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Annihilating branching processes

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We consider Markov processes $\eta_t \subset \mathbb{Z}^d$ in which (i) particles die at rate $\delta \geq 0$, (ii) births from x to a neighboring y occur at rate 1, and (iii) when a new particle lands on an occupied site the particles annihilate each other and a vacant site results. When $\delta = 0$ product measure with density $\frac{1}{2}$ is a stationary distribution; we show it is the limit whenever $P(\eta_0 \neq \emptyset) = 1$. We also show that if δ is small there is a nontrivial stationary distribution, and that for any δ there are most two extremal translation invariant stationary distributions.

1. Introduction

In this paper we will study annihilating branching processes, or ABP for short. These systems are Markov processes whose state at time t is $\eta_t \subset \mathbb{Z}^d$. Sites $x \in \eta_t$ are considered to be occupied by particles and the system evolves according to the following rules:

(i) Particles die at rate $\delta \geq 0$.

(ii) If x is occupied and $|x-y| = 1$ then births occur from x to y at rate 1.

(iii) If y is occupied the two particles annihilate each other and an empty site results.

If (iii) were changed so that instead of annihilating, the two particles coalesced to one, we would have the contact process. Usually, in the contact process births occur at rate λ and deaths at rate 1. We have changed the time scale because we will be particularly interested in the case $\delta = 0$.

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If we let ξ_t^0 denote the contact process with $\xi_0^0 = \{0\}$, then it is known that

$$P(\xi_t^0 \neq \emptyset \text{ for all } t) \begin{cases} = 0 & \text{for large } \delta, \\ > 0 & \text{for small } \delta. \end{cases}$$

The first result extends immediately to the ABP for if η_t^0 is the ABP with $\eta_0^0 = \{0\}$ then the two systems can be constructed on the same space with $\xi_t^0 \supset \eta_t^0$. Our first result shows that the second conclusion is true as well:

Theorem 1.1. *If δ is small then $P(\eta_t^0 \neq \emptyset$ for all $t) > 0$.*

Theorem 1.1 is proved using a general method that the first and third authors have developed and that is surveyed in Durrett (1989). The key to the proof is proving that if $\varepsilon > 0$ and δ is small then the ABP dominates oriented percolation with parameter $1-\varepsilon$. The first step in explaining the last sentence is to introduce the oriented percolation process. Let $\mathscr{L} = \{(m, n) \in \mathbb{Z}^2 \colon m+n \text{ is even}\}$, and for $(m, n) \in \mathscr{L}$, let $\omega_{m,n}$ be i.i.d. with $P(\omega_{m,n} = 1) = 1-\varepsilon$ and $P(\omega_{m,n} = 0) = \varepsilon$. We say there is an open path from $(x, 0)$ to (y, n) if there is a sequence of points $m_0 = x, m_1, \ldots, m_n = y$ so that $|m_{k+1} - m_k| = 1$ and $\omega(m_k, k) = 1$ for $0 \leqslant k < n$. Let

$$W_n^0 = \{y \colon \text{there is an open path from } (0,0) \text{ to } (y, n)\}$$

and think of W_n^0 as the set of wet points at time n when the origin is wet at time 0.

To compare η_t^0 with the percolation process W_n^0, we map $\mathscr{L}$ into $\mathbb{R}^d \times [0, \infty)$ using $\varphi(m, n) = (2mLe_1, nT)$, where $e_1 = (1, 0, \ldots, 0)$, $T = \kappa_d L$, and κ_d is a constant that depends on the dimension and has to be chosen appropriately. Let $I = [-L, L]^d$, $I_m = 2mLe_1 + I$ and

$$\chi_n^0 = \{m \colon \eta_{nT}^0 \cap I_m \neq \emptyset, (m, n) \in \mathscr{L}\}.$$

With all this notation introduced we can now make a precise statement:

(*) Let $\varepsilon > 0$. If we pick L large enough and then δ small, the two processes can be defined on the same space with $\chi_n^0 \supset W_n^0$ for all $n \geqslant 0$.

If ε is small enough (e.g., $\varepsilon < \frac{1}{81}$) then it follows from known results about oriented percolation (see Durrett, 1984, Section 10) that $P(W_n^0 \neq \emptyset$ for all $n) > 0$ and we have proved Theorem 1.1.

To prove (*) we let $B = [-2L+1, 2L-1]^d \times [0, T)$, define disjoint boxes $B_{m,n} = \varphi(m, n) + B$, $(m, n) \in \mathscr{L}$, and prove:

(**) Given $\eta_{nT}^0 = A$ with $A \cap I_m \neq \emptyset$ there is a 'good' event $G_{m,n,A}$ determined by the values of the ABP in the space time box $B_{m,n}$ so that:

(i) On $G_{m,n,A}$, $\eta_{(n+1)T}^0 \cap I_{m-1} \neq \emptyset$ and $\eta_{(n+1)T}^0 \cap I_{m+1} \neq \emptyset$.

