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Are there bushes in a forest?

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In this paper we consider a process in which each site $x \in \mathbb{Z}^d$ can be occupied by grass, bushes or trees and ask the question: Are there equilibria in which bushes and trees are both present? The answer is sometimes yes and sometimes no.

contact process * equilibria * coexistence * Markov process

1. Introduction

In this paper we consider Markov processes in which the state at time t is $\xi_t : \mathbb{Z}^d \to \{0, 1, 2\}$. We think of $0 =$ grass, $1 =$ bushes, and $2 =$ trees, and formulate the evolution as follows: (i) 1's and 2's each die (i.e. become 0) at rate 1. (ii) 1's (resp. 2's) give birth at rate λ_1 (resp. λ_2). (iii) If the birth occurs at x the offspring is sent to a site chosen at random from $\{y: y-x \in \mathcal{N}\}$, $\mathcal{N} =$ the set of neighbors of 0. (iv) If $\xi_t(y) \geq \xi_t(x)$ then the birth is suppressed. The last rule reflects the fact that grass, bushes, trees is a successional sequence, i.e. each plant can displace its predecessor. Since 2's can replace 1's or 0's, it should be clear that $\zeta_t = \{y: \xi_t(y) = 2\}$ is a Markov process. In the terminology of Liggett (1985) or Durrett (1988), it is the contact process with neighborhood set $\mathcal{N}$.

Let $\lambda_c = \inf\{\lambda_2 : P(\zeta_t^0 \neq \emptyset \text{ for all } t \geq 0) > 0\}$, where $\zeta_t^0 = \{y: \xi_t^{20}(y) = 2\}$ when $\xi_0^{20}(0) = 2$ and $\xi_0^{20}(x) = 0$ for $x \neq 0$. Here the superscript 20 on ξ suggests a 2 in a sea of 0's, and the superscript 0 on ζ indicates that this set valued process has $\zeta_0^0 = \{0\}$. If $\lambda_2 < \lambda_c$ then the 2's die out and the process reduces to a one type contact process, so we will only be interested in what happens when $\lambda_2 > \lambda_c$. Our first result shows that the one dimensional nearest neighbor case is not interesting.

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Theorem 1.1. *Suppose $d=1$ and $\mathcal{N}=\{-1,1\}$. If $\lambda_2>\lambda_c$ then the '1's die out', that is, if $|\zeta_t|=|\{y: \xi_t(y)=2\}|=\infty$, then $\xi_t\Rightarrow\mu_2$ the limit starting from all sites $=2$.*

Here $\Rightarrow$ denotes weak convergence, which in this setting is just convergence of finite dimensional distributions. The last result is easy to prove. Generalizing Lemma 3.1 of Durrett (1980) we conclude that if ξ_t^2 is the process starting from $\xi_0^2(x)\equiv 2$, then $\xi_t^{20}(x)=\xi_t^2(x)$ for $x\in[l_t, r_t]$ where $l_t=\inf\zeta_t^0$, and $r_t=\sup\zeta_t^0$. Well known results for the contact process (see the paper cited; Liggett, 1985, Chapter VI; Durrett, 1988, Chapter 4; for this and other well known results cited below) imply that on $\Omega_\infty=\{\zeta_t^0\neq\emptyset$ for all $t\}$, $l_t\to-\infty$, and $r_t\to\infty$ a.s. When there are infinitely many 2's in the initial configuration one of them must start a process that lives forever and the result follows.

If $d=1$ and $\mathcal{N}=\{y: 0<|y|\le M\}$ then breaking $\mathbb{Z}$ into blocks of length M, calling a block occupied if all the sites have state 2, and comparing with oriented percolation, it is easy to see that there is a constant $c_M<\infty$ so that if $\lambda_2>c_M$ then the 1's die out. It is harder to show that the other alternative can occur.

Theorem 1.2. *Suppose $\mathcal{N}=\{y\in\mathbb{Z}^d: \|y\|_\infty\le M\}$ where $\|y\|_\infty=\sup|y_i|$. If $\lambda_1>\lambda_2^2>1$ then 'coexistence' occurs for large M, that is, there is a translation invariant stationary distribution μ_{12} with $\mu_{12}(A_\infty^1\cap A_\infty^2)=1$ where $A_\infty^i=\{\eta: \eta(x)=i$ for infinitely many $x\}$.*

Note. To simplify a calculation in the proof we have $0\in\mathcal{N}$, even though births from x to x can have no effect. The model with $0\in\mathcal{N}$ is equivalent to the one with 0 removed and the λ_i reduced by a factor of $1-(2M+1)^{-d}$.

To explain the condition in Theorem 1.2, let $Y=\mathbb{Z}^d/M$, $\mathcal{N}=\{y\in Y: \|y\|_\infty\le 1\}$. Results of Bramson, Durrett and Swindle (1989) imply that if $\lambda_2>1$ then as $M\to\infty$, μ_2 approaches a product measure with density $(\lambda_2-1)/\lambda_2$. Setting $M=\infty$, we define a *mean field* version of the set of sites occupied by 1's, Z_t in which:

(i) Each particle (i.e. point of Z_t) dies at rate 1 and gives birth at rate λ_1.

