\alpha p_j \log(j^\varepsilon p_j) \leq r e^{-1} \sum_j j^\alpha (j^\varepsilon p_j)^{1/s} / j^\varepsilon \leq r e^{-1} \left(\sum_j j^{\alpha s + \varepsilon} p_j \right)^{1/s} \left(\sum_j j^{-r\varepsilon} \right)^{1/r},$$

and the proof is complete.

(4.2) **Lemma.** *Let μ be a probability measure on $X(A) = (\mathbb{Z}_+)^A$, where A is a finite subset of \mathbb{Z} . For each $x \in A$ let μ_x denote the distribution of $\eta(x)$. Then for $\alpha \geq 0$ and $\delta > 0$*

$$\sum_j j^{\alpha+\delta} \mu_x(j) < \infty \quad \text{for all } x \in A$$

implies
$$-\sum_{\zeta} \zeta(x)^2 \mu(\zeta) \log \mu(\zeta) < \infty,$$

where the second sum is over all $\zeta \in X(A)$.

Proof. Let $F: (\mathbb{Z}_+)^A \rightarrow \mathbb{Z}_+$ be any function that is 1-1, onto, and has

$$F(x) < F(y) \quad \text{when} \quad \sum_{i \in A} x_i < \sum_{i \in A} y_i.$$

If we let
$$\Gamma_n = \left\{ x \in (\mathbb{Z}_+)^A: \sum_{i \in A} x_i \leq n \right\}$$

and
$$\Delta_n = \left\{ x \in (\mathbb{Z}_+)^A: \sum_{i \in A} x_i = n \right\},$$

then
$$|\Gamma_{n-1}| \leq F(x) \leq |\Gamma_n| \quad \text{for} \quad x \in \Delta_n.$$

To compute $|\Gamma_n|$, notice that if $\ell = |A|$ then mapping $(x_1, \dots, x_\ell) \rightarrow (s_1, \dots, s_\ell)$ where $s_k = (1 + x_1) + \dots + (1 + x_k)$ shows

$$|\Gamma_n| = |\{(s_1, \dots, s_n): 1 < s_1 \dots < s_\ell \leq n + \ell\}| = \binom{n + \ell}{\ell}$$

so

$$n^\ell / \ell! \leq F(x) \leq (n + \ell)^\ell \quad \text{for} \quad x \in \Delta_n. \quad (4.3)$$

To get from the last inequality to the desired conclusion, observe that if μ^* is the distribution of $F(\zeta)$ then changing variables $k = F(\zeta)$ and using (4.3) gives

$$\sum_k k^{(\alpha+\delta)/\ell} \mu^*(k) = \sum_{\zeta} F(\zeta)^{(\alpha+\delta)/\ell} \mu(\zeta) \leq \sum_{\zeta} \left(\ell + \sum_{x \in A} \zeta(x) \right)^{\alpha+\delta} \mu(\zeta) < \infty,$$

by hypothesis since

$$\left(\ell + \sum_x \zeta(x) \right)^{\alpha+\delta} \leq (\ell(1 + \max_x \zeta(x)))^{\alpha+\delta} \leq \sum_x (\ell(1 + \zeta(x)))^{\alpha+\delta}.$$

Using (4.1) and (4.3) now gives

$$\infty > (\ell!) \sum_k k^{\alpha/\ell} \mu^*(k) \log \mu^*(k) \geq \sum_{\zeta} \left(\sum_x \zeta(x) \right)^{\alpha} \mu(\zeta) \log \mu(\zeta).$$

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