(ii) If L is large then $P(G_{m,n,A} \mid \eta_{nT}^0 = A) \geqslant 1 - \varepsilon$ for all A with $A \cap I_m \neq \emptyset$.

Once (**) is established (*) follows easily by induction. Details are given at the end of Section 4. To prove (**) it suffices to consider the case $\delta = 0$. For if we can pick L and T so that the process with $\delta = 0$ dominates oriented percolation with parameter $p=1-\varepsilon$, then we can pick δ_0 so that the probability of a death in the space time box B is $<\varepsilon$ and it follows that for $\delta\leq\delta_0$, the process with deaths at rate δ dominates oriented site percolation with parameter $1-2\varepsilon$.

Two pleasant features of the above approach are (a) the hard work is done for $\delta=0$ and (b) the proof immediately generalizes to cover perturbation by any mechanism that is translation invariant and has bounded rates. For example, suppose that instead of adding spontaneous deaths at rate δ we change the rule (i) to:

(i′) Particles jump from x to y at rate $\delta p(x,y)$ where $p(x,y)$ is the transition probability of a random walk, i.e., $p(x,y)=f(y-x)$.

A trivial modification of the argument just sketched shows that for small δ, the new system has $P(\eta_t^0\neq\emptyset$ for all $t\geq 0)>0$. Bramson and Gray (1985) used the 'contour method' to prove the last result for the special case in which $f(z)=1/(2d)$ for $z\in\mathbb{Z}^d$ with $|z|=1$ (and only gave the details for the case $d=1$). To fully appreciate the advantages of our new approach, the reader should try to use the contour method to prove Theorem 1.1 or even to extend their result to a general random walk.

Theorem 1.1 demonstrates that the process has positive probability of not dying out when δ is small. Our next goal is to describe the set of stationary distributions. One is trivial to find: $\delta_\emptyset$, the pointmass on the empty set. The key to identifying the other(s) is a duality equation:

$$P(|\eta_t^A\cap B| \text{ is odd})=P(|A\cap\eta_t^B| \text{ is odd}), \tag{1.1}$$

similar to the one for the contact process:

$$P(\xi_t^A\cap B\neq\emptyset)=P(A\cap\xi_t^B\neq\emptyset). \tag{1.2}$$

Here we assume that B is finite and the superscript indicates the initial set, e.g., $\eta_0^A=A$. To analyze the contact process one starts with the observation that if we let $A=\mathbb{Z}^d$ in (1.2) then

$$P(\xi_t^{\mathbb{Z}^d}\cap B\neq\emptyset)=P(\xi_t^B\neq\emptyset)\downarrow P(\xi_s^B\neq\emptyset \text{ for all } s)\quad \text{as } t\uparrow\infty, \tag{1.3}$$

since $\emptyset$ is an absorbing state. The analogue for the ABP is to let A be a random set with distribution $\nu_{1/2}=$ product measure with density $\frac{1}{2}$, i.e., the events $\{x\in A\}$ are independent and have probability $\frac{1}{2}$. Writing $\eta_t^{1/2}$ for η_t^A in this case,

$$\begin{aligned} P(|\eta_t^{1/2}\cap B| \text{ is odd}) &= P(|\eta_0^{1/2}\cap\eta_t^B| \text{ is odd}) \\ &= \tfrac{1}{2}P(\eta_t^B\neq\emptyset) \\ &\downarrow \tfrac{1}{2}P(\eta_t^B\neq\emptyset \text{ for all } t)\quad \text{as } t\uparrow\infty, \end{aligned} \tag{1.4}$$

since the probability of an odd number of heads in any positive number of flips of a fair coin is $\frac{1}{2}$, and $\emptyset$ is an absorbing state.

The probabilities in (1.4) determine the distribution of $\eta_t^{1/2}$ (see Griffeath, 1979, p. 69), so we have shown that $\eta_t^{1/2}$ converges weakly to a limit $\eta_\infty^{1/2}$. General results (see Liggett, 1985, part (d) of Proposition 1.8 on p. 10) imply that $\eta_\infty^{1/2}$ is a stationary distribution. (Here and in what follows we will avoid linguistic contortions by using the same symbol $\eta_t^{1/2}$ for the random variable and its distribution.) The next result implies that all translation invariant stationary distributions are a convex combination of $\delta_\emptyset$ and $\eta_\infty^{1/2}$.

Theorem 1.2. *Suppose $\delta > 0$. For any translation invariant initial distribution μ with $\mu(\{\emptyset\}) = 0$, $\eta_t^\mu \Rightarrow \eta_\infty^{1/2}$.*

Here η_t^μ denotes a version of the process with initial distribution μ and $\Rightarrow$ denotes weak convergence, which in this setting is just convergence of finite dimensional distributions.