(ii) The offspring of a particle at x is sent to a point y chosen at random from $\{y: \|y-x\|_\infty\le 1\}$. We flip a coin with probability $(\lambda_2-1)/\lambda_2$ of heads to see if y is occupied by a 2. If it is, the birth is suppressed.

(iii) To simulate births from sites occupied by 2's, each x in Z_t^0 is at rate λ_2 'attacked' by a randomly chosen y with $\|y-x\|_\infty\le 1$. We flip another coin to see if y is occupied by a 2. If it is, we remove x from Z_t^0.

Z_t^0 is just a branching random walk in which 1's die at rate $1+\lambda_2\cdot(\lambda_2-1)/\lambda_2=\lambda_2$, and births occur at rate $\lambda_1\cdot 1/\lambda_2$, so in order for the 1's to survive we must have $\lambda_1/\lambda_2>\lambda_2$.

To prove Theorem 1.2 we use an idea of Bramson and Durrett (1989). We first show that when viewed on suitable length and time scales, the mean field process Z_t dominates oriented site percolation with $p=1-\varepsilon$. In the construction the site

(m, n) is declared to be open if a certain good event happens in the space-time box:

$$B_{m,n} = (2mL, 0, \ldots, 0, nT) + \{(-2L, 2L)^d \times [0, T]\}.$$

Since the event in the construction depends on what happens in a finite space time box, it follows from 'continuity' that when M is large, the set of 1's in the real system dominates the percolation process with $p = 1 - 2\varepsilon$. If $2\varepsilon < \frac{1}{81}$, known results for percolation give a positive lower bound on the deensity of 1's and standard techniques take over (take Cesaro averages and extract a convergent subsequence) to produce the desired stationary distribution.

Theorem 1.2 suggests a number of questions. The first, and easiest to answer, is what happens if 1's die at rate δ_1 and 2's die at rate δ_2. Repeating the heuristic proof we see that in the limit $M \to \infty$: (a) 1's die at rate δ_1 and are replaced by 2's at rate $\lambda_2 \cdot (\lambda_2 - \delta_2)/\lambda_2$ and (b) 1's give birth onto sites not occupied by 2's at rate $\lambda_1 \cdot \delta_2/\lambda_2$, so for survival we need

$$\lambda_1 \cdot \delta_2 > \lambda_2(\delta_1 + \lambda_2 - \delta_2).$$

The proof of Theorem 1.2 that we give works for this case as well, but for simplicity we will restrict our attention to the case $\delta_1 = \delta_2 = 1$.

A second natural question is to describe the set of stationary distributions. There are three trivial ones: let μ_i be the limit starting from $\xi_0^i(x) \equiv i$. Durrett and Møller (1991) have recently shown that:

Theorem. *If $\lambda_1 > \lambda_2^2 > 1$ and M is large then μ_{12} is the limit starting from any initial η in $A_\infty^1 \cap A_\infty^2$. Therefore all stationary distributions are convex combinations of μ_0, μ_1, μ_2 and μ_{12}.* □

The proof of this theorem provides the following quantitative information about μ_{12}.

Theorem. *As $M \to \infty$, μ_{12} approaches a product measure in which 2's, 1's, and 0's appear with densities $(\lambda_2 - 1)/\lambda_2$, $(\lambda_1 - \lambda_2^2)/(\lambda_1\lambda_2)$ and $\lambda_2^2/(\lambda_1\lambda_2)$ respectively.* □

Notice that the density of 1's in the limit approaches 0 as $\lambda_1 \downarrow \lambda_2^2$. This supports our conjecture that the condition in Theorem 1.2 is sharp.

Conjecture 1.1. If $\lambda_1 < \lambda_2^2$ then the 1's die out for large M.

Ironically, this seems much more difficult to prove than Theorem 1.2. The problem is that if A is the set of sites occupied by 1's at 0 at time 0, we can as a worst case suppose that all sites outside A are occupied by 2's and prove survivial. This type of reasoning cannot be used to prove Conjecture 1.1. In the best case there would be no 2's but then the 1's would survive for $\lambda_1 > \lambda_c$.

Having seen that in one dimension coexistence can happen for $\mathcal{N} = \{y: 0<|y|\leq M\}$ when M is large but not when M is 1, it is natural to ask where the changeover occurs. Our guess is:

Conjecture 1.2. Coexistence is possible when $d=1$ and $\mathcal{N}=\{-2,-1,1,2\}$.

This result is difficult to prove because coexistence can occur only for λ_2 near the critical value. Computer simulations indicate that the critical value is about 2.6 and coexistence is no longer possible when $\lambda_2 > 3.2$.

One method for proving the last conjecture would be to find a way of characterizing the values of λ_2 for which coexistence is possible for some value of λ_1. One approach to this problem is to consider the system with $\lambda_1=\infty$ starting from $\xi_0(x)=2$ for $x\leq 0$ and $\xi_0(x)=1$ for $x>0$. By $\lambda_1=\infty$, we mean that if there is a 0 within range of a site occupied by a 1, it immediately becomes a 1. Let $r_t=\sup\{x: \xi_t(x)=2\}$. It is known that $r_t/t\to\alpha(\lambda_2)$ a.s., $\lambda_c=\inf\{\lambda: \alpha(\lambda)>0\}$ and $\alpha(\lambda_c)=0$.

Conjecture 1.3. Let $l_t=\inf\{x: \xi_t(x)=1\}$. $l_t/t\to\beta(\lambda_2)$ a.s. Ones die out for all λ_1 if $\beta(\lambda_2)>0$, and can coexist for large λ_1 if $\beta(\lambda_2)<0$. Furthermore, $\alpha(\lambda_2)>\beta(\lambda_2)$ for $M\geq 2$, so coexistence is possible for λ_2 near λ_c.

One can also ask if coexistence is possible when $d=2$ and $\mathcal{N}=\{y: \|y\|_1=1\}$ where $\|y\|_1=|y_1|+\cdots+|y_d|$. We conjecture that the answer is yes, and we can prove that it is if the dimension is large enough. For simplicity (and variety) consider a discrete time model in which there can be a '1 bond' with probability p_1 and a '2 bond' with probability p_2 from $(x,n)\to(x+y,n+1)$ for $x,y\in\mathbb{Z}^d$ with $\|y\|_1=1$, and the existence of these bonds is determined by independent coin flips. (In particular, the probability bonds of both types are present is p_1p_2.) As the reader can probably guess: (a) If there is a 2 at x at time n and a 2 bond from $(x,n)\to(x+y,n+1)$ then there will be a 2 at $x+y$ at time $n+1$. (b) If there is a 1 at x at time n, a 1 bond from $(x,n)\to(x+y,n+1)$, and no site occupied by a 2 at time n gives birth onto $(x+y,n+1)$ then there will be a 1 at $x+y$ at time $n+1$. By using results of Cox and Durrett (1983) we can show:

Theorem 1.3. *If $d\geq 4$ and $\mathcal{N}=\{y: \|y\|_1=1\}$ then coexistence is possible in the discrete time model.*

To prove this result we consider a percolation model in which sites are called open if there are no 2 bonds that end at that site, and all the 1 bonds out of the site are open. Estimates on critical values in Cox and Durrett (1983) (supplemented by a numerical computation of the return probability for four dimensional simple random walk from Kondo and Hara, 1987) imply that there is a $p_2>$ the critical value for oriented bond percolation and a $p_1<1$, so that the open sites percolate. It follows easily that the two species can coexist in equilibrium.

Theorem 1.n is proved in Section $n+1$. We construct the process in Section 2. Once this is done the remainder of the paper can be read in any order. We would like to thank Simon Levin for discussions that led to these investigations. We are indebted to Gordon Slade for the reference to Kondo and Hara.

2. Proof of Theorem 1.1

We begin by constructing the process from a collection of Poisson processes. For $i \in \{1,2\}$ and $x, y \in \mathbb{Z}^d$ with $y - x \in \mathcal{N}$, let $\{T_n^{i,x,y}: n \geq 1\}$ and $\{U_n^{i,x}: n \geq 1\}$ be the arrival times of Poisson processes with rates $\lambda_i/|\mathcal{N}|$ and 1. As the reader can probably guess from the rates: at times $U_n^{i,x}$ we kill the particle at x if it is of type i, and at times $T_n^{i,x,y}$ there is a birth from x to y is in state i and y is in state $j < i$. Generalizing the usual practice in the graphical representation of the contact process we write a δ_i at $(x, U_n^{i,x})$ and draw an arrow of type i from $(x, T_n^{i,x,y})$ to $(y, T_n^{i,x,y})$. If, for example $i = 2$, we will call the last object a 2-arrow. To distinguish the two ends of the arrow we will say that it attacks y and that its source is x.

Even though there are infinitely many Poisson processes, and hence no first arrival, it is easy to show that the recipe above allows us to construct the process starting from any $\xi_0 \in \{0,1,2\}^{\mathbb{Z}^d}$. To prove this we use an idea of Harris (1972). Consider a random graph in which x and y are connected if

$$\min(T_1^{1,x,y}\, T_1^{2,x,y}\, T_1^{1,y,x}\, T_1^{2,y,x}) \leq \tau.$$

If τ is chosen small enough so that the probability of connection is $<1/|\mathcal{N}|$ then a simple argument (compare with a branching process) shows that all the components of our random graph are finite. The evolution of each component is unaffected by the others and can be computed separately. In this way we can construct the process up to time τ and iterating we can construct the process for all time.