We have ignored the case $\delta = 0$ in Theorem 1.2 because we can prove a better result in that case. When $\delta = 0$, an isolated particle cannot die, so if $B \neq \emptyset$ then $P(\eta_t^B \neq \emptyset) = 1$ for all $t \geq 0$. Using this observation in (1.4) it follows that

$$P(|\eta_t^{1/2} \cap B| \text{ is odd}) = \tfrac{1}{2} \quad \text{for all } B \neq \emptyset. \tag{1.5}$$

As remarked earlier, the probabilities in (1.5) determine the distribution of $\eta_t^{1/2}$, so we have shown that for $\delta = 0$, we have $\eta_t^{1/2} \stackrel{\text{d}}{=} \nu_{1/2}$ for all t. Our final result shows that $\nu_{1/2}$ is the only interesting stationary distribution in that case.

Theorem 1.3. *Suppose $\delta = 0$. If $P(\eta_0 \neq \emptyset) = 1$ then $\eta_t \Rightarrow \nu_{1/2}$ as $t \to \infty$.*

Remark. Let $\tau = \inf\{t\colon \eta_t = \emptyset\}$. When $\delta = 0$, $\eta_\infty^{1/2} \stackrel{\text{d}}{=} \nu_{1/2}$ and Theorem 1.3 generalizes immediately to

$$\eta_t \Rightarrow \delta_\emptyset P(\tau < \infty) + \eta_\infty^{1/2} P(\tau = \infty).$$

Using the results of Section 3 and the proof of Theorem 1.3, it is not hard to show that the last conclusion holds for small δ. It should hold for all δ but we have no idea how to show this.

The key to the proof of Theorem 1.3 is an observation of Griffeath (1978):

Proposition 1.1. *Let B be a finite set and $\tilde{\eta}_t^B$ be an independent copy of the ABP with initial state B. Then*

$$P(|\eta_{s+t} \cap B| \textit{ is odd}) = P(|\eta_s \cap \tilde{\eta}_t^B| \textit{ is odd}). \qquad \square$$

This generalization of (1.1), which is valid for $\delta \geq 0$, follows from the construction given in Section 2. To prove Theorem 1.3 it then suffices to show:

Proposition 1.2. *If $P(\eta_0 \neq \emptyset) = 1$ and $B \neq \emptyset$ is finite then*

$$P(|\eta_t \cap \tilde{\eta}_t^B| \textit{ is odd}) \to \tfrac{1}{2} \quad \textit{as } t \to \infty. \qquad \square$$

To prove this we use the fact that η_t and $\tilde{\eta}_t^B$ each dominate oriented percolation to conclude that $|\eta_t \cap \tilde{\eta}_t^B| \to \infty$ in probability. Once we know that $|\eta_t \cap \tilde{\eta}_t^B|$ is large with high probability, it is easy to conclude that $P(|\eta_{t+1} \cap \tilde{\eta}_{t+1}^B| \text{ is odd}) \approx \tfrac{1}{2}$.

In Section 2 we construct the process, derive the duality equation (1.1), and prove Theorem 1.2. In Section 3 we introduce and study a tagged particle process that is the key to the proof of (**) given in Section 4. Finally, Theorem 1.3 is proved in Section 5. Theorem 1.3 has been discovered and proved independently by Sudbury (1990).

2. Construction of the process, duality equation, proof of Theorem 1.2

We will construct the process from a graphical representation, as in Section 1 of Chapter 3 of Griffeath's (1979) book. For each $x, y \in \mathbb{Z}^d$ with $|x-y|=1$ let $\{T_n^{(x,y)}: n \geq 1\}$ and $\{U_n^x: n \geq 1\}$ be the arrival times of independent Poisson processes with rates 1 and $\delta \geq 0$ respectively. We draw an arrow from $(x, T_n^{(x,y)})$ to $(y, T_n^{(x,y)})$ to indicate that if x is occupied at time $T_n^{(x,y)}$, the particle at x will send an offspring to y. We write a δ at (x, U_n^x) to indicate that the site x will become vacant at time U_n^x. We say there is a path from $(x, 0)$ to (y, t) if there is a sequence of times $s_0 = 0 < s_1 < \cdots < s_n < s_{n+1} = t$, and spatial locations $x_0 = x, x_1, \ldots, x_n = y$, so that:

(i) For $i = 1, 2, \ldots, n$ there is an arrow from (x_{i-1}, s_i) to (x_i, s_i).

(ii) The vertical segments $\{x_i\} \times [s_i, s_{i+1}]$, $i = 0, 1, \ldots, n$, do not contain any δ's.