Lemma 2.1. *Suppose $\xi_0(0) = 2$. Let ξ_t, ξ_t^{20} and ξ_t^2 be three copies of the process on $\mathbb{Z}$ with $\mathcal{N} = \{-1, 1\}$ and initial configurations that have $\xi_0^{20}(0) = 2$, $\xi_0^{20}(x) = 0$ for $x \neq 2$, and $\xi_0^2(y) = 2$ for all y. Let $\zeta_t^0 = \{y: \xi_t^{20}(y) = 2\}$, $l_t = \inf \zeta_t^0$ and $r_t = \sup \zeta_t^0$. Then*

$$\xi_t(x) = \xi_t^{20}(x) = \xi_t^2(x) \quad \text{for } x \in [l_t, r_t].$$

Proof. We check that each transition preserves the desired equalities. Here one picture is worth a hundred words.

0	2	2	0	2	0	2	2	2	0	0	2	2	2	0	0	ξ_t^2
0	2	1	0	2	0	2	2	2	0	0	2	1	2	1	0	ξ_t
0	0	0	0	2	0	2	2	2	0	0	2	0	0	0	0	ξ_t^{20}
				l_t							r_t					

At the sites $x \in (l_t, r_t)$, the three processes have the same state at $x-1$, x, and $x+1$, so arrivals $U_n^{i,x}$ and $T_n^{i,y,x}$ will have the same effect. At $x = r_t$, the state is 2 in all three processes so a $T_n^{2,x,x+1}$ arrival causes $\xi_t^2(x+1)=\xi_t^{20}(x+1)=\xi_t(x+1)=2$, r_t increases by 1 and the inequality is preserved. The remaining cases for $x=r_t$ ($U_n^{i,x}$, $T_n^{2,x,x-1}$ and $T_n^{i,x+1,x}$) are easier to check and the proof is complete. □

Proof of Theorem 1.1. Let $\zeta_t=\{x: \xi_t(x)=2\}$ and suppose without loss of generality that $0\in\zeta_t$. Let ξ_t^{20}, ζ_t^0, l_t and r_t be as in Lemma 2.1. Results in Durrett (1980) show that

$$l_t\to-\infty \text{ and } r_t\to\infty \quad \text{a.s. on } \Omega_\infty=\{\zeta_t^0\neq\emptyset \text{ for all } t\}. \tag{2.1}$$

On Ω_∞^c, let $\tau=\inf\{t: \zeta_t^0=\emptyset\}$. At time τ we pick another particle in ζ_t (which is $\neq\emptyset$ since we have supposed $|\zeta_0|=\infty$) and try again. Eventually, we find a particle that lives forever and the desired result follows from Lemma 2.1 and (2.1). For more details see Durrett (1980, pp. 902–904) where a similar 'restart argument' is used to prove the analogous conclusion for the contact process. □

3. Proof of Theorem 1.2

We consider $\xi_t:\mathbb{Z}^d/M\to\{0,1,2\}$ so that we can more easily let $M\to\infty$. Let $\mathscr{L}=\{(m,n)\in\mathbb{Z}^2: m+n \text{ is even}, n\geq 0\}$. Let $T=L^2$, $e_1=(1,0,\ldots,0)$, and define

$$\varphi(m,n)=(2mLe_1, nT) \quad \text{for } (m,n)\in\mathscr{L},$$
$$B=(-2L,2L)^d\times[0,T], \qquad B_{m,n}=\varphi(m,n)+B,$$
$$I=[-L,L]^d, \qquad I_m=2mLe_1+I.$$

Our approach will be to show:

(∗) *Let $\varepsilon>0$. If L and N are chosen appropriately and we have $A\subset I_m$ with $|A|\geq N$ occupied by 1's then, even if there are 2's at all the sites in A^c at time nT and at all sites in $B_{m,n}^c$ at times $t\in[nT,(n+1)T]$, we will with probability $\geq 1-\varepsilon$ have at least N sites occupied by 1's in I_{m+1} and I_{m-1}.*

Notice that $B_{m,n}\cap B_{(m+2),n}=\emptyset$. We assume "there are 2's at all the sites in A^c at time nT and at all sites in $B_{m,n}^c$ at times $t\in[nT,(n+1)T]$" to get enough independence to conclude that our system dominates oriented site percolation. Before embarking on the proof of (∗) we would like to observe that by translation invariance it suffices to prove the result when $m=0$ and $n=0$.