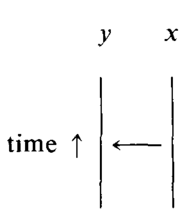
Let $N_t^x(y)$ be the number of paths from $(x, 0)$ to (y, t), and let

$$N_t^A(y) = \sum_{x \in A} N_t^x(y),$$

$$\xi_t^A(y) = N_t^A(y) \wedge 1,$$

$$\eta_t^A(y) = N_t^A(y) \mod 2.$$

Here and in what follows we take 0 and 1 to be our representatives of the two equivalence classes of integers mod 2. If we let $\xi_t^A = \{y: \xi_t^A(y) = 1\}$ then the result is the contact process. (For more details, see Durrett, 1988, Chapter 4; Liggett, 1985, Chapter VI.) We claim that $\eta_t^A = \{y: \eta_t^A(y) = 1\}$ is the ABP. To verify this, notice that if a δ occurs at y at time t then $N_t^A(y) = 0$ so y is vacant. As for the arrows, checking the various cases:



before		after	
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	1

shows that they have the desired effect.

The reason for interest in the above construction is that it allows us to define a dual process by declaring that dual paths can go (i) downward in time (but not through δ's) and (ii) across arrows in a direction *opposite* to their orientation, and setting for $0 \leq s \leq t$,

$$\hat{N}_s^{(y,t)}(x) \;=\; \text{the number of dual paths from } (y, t) \text{ to } (x, t-s),$$

$$\hat{N}_s^{(A,t)}(x) = \sum_{y \in A} \hat{N}_s^{(y,t)}(x),$$

$$\hat{\eta}_s^{(A,t)}(x) = \hat{N}_s^{(A,t)}(x) \mod 2.$$

It is easy to see that $N_t^x(y) = \hat{N}_t^{(y,t)}(x)$. So summing over $x \in A$ and $y \in B$,

$$\sum_{y \in B} N_t^A(y) = \sum_{x \in A} \hat{N}_t^{(B,t)}(x).$$

If either A or B is finite, both sums are, and

$$\sum_{y \in B} \eta_t^A(y) = \sum_{x \in A} \hat{\eta}_t^{(B,t)}(x) \mod 2.$$

A little thought reveals

$$\{\hat{\eta}_s^{(B,t)}(x): 0 \leq s \leq t\} \stackrel{d}{=} \{\eta_s^B(x): 0 \leq s \leq t\},$$

and we have proved the duality equation

$$P(|\eta_t^A \cap B| \text{ is odd}) = P(|A \cap \eta_t^B| \text{ is odd}).$$

The last proof generalizes easily to give Griffeath's observation (Proposition 1.1). Just observe

$$\{|\eta_{s+t}^A \cap B| \text{ is odd}\} = \{|\eta_s^A \cap \hat{\eta}_t^{(B,s+t)}| \text{ is odd}\}.$$

With the duality equation established we turn now to the proof of Theorem 1.2. By Griffeath's observation (Proposition 1.1) with $s = 2$ and the definition of the limit in (1.4), it is enough to show that if B is finite and $\tilde{\eta}_t^B$ is an independent copy of the process with initial state B then as $t \to \infty$,

$$P(|\eta_2^\mu \cap \tilde{\eta}_t^B| \text{ is odd}) \to \tfrac{1}{2} P(\tilde{\eta}_s^B \neq \emptyset \text{ for all } s). \tag{2.1}$$

We begin with a simple fact:

Lemma 2.1. *If $\delta > 0$ then on $\Omega_\infty = \{\eta_s \neq \emptyset$ for all $s\}$, $|\eta_t| \to \infty$ a.s.*

Proof. Let $h(\eta) = P_\eta(\Omega_\infty)$, i.e., the probability of Ω_∞ when the initial configuration is η. If $\mathscr{F}_t = \sigma(\eta_s\colon s \leq t)$ then Lévy's 0-1 law (Chung, 1974, p. 341) implies

$$h(\eta_t) = E_\eta(1_{\Omega_\infty} | \mathscr{F}_t) \to 1_{\Omega_\infty} \quad \text{a.s.} \quad \text{as } t \to \infty,$$

i.e., $h(\eta_t) \to 1$ a.s. on Ω_∞. If $|\eta_t| \leq n$, then the probability that all particles will die before they give birth is at least $(\delta/(2d+\delta))^n$, so $h(\eta_t) \leq 1 - (\delta/(2d+\delta))^n$, and the desired conclusion follows. □

The last result implies that if t is large and $\tilde{\eta}_t^B \neq \emptyset$ then $|\tilde{\eta}_t^B|$ will be large. The next ingredient is Lemma 9.14 of Harris (1976). The assumptions of that result are lengthy so we will not state them here. It is easy to check that they are satisfied for the ABP.