3.1. Preliminaries

To get the real system from the mean field system defined in the introduction, we will have to replace the coin flips by '2-*dual processes*', so our next step is to describe that notion. Suppose $M<\infty$ and let $\hat{\zeta}_s^{(x,t)}=$ the set of points at time $t-s$ that can be reached by a '*dual* 2 *path*' starting from (x,t). These paths can go down the graphical representation (but not through δ_2's) and across 2-arrows in the direction opposite to their orientation. It is easy to check (see Durrett, 1988, Chapter 4) that

$$\{x \in \zeta_t^A\} = \{\hat{\zeta}_t^{(x,t)} \cap A \neq \emptyset\},$$

$$\{\hat{\zeta}_s^{(x,t)}: 0 \leq s \leq t\} \stackrel{d}{=} \{\zeta_s^x: 0 \leq s \leq t\}.$$

We will decide that x is occupied by a 2 at time t if $\hat{\zeta}_s^{(x,t)}$ survives until time K or reaches $(x+[-K, K]^d)^c$. In Subsection 3.3 we will show:

Proposition 3.1. *As $M \to \infty$, $\hat{\zeta}_s^{(x,t)}$ approaches a branching random walk Y_s^x in which: (i) Particles die at rate 1 and give birth at rate λ_2. (ii) The offspring of a particle at x is sent to a point y chosen at random from $x+[-1, 1]^d$.*

Pick $\rho > (\lambda_2 - 1)/\lambda_2$ so that $1 + \lambda_2 \cdot \rho < \lambda_1/\lambda_2$. The survival probability $P(Y_s^x > 0$ for all $s \geq 0) = (\lambda_2 - 1)/\lambda_2$ so:

Proposition 3.2. *If we pick K large then the probability Y_s^x survives until time K or reaches a point in $(x+[-K+1, K-1]^d)^c$ is $<\rho$.* □

The $+1$ and -1 are to leave room for the limit $M \to \infty$. The last observation will imply that after time K or for sites in $[-2L+K, 2L-K]^d$ the collection of 2's that we see is, for large M, not too much thicker than a product measure with density ρ. We can avoid the sites near the boundary of $[-2L, 2L]^d$ by not using them. We take a rather drastic approach to cope with the first K units of time. We ignore births and observe that each 1 dies at rate $\leq \lambda_2 + 1$, so:

Proposition 3.3. *Let $\varepsilon > 0$. If N is large then with probability $\geq 1 - \frac{1}{3}\varepsilon$ the number of surviving ones at time K is at least $N' \equiv [\frac{1}{2}N \cdot \exp(-(\lambda_2 + 1)K)] + 1$.* □

The first phase of the construction decimates the set of 1's, but after that phase the set of 1's will almost be a supercritical branching random walk, so we can recover our losses. We begin by considering what happens when $M = \infty$. Let $\bar{Z}_t^x$ be a modification of the mean field process starting from a single 1 at x, in which particles die at rate $1 + \lambda_2 \rho$ and those that land outside $[-2L+K+1, 2L-K-1]^d$ are killed. It is easy to check that

$$E|\bar{Z}_t^x \cap A| = e^{\kappa t} P(\bar{S}_t^x \in A) \tag{3.1}$$

where $\kappa = (\lambda_1/\lambda_2) - (1 + \lambda_2 \rho) > 0$ and $\bar{S}_t^x$ is a random walk that starts at x, takes steps at rate (λ_1/λ_2), and is killed when it lands outside $[-2L+K+1, 2L-K-1]^d$. (Observe that both sides of (3.1) satisfy the same differential equation.)

Let $I_1' = 2Le_1 + [-L+1, L-1]^d$, i.e. I_1 shrunk by a little bit. Donsker's theorem implies that if $T' = L^2 - K$ and $x/L \to \theta \in [-1, 1]^d$ then

$$P(\bar{S}_{T'}^x \in I_1') \to \psi(\theta), \tag{3.2}$$

where $\psi(\theta)=P_\theta(B_t\in[-2,2]^d$ for $t\leqslant 1$, $B_1\in 2e_1+[-1,1]^d)$, B_t is a constant multiple of the standard d-dimensional Brownian motion, and P_θ is the distribution of the process starting from $B_0=\theta$. $\psi(\theta)>0$ and continuous, so a simple argument (suppose not and extract a convergent subsequence) shows

$$\liminf_{L\to\infty}\left(\inf_{x\in[-L,L]^d} P(\bar{S}^x_{T'}\in I_1')\right)\geqslant \inf_{\theta\in[-1,1]^d}\psi(\theta)>0. \tag{3.3}$$

It follows from (3.1) and (3.3) that we can pick L large enough so that

$$\inf_{x\in[-L,L]^d} E|\bar{Z}^x_{T'}\cap I_1'|\geqslant 4\exp((\lambda_2+1)K), \tag{3.4}$$

$$E|\bar{Z}^x_{T'}\cap I_1'|^2\leqslant E|Z^x_{T'}|^2\equiv C<\infty. \tag{3.5}$$

Now if $|A'|=N'$ (defined in Proposition 3.3), and we let $Y=|\bar{Z}^{A'}_{T'}\cap I_1'|$ where $\bar{Z}^{A'}_t$ is the modified branching process with $\bar{Z}^{A'}_0=A'$, then $EY\geqslant 2N$, and it follows from Chebyshev's inequality that

$$N^2P(Y<N)\leqslant \mathrm{var}(Y)\leqslant CN'. \tag{3.6}$$

Combining Proposition 3.3 and (3.6) we see that (*) holds for the system with $M=\infty$.