Lemma 2.2. *Let μ be translation invariant with $\mu(\{\emptyset\})=0$. Given $\varepsilon>0$, there is an integer $K(\varepsilon)$ so that $|\zeta| \geq K(\varepsilon)$ implies $P(\eta_1^\mu \cap \zeta \neq \emptyset) \geq 1-\varepsilon$.* □

It is trivial to strengthen Lemma 2.2 to:

Corollary 2.1. *Let L be a positive integer. If $|\zeta| \geq L \cdot K(\varepsilon/L)$ then $P(|\eta_1^\mu \cap \zeta| \geq L) \geq 1-\varepsilon$.*

Proof. Divide ζ into disjoint sets $\zeta_1, \ldots, \zeta_L$ with $|\zeta_i| \geq K(\varepsilon/L)$ and use Lemma 2.2 to conclude

$$P(|\eta_1^\mu \cap \zeta_i| \geq 1) \geq 1-\varepsilon/L. \qquad \square$$

Lemma 2.1 and Corollary 2.1 imply

$$|\eta_1^\mu \cap \tilde{\eta}_t^B| \to \infty 1_{\tilde{\Omega}_\infty} \quad \text{in probability,} \tag{2.2}$$

where $\tilde{\Omega}_\infty = \{\tilde{\eta}_t^B \neq \emptyset \text{ for all } t\}$, and the right-hand side is ∞ on $\tilde{\Omega}_\infty$, and 0 on $\tilde{\Omega}_\infty^c$. To get from this to the desired result

$$P(|\eta_2^\mu \cap \tilde{\eta}_t^B| \text{ is odd}) \to \tfrac{1}{2}P(\tilde{\Omega}_\infty) \quad \text{as } t \to \infty, \tag{2.1}$$

we will find a lot of independent events that can change the parity. Let $U_t = \eta_1^\mu \cap \tilde{\eta}_t^B$. We say that $x \in U_t$ is *isolated* if in the graphical representation of η_s^μ,

$$\{T_n^{(y,x)}: n \geq 1, |x-y|=1\} \cap [1,2] = \emptyset,$$

i.e., the outside world does not influence x. Let V_t be the set of isolated $x \in U_t$. Since $\{x \text{ is isolated}\}$ are i.i.d. events that are independent of η_1^μ and $\{\tilde{\eta}_s^B: s \geq 0\}$, it follows easily that

$$|V_t| \to \infty 1_{\tilde{\Omega}_\infty} \quad \text{in probability.} \tag{2.3}$$

To get from (2.3) to (2.1) we observe that the events {a death occurs at x during $[1,2]$}, $x \in V_t$, are i.i.d. events that change the parity and are independent of what happens in the rest of the process. The first step in translating our intuition into a proof is:

Lemma 2.3. *Let $X_1, X_2, \ldots$ be independent r.v.'s with $P(X_m=1)=1-P(X_m=0)=\theta_m$ where $0<\beta\leq\theta_m\leq 1-\beta<1$, and let $S_n = X_1+\cdots+X_n$. Then*

$$|P(S_n \text{ is odd}) - \tfrac{1}{2}| \leq \tfrac{1}{2}(1-2\beta)^n.$$

Proof. Let $p_n = P(S_n \text{ is odd})$, $p_0 = 0$. For $n \geq 1$,

$$p_n = p_{n-1}(1-\theta_n) + (1-p_{n-1})\theta_n.$$

Subtracting $\frac{1}{2} = \frac{1}{2}(1-\theta_n) + \frac{1}{2}\theta_n$ gives

$$p_n - \tfrac{1}{2} = (p_{n-1} - \tfrac{1}{2})(1-\theta_n) + (\tfrac{1}{2} - p_{n-1})\theta_n.$$

So

$$|p_n - \tfrac{1}{2}| = |p_{n-1} - \tfrac{1}{2}| \cdot |\theta_n - (1-\theta_n)| \leq (1-2\beta)|p_{n-1} - \tfrac{1}{2}|,$$

and the result follows by induction. □

Let $\mathcal{G}_t$ be the σ-field generated by $\tilde{\eta}_t^B$, η_0^μ, V_t, and all the Poisson points in the graphical representation of η_t^μ in $\mathbb{Z}^d \times [1, 2]$ except those concerning deaths at $x \in V_t$. It is easy to see that

$$P(|\eta_2^\mu \cap \tilde{\eta}_t^B| \text{ is odd} \,|\, \mathcal{G}_t) = P\left(\sum_{x \in V_t} g_x = h \bmod 2 \,\middle|\, \mathcal{G}_t \right) \quad \text{a.s.} \tag{2.4}$$

where

$$g_x = 1_{\{\text{there is no death at } x \text{ in } [1,2]\}},$$

and

$$h = 1 - \{|\eta_2^\mu \cap \tilde{\eta}_t^B \cap V_t^c| \bmod 2\}.$$

Now V_t and h are measurable with respect to $\mathcal{G}_t$, and conditional on $\mathcal{G}_t$, g_x, $x \in V_t$, are independent, so it follows from (2.3) and Lemma 2.3 that

$$P\left(\sum_{x \in V_t} g_x = h \bmod 2 \,\middle|\, \mathcal{G}_t \right) \to \tfrac{1}{2} \cdot 1_{\tilde{\Omega}_\infty} \quad \text{in probability.} \tag{2.5}$$

Combining (2.4) and (2.5), taking expected values, and using the bounded convergence theorem we have proved

$$P(|\eta_2^\mu \cap \tilde{\eta}_t^B| \text{ is odd}) \to \tfrac{1}{2} P(\tilde{\Omega}_\infty), \tag{2.1}$$

and the proof of Theorem 1.2 is complete. □

3. Motion of a tagged particle

Throughout this section we will assume that $\delta = 0$. In this section we will define and study the motion of a tagged particle $r_t \in \eta_t$ that is the key to the proof of (**) given in the next section. In defining r_t we want the first coordinate to increase at a linear rate and to keep the other coordinates close to 0. In what follows, it is convenient to use function notation for the process, i.e., $\eta_t(x) = 1$ if $x \in \eta_t$ and $=0$ otherwise. Things will be arranged so that at all times:

(C0): $\eta_t(r_t)=1$.
(C1): $\eta_t(r_t+e_1)=0$.
(Ci) $2\le i\le d$: $\eta_t(r_t-e_i)=0$ if $r_t^i>0$, $\eta_t(r_t+e_i)=0$ if $r_t^i<0$.

Here e_i is the ith unit vector and r_t^i is the ith coordinate of r_t.

We do not move our particle until one of the conditions becomes violated. If it is one of the conditions (C1)-(Cd) that fails we will use:

Repositioning Algorithm. Repeatedly apply the following rules until (C1)-(Cd) hold:

(R1): If $\eta_t(r_t+e_1)=1$ then move to r_t+e_1.
(Ri) $2\le i\le d$: Let $\alpha_i(x)=x-e_i$ if $x^i>0$ and $=x+e_i$ if $x^i<0$.
If $\eta_t(\alpha_i(r_t))=1$ then we move to $\alpha_i(r_t)$.

If several rules (Rj) can be applied, use the one with the smallest number j.

For the discussion below it is useful to note that if we define $\alpha_1(x)=x+e_1$ then the first rule is the same as the others.

When (C0) fails, the particle at r_t was killed by a particle

$$y\in\bigcup_{i=1}^{d}A_i(x)$$

where $A_1(x)=\{x-e_1\}$ and for $2\le i\le d$,

$$A_i(x)=\begin{cases}\{x+e_i\} & \text{if } x^i>0,\\ \{x-e_i\} & \text{if } x^i<0,\\ \{x+e_i,x-e_i\} & \text{if } x^i=0.\end{cases}$$

If this happens we move our tagged particle to y and apply the repositioning algorithm. To check your understanding of the rules try the following example in $d=2$:

0	1	c	0	0	0
1	*	0	1	1	0
1	0	0	0	a	0
1	1	b	0	0	1
0	1	0	0	1	0

Here * indicates the position of the tagged particle r_t, which for concreteness we imagine to be at (2, 4). The sites a, b and c are occupied by 1's and are the possible new locations of the tagged particle.

event	new position
birth occurs at r_t+e_1	a
birth occurs at r_t-e_2	b
r_t killed by r_t-e_1	b
r_t killed by r_t+e_2	c

Intuitively r_t^1 has positive drift for the following reasons:

(i) When $\eta_t(r_t - e_1) = 1$, $r_t^1 \to r_t^1 - 1$ at rate $\leqslant 1$ and $r_t^1 \to r_t^1 + k$ with $k \geqslant 1$ at rate $\geqslant 1$.

(ii) When $\eta_t(r_t - e_1) = 0$, $r_t^1 \to r_t^1 - 1$ at rate 0 and $r_t^1 \to r_t^1 + k$ with $k \geqslant 1$ at rate $\geqslant 1$.

(iii) We move from case (i) to case (ii) at rate $\geqslant 1$ (i.e., when the particle at r_t kills the one at $r_t - e_1$) and from case (ii) to case (i) at rate $\leqslant 2d(1+d) + 2d - 1 = 2d(d+2) - 1$.

To explain the last conclusion, we observe that (a) $r_t - e_1$ gets filled in at rate $\leqslant 2d$ but if the tagged particle moves because one of the conditions (C0)-(Cd) becomes violated we may also end up in case (i); (b) each of the $\leqslant d$ points $\alpha_i(r_t)$ gets filled in at rate $\leqslant 2d$; (c) while we are in case (ii), the particle at r_t gets killed at rate $\leqslant 2d-1$, equality occurring when $r_t^i = 0$ for $i = 2, \ldots, d$.