3.2. Block construction

Given a subset A of I_m, we will define a process $\bar{\eta}^{m,n,A}_t$ that is a subset of the sites occupied by 1's at time t when we start with A occupied by 1's at time nT. $\bar{\eta}^{m,n,A}_{nT}=A$ and evolves as follows.

(i) For $nT\leqslant t\leqslant nT+K$ if a δ_1 lands at x, or x is attacked by a 2-arrow, it is removed from the set. In this phase births of 1's are ignored.

(ii) For $nT+K\leqslant t\leqslant (n+1)T$, if there is a 1-arrow from $x\in\bar{\eta}^{m,n,A}_t$ to y, we look at $\hat{\zeta}^{(y,t)}_s$ to see if y is added to $\bar{\eta}^{m,n,A}_t$. If there is a 2-arrow from y to $x\in\bar{\eta}^{m,n,A}_t$ then we look at $\hat{\zeta}^{(y,t)}_s$ to see if the 1 at x will be replaced by a 2. In either case if the dual process $\hat{\zeta}^{(y,t)}_s$ survives until time K or reaches $(y+[-K,K]^d)^c$ we decide that y is occupied by a 2.

Let $A'=\bar{\eta}^{m,n,A}_{nT+K}$. If $|A|\geqslant N$ then with high probability $|A'|\geqslant \frac{1}{2}N\cdot\exp(-(\lambda_2+1)K)$. In Subsection 3.3 we will show:

Proposition 3.4. *As $M\to\infty$ the behavior of $\bar{\eta}^{m,n,A}_{nT+K+t}$ approaches that of $\bar{Z}^A_t$, so (*) holds.*

As we will now explain, this will allow us to conclude that if $H=\{(1/M)e_1,\ldots,(N/M)e_1\}$, and $\eta^H_t=\{y\colon \zeta^H_t(y)=1\}$, where ξ^H_t is the system starting with 1's on H and 2's on H^c, then η^H_t dominates oriented site percolation on $\mathcal{L}$ with $p=1-\varepsilon$. First we recall the definition of the percolation process. Given random variables $\omega(m,n)\in\{0,1\}$ that indicate whether (m,n) is open (1) or closed (0), we say (y,n) can be reached from (x,m) and write $(x,m)\to(y,n)$ if there is a sequence of points $x_m = x, \ldots, x_n = y$ so that for $m \leq l < n$, $|x_l - x_{l+1}| = 1$ and $\omega(x_l, l) = 1$. (Notice that $\omega(x_n, n)$ is allowed to be 0.) Let $\mathscr{C} = \{z: (0,0) \to z\}$ be the cluster containing 0 and let $\Omega_\infty = \{|\mathscr{C}| = \infty\}$ be the event that '*percolation occurs*'. In determining whether or not Ω_∞ occurs we only need to look at $\omega(m, n)$ for $(m, n) \in \mathscr{L}$ with $-n \leq m \leq n$, so we will only defined those variables.

Let $A_{0,0} = H$. Assuming the $A_{m,n}$ $-n \leq m \leq n$ have been defined, we will now define the $\omega(m, n)$. There are two cases to consider:

Case 1. $A_{m,n} \neq \emptyset$. We set $\omega(m, n) = 1$ if

$$|\bar{\eta}^{m,n,A(m,n)}_{(n+1)T} \cap I_{m+i}| \geq N \quad \text{for } i = +1, -1$$

and let $A^i_{m,n} \subset \bar{\eta}^{m,n,A(m,n)}_{(n+1)T} \cap I_{m+i}$ have cardinality N. Otherwise set $\omega(m, n) = 0$ and $A^i_{m,n} = \emptyset$.

Case 2. $A_{m,n} = \emptyset$. We let $H(m) = 2mLe_1 + H$ and set $\omega(m, n) = 1$ if

$$|\bar{\eta}^{m,n,H(m)}_{(n+1)T} \cap I_{m+i}| \geq N \quad \text{for } i = +1, -1$$

and $\omega(m, n) = 0$ otherwise. In either case we set $A^i_{m,n} = \emptyset$ for $i = +1, -1$.

To make the next generation of A's we set

$$A^{+1}_{-n-1,n} = A^{-1}_{n+1,n} = \emptyset \quad \text{and} \quad A_{m,n+1} = A^{+1}_{m-1,n} \cup A^{-1}_{m+1,n}.$$

To see the reason for these definitions observe that it follows by induction that

$$A_{m,n} \subset \eta^H_{nT} \quad \text{and} \quad \mathscr{C} = \{(m, n): A_{m,n} \neq \emptyset\}. \tag{3.7}$$

The definitions of the blocks imply that if we let $\mathscr{F}_t$ be the σ-field generated by the graphical representation up to time t then for $(m, n) \in \mathscr{L}$,

$$P(\omega(m, n) = 1 \mid \mathscr{F}_{nT}) \geq 1 - \varepsilon,$$

and given $\mathscr{F}_{nT}\omega(m, n) m \in \{-n, -n+2, \ldots, n\}$ are independent. Known results (see Durrett, 1988, Chapter 6) now imply that if $\varepsilon < \frac{1}{81}$ then $P(\Omega_\infty) > 0$.

3.3. *Continuity argument*

In this section we will prove Propositions 3.1 and 3.4.

Proof of Proposition 3.1. The number of points in

$$V_r \equiv \bigcup_{0 \leq s \leq r} \hat{\zeta}^{(x,t)}_s$$

is dominated by a branching process W_r in which births occur at rate λ_2 and deaths at rate 0. $EW_T = \exp(\lambda_2 t) < \infty$, so $P(W_K > M^{1/3}) \to 0$. If there are $\leq M^{1/3}$ points in V_K then the probability of having a birth land on an occupied site is smaller than

$$M^{1/3}(M^{1/3}/(2M+1)^d) \to 0. \tag{3.8}$$

To deal with the spatial location of the particles, we observe that given a realization of the branching random walk Y^x_s, $s \geq 0$, defined in the proposition, we can construct a realization of the 2-dual by replacing the displacements U_i, which are uniform on $[-1,1]^d$ by $\pi((M+\frac{1}{2})x)/M$ where $\pi(y)$ is the closest point in $\mathbb{Z}^d$ to x (with some convention for breaking ties). We will not worry about what to do after a collision (a birth onto an occupied site) occurs because (3.8) implies that the probability of a collision before time K approaches 0 when $M\to\infty$. When there are $\leq M^{1/3}$ points in V_K and no collision occurs, then all the particles in $\hat{\zeta}_K^{(x,t)}$ end up within $M^{1/3}/M$ (in L^∞ norm) of their counterparts in Y_K^x. This proves the proposition. □

The last observation in the above proof in combination with Proposition 3.2 proves:

Proposition 3.5. *If M is large the probability that $\hat{\zeta}_s^{(x,t)}$ survives until time K or reaches $(x+[-K,K]^d)^c$ is $<\rho$.* □

Proof of Proposition 3.4. The last argument contains all the ideas for showing that the behavior of $\eta_{nT+K+t}^{m,n,A'}$ approaches that of $Z_t^{A'}$ as $M\to\infty$. In the event of interest we start with $\leq 2N$ particles, so comparing with a branching process W_t in which particles are born at rate λ_1 and die at rate 0, we conclude that the probability of a newborn 1 landing on a site in $\eta_{nT+K+t}^{m,n,A}$ approaches 0 as $M\to\infty$. When $W_{T'}\leq M^{1/5}$ new 2-duals which we have to follow backwards are generated at rate $\leq\lambda_2 M^{1/5}$, so with high probability $\leq 2\lambda_2 TM^{1/5}$ duals are generated. By computing second moments and using Chebyshev's inequality we see that if M is large then

$$\left|\bigcup_{s=0}^{T}\hat{\zeta}_s^{(x,t)}\right|\leq M^{1/5}$$

for all these duals. The total number of sites we have to look at is $\leq CM^{2/5}$ so repeating the proof of (3.8) shows that the probability of a collision is small when M is large. By considering the locations of the particles involved as before, one concludes that the proposition holds and (*) follows. □

3.4. Denouement

The last detail is to explain how "known results for percolation give a positive lower bound on the density of 1's and standard techniques take over to produce the desired stationary distribution." The first part of the sentence refers to:

Proposition 3.6. *If $\varepsilon<\frac{1}{81}$ and $W_n=\{m:(m,n)\in\mathscr{C}\}$ then $P(0\in W_{2n})\geq P(\Omega_\infty)>0$ for all n.* □

(See Durrett, 1988, Chapter 5.) The second half of the sentence takes longer to explain. The first step is to take Cesaro averages of the distributions of η_s^H, $0\leq s\leq 2nT$, where $H=\{1/Me_1,\ldots,N/Me_1\}$, let $n\to\infty$ and extract a convergent subsequence. This is possible since the set of probability measures on $\{0,1,2\}^{\mathbb{Z}^d}$ is compact (in the obvious topology). Let μ be a subsequential limit. Since our process has the Feller property, μ is stationary distribution. (See Liggett, 1985). Let

A_0, A_f, A_∞ be the set of configurations in $\{0,1,2\}^{\mathbb{Z}^d}$ with 0, finitely many, and ∞ many ones respectively. Since μ is a stationary distribution $\mu(A_f)=0$, and so it follows from Proposition 3.6 that $\mu(A_\infty)>0$. Let $\mu_{12}(B)=\mu(B\cap A_\infty)/\mu(A_\infty)$. Since A_0 and A_∞ are invariant for the process, μ_{12} is a stationary distribution. By definition $\mu_{12}(A_\infty)=1$. To see that μ_{12} concentrates on configurations with infinitely many 2's we observe that an easy argument shows that $\zeta_t=\{y: \xi_t^H(y)=2\}\Rightarrow\mu_2$. (Use the self-duality of the contact process and observe that the probability the number of particles in the dual $\in[1, N]$ goes to 0.) □

4. Proof of Theorem 1.3

Let $\mathscr{L}=\{z\in\mathbb{Z}^{d+1}: z_1+\cdots+z_{d+1}$ is even$\}$ and make $\mathscr{L}$ into a graph by drawing bonds from $(x,n)\to(x+y,n+1)$ for $x,y\in\mathbb{Z}^d$ with $\|y\|_1=1$. Bond (resp. site) percolation on $\mathscr{L}$ is defined by flipping independent coins to determine the state (open or closed) of the bonds (resp. sites). We write $(x,0)\to(y,n)$ (and say (y,n) can be reached from $(x,0)$) if there is a sequence $x_0=x, x_1,\ldots,x_n=y$ so that for $0\le i<n$, $\|x_{i+1}-x_i\|=1$ and the bond from (x_i,i) to $(x_{i+1},i+1)$ (resp. the site (x_i,i)) is open. We let $\mathscr{C}_0=\{(y,n): (0,0)\to(y,n)\}$, $\Omega_\infty=\{|\mathscr{C}_0|=\infty\}$ be the event that '*percolation occurs*', and let $p_c=\inf\{p: P(\Omega_\infty)>0\}$.

Let S_n be a random walk in which $P(S_{n+1}-S_n=y)=1/(2d)$ when $\|y\|_1=1$. Let S_n and S'_n be two independent copies of the random walk starting at 0. Let

$$\pi=P(S_n=S'_n \text{ for some } n\ge1),$$

$$\sigma=P(S_n=S'_n \text{ and } S_{n-1}=S'_{n-1} \text{ for some } n\ge1).$$

π and σ are respectively the probability that the two random walks have a site or bond in common. Cox and Durrett (1983) proved

$$p_c(\text{bond})\le\sigma. \tag{4.1}$$

The same argument shows

$$p_c(\text{site})\le\pi. \tag{4.2}$$

In view of the discussion in the introduction, Theorem 1.3 follows once we can show it is possible to pick p_2 so that

$$p_2>p_c(\text{bond}) \quad\text{and}\quad (1-p_2)^{2d}>p_c(\text{site}), \tag{4.3}$$

for then it follows that $(1-p_2)^{2d}p_1^{2d}>p_c(\text{site})$ for p_1 close to 1. As $d\to\infty$, $\sigma,\pi\sim 1/(2d)$ so $(1-\sigma)^{2d}\to e^{-1}$ and the two inequalities can both be satisfied when d is large. To see that $d=4$ is large enough requires more work.

Form a random walk $R_n=Y_1+\cdots+Y_n$ with $Y_{2i-1}=X_i$ and $Y_{2i}=-X'_i$ where X_i (resp. X'_i) are the increments of S_n (resp. S'_n). It should be clear that

$$\pi=P(h_0) \quad\text{where } H_0=\{R_n=0 \text{ for some } n\ge1\}. \tag{4.4}$$

To get a formula for σ in terms of π let

$$\pi' = P(H_0, R_2 \neq 0) = P(H_0) - P(R_2 = 0) \tag{4.5}$$

and observe that either $R_2 = 0$ or $R_2 \neq 0$, so

$$\sigma = 1/(2d) + \pi'\sigma \quad \text{or} \quad \sigma = 1/(2d(1-\pi')). \tag{4.6}$$

In three dimensions the value $\pi \approx 0.340\,537\,33$ can be found in Spitzer (1976, p. 103). Plugging into (4.5) and (4.6) gives $\pi' < 0.1740$ and $\sigma < 0.2018$. Unfortunately $(0.7982)^6 = 0.2587 < 0.3405$ so the inequalities in (4.3) are inconsistent. To compute the value of π in four dimensions we turn to Kondo and Hara (1987). It is well known that

$$\frac{1}{1-\pi} = \sum_{n=0}^{\infty} P(R_n = 0),$$

and

$$P(R_n = 0) = \int_{(-\pi,\pi)^d} (2\pi)^{-d} \varphi(\theta)^n \, \mathrm{d}\theta,$$

where $\varphi(\theta) = E \exp(i\theta \cdot Y_1) = d^{-1} \sum_{i=1}^d \cos \theta_i$. Summing gives

$$\sum_{n=0}^{\infty} P(R_n = 0) = \int_{(-\pi,\pi)^d} (2\pi)^{-d} (1 - \varphi(\theta))^{-1} \, \mathrm{d}\theta.$$

Since

$$1 - \varphi(\theta) = d^{-1} \sum_{i=1}^{d} (1 - \cos \theta_i),$$

the last quantity is d times Kondo and Hara's (1987, (17) on p. 1208) $I(d; 1)$. They compute $I(4; 1) = 0.309\,866\,780\,4621$ which corresponds to $\pi < 0.1934$. Plugging into (4.5) and (4.6) gives $\pi' < 0.0684$, and $\sigma < 0.1342$. This time $(0.8658)^8 > 0.3157 > 0.1934$ so the inequalities in (4.3) are consistent and we have shown that coexistence is possible in $d = 4$. □

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