To translate the intuition contained in (i)-(iii) into a proof, we will define point processes on $\{+\} \times [0, \infty)$ and $\{-\} \times [0, \infty)$. When there is a birth from r_t to $r_t + e_1$ we put a point at $(+, t)$. Let

$$\varphi(t) = \int_0^t 1_{(\eta_s(r_s - e_1) = 1)} \, ds$$

and when the particle at r_t is killed by one at $r_t - e_1$ put a point at $(-, \varphi(t))$. It is easy to see that the processes just defined are rate one Poisson processes. The + and − are to indicate that at the corresponding times r_t^1 changes by $\geqslant +1$ and $\geqslant -1$ respectively. If we let N_s^+ and N_s^- be the number of points in $[0, s]$ in the two processes then

$$r_t^1 - r_0^1 \geqslant S_t \equiv N_t^+ - N_{\varphi(t)}^-.$$

Remark. For the proof of Theorem 1.1, we will also need to control the behavior of r_t^i, $2 \leqslant i \leqslant d$. To prepare for that we would like the reader to observe that the arguments for r_t^1 generalize immediately to the behavior of r_t^i when it is <0. Replacing e_1 by e_i, the intuition in (i)-(iii) and the construction for making it rigorous give

$$r_t^i \geqslant r_0^i + N_t^+ - N_{\varphi(t)}^- \quad \text{as long as } r_t^i < 0.$$

Our first step in getting a lower bound on S_t is to get an upper bound on $\varphi(t)$. From (iii), we see that $\eta_t(r_t - e_1)$ stays 1 for an amount of time $\leqslant$ an exponential with mean 1 and stays 0 for an amount of time $\geqslant$ an exponential with mean $1/\{2d(d+2) - 1\}$. A routine argument shows:

Proposition 3.1. *If $a > 1 - (1/(2d(d+2)))$, there are constants C, γ that depend on a so that*

$$P(\varphi(t) > at) \leqslant C \, e^{-\gamma t} \quad \textit{for all } t.$$

(We will give the proof of this and the next two results in a minute.) Combining the last result with large deviations results for the Poisson process gives:

Proposition 3.2. *If $b < 1/(2d(d+2))$, there are constants C, γ that depend on b so that*

$$P(S_t < bt) \leqslant C \, e^{-\gamma t}.$$

Using the fact that $S_t \geq -N^-_{M/2}$ for all $t \leq \frac{1}{2}M$, any M, and summing the estimate in Proposition 3.2 with $b=0$ over integers $t \geq \frac{1}{2}M$ leads easily to:

Proposition 3.3.

$$P\left(\inf_t S_t < -M\right) \leq C\,\mathrm{e}^{-\gamma M}.$$

Here and in what follows C, $\gamma \in (0,\infty)$ but change from line to line.

We will now prove Propositions 3.1–3.3. Readers who are willing to believe these results can skip their proofs.

Proof of Proposition 3.1. Let $U_0 = 0$ and for $k \geq 1$ let

$$V_k = \inf\{t \geq U_{k-1} : \eta_t(r_t - e_1) = 0\},$$
$$U_k = \inf\{t \geq V_k : \eta_t(r_t - e_1) = 1\}.$$

(V_1 will be 0 if $\eta_0(r_t - e_1) = 0$.) As explained in the discussion of (iii), $V_k - U_{k-1} \leq$ the time we have to wait after U_{k-1} until the first birth from r_t to $r_t - e_1$, and $U_k - V_k \geq$ the time we have to wait after V_k until the first birth lands on $r_t - e_1$, or on one of the sites $\alpha_i(r_t)$, $1 \leq i \leq d$, or on r_t (ignoring births from $r_t - e_1$). A little thought reveals that we can construct independent random variables $v_1, v_2, \ldots$ and $u_1, u_2, \ldots$ with $P(v_k > t) = \mathrm{e}^{-t}$ and $P(u_k > t) = \mathrm{e}^{-t/\mu}$ where $\mu = 1/(2d(d+2)-1)$ so that

$$V_k - U_{k-1} \leq v_k \quad \text{and} \quad U_k - V_k \geq u_k.$$

Let $c = 1 - a < 1/(2d(d+2)) = \mu/(\mu+1)$ and pick $\alpha < 1/(\mu+1)$ so that $\mu\alpha > c > 0$. Standard large deviations results (see e.g. Billingsley, 1979, Theorem 9.3 on p. 124) imply that if $\delta > 0$,

$$P(v_1 + \cdots + v_{[\alpha t]} > (1+\delta)\alpha t) \leq C\,\mathrm{e}^{-\gamma t}$$

and

$$P(u_1 + \cdots + u_{[\alpha t]} < (1-\delta)\mu\alpha t) \leq C\,\mathrm{e}^{-\gamma t},$$

where $[x] =$ the largest integer $\leq x$. Pick δ so that $((1+\delta)\alpha + (1-\delta)\mu\alpha) < 1$ and $(1-\delta)\mu\alpha > c$. This is possible by the choice of α. To complete the proof of the proposition, we observe that when $v_1 + \cdots + v_{[\alpha t]} \leq (1+\delta)\alpha t$ and $u_1 + \cdots + u_{[\alpha t]} \geq (1-\delta)\mu\alpha t$, we have

$$|\{s \leq t : \eta_s(r_s - e_1) = 0\}| \geq ct, \tag{3.1}$$

and hence $\varphi(t) \leq at$. To prove (3.1), we let $N_t = \sup\{k : U_k \leq t\}$ and consider two cases:

Case 1: $N_t \geq [\alpha t]$.

$$|\{s \leq t : \eta_s(r_s - e_1) = 0\}| \geq u_1 + \cdots + u_{[\alpha t]} \geq (1-\delta)\mu\alpha t \geq ct.$$

Case 2: $N_t < [\alpha t]$.

$$\begin{aligned}|\{s \leq t : \eta_s(r_s - e_1) = 1\}| &\leq v_1 + \cdots + v_{[\alpha t]} \leq (1+\delta)\alpha t \\ &\leq \{1 - (1-\delta)\mu\alpha\}t \leq (1-c)t. \qquad \square\end{aligned}$$

Proof of Proposition 3.2. Standard large deviations results imply that if $\delta>0$,

$$P(N_t^+<(1-\delta)t)\leqslant C\,\mathrm{e}^{-\gamma t}$$

and

$$P(N_{at}^->(1+\delta)at)\leqslant C\,\mathrm{e}^{-\gamma t}.$$

If $b<1/(2d(d+2))$ we can pick $a>1-(1/(2d(d+2)))$ and $\delta>0$ so that

$$(1-\delta)-(1+\delta)a>b.$$

The desired result then follows from Proposition 3.1. □

Proof of Proposition 3.3. We begin by observing

$$P\left(\inf_{t\leqslant M/2} S_t<-M\right)\leqslant P(N_{M/2}^-\geqslant M)\leqslant C\,\mathrm{e}^{-\gamma M}.$$

To handle times $>\frac{1}{2}M$ we observe that by considering the first time $\tau>n-1$ at which $S_\tau\leqslant 0$,

$$P(S_n\leqslant 0)\geqslant \mathrm{e}^{-2}P(S_t\leqslant 0 \text{ for some } t\in(n-1,n])$$

since the probability of no arrivals in N_t^+ or N_t^- in one unit of time is e^{-2}. Using the last observation and Proposition 3.2 with $b=0$ it follows that

$$P\left(\inf_{t\geqslant M/2} S_t\leqslant 0\right)\leqslant \mathrm{e}^2\sum_{n\geqslant M/2} P(S_n\leqslant 0)\leqslant \sum_{n\geqslant M/2} C\,\mathrm{e}^{-\gamma n}. \qquad \square$$

4. Comparison with oriented percolation, proof of Theorem 1.1

To prove Theorem 1.1 we begin by showing the following for $\delta=0$:

(**) Given $\eta_{nT}=A$ with $A\cap I_m\neq\emptyset$ there is an event $G_{m,n,A}$ measurable with respect to the graphical representation in $B_{m,n}$ so that:

(i) On $G_{m,n,A}$, $\eta_{(n+1)T}\cap I_{m-1}\neq\emptyset$ and $\eta_{(n+1)T}\cap I_{m+1}\neq\emptyset$.

(ii) If L is large then $P(G_{m,n,A}\,|\,\eta_{nT}^0=A)\geqslant 1-\varepsilon$ for all A with $A\cap I_m\neq\emptyset$.

Note. With the proof of Theorem 1.3 in mind, we ignore the fact that $\eta_0=\{0\}$ in Theorem 1.1.

The good event $G_{m,n,A}$ is the success of a procedure designed to 'move' a particle from I_m to I_{m+1} and one from I_m to I_{m-1} in $[nT,(n+1)T)$. Before entering into the somewhat unpleasant details, we would like to point out the sources of our troubles. The arguments in the last section gives us a lower bound on the drift of r_t^1 to the right but no upper bound, and the argument has to work when A is a single point or all of $\mathbb{Z}^d$.

By the Markov property and translation invariance in time and space it suffices to prove (**) when $m=0$, $n=0$, and $\eta_0=A$ has $A\cap I_0\neq\emptyset$. The first step in the construction of our moving particle ρ_t is to find a starting point ρ_0 so that $\alpha_i(\rho_0)$ is vacant for $1\leq i\leq d$. Let $x_0\in A$ and $j\geq 0$. If all the points $\alpha_i(x_j)$ are vacant then stop and set $\rho_0=x_j$. If at least one of the points $\alpha_i(x_j)$ is occupied let x_{j+1} be the one with the smallest value of i and try again. One of two things can happen: (a) the construction terminates at a point ρ_0 with the desired properties without leaving $[-\frac{3}{2}L, \frac{3}{2}L]^d$ or (b) not. In the second case let y_k be the first x_j not in $(-L-4k, L+4k)^d$, let $\theta_k=\{\alpha_i(y_k)\colon 1\leq i\leq d\}\cap A$, and let $F_k=\{$from time 0 to 1 there is exactly one arrow from y_k to each point in θ_k and no other arrows land on $\{y_k\}\cup\theta_k\}$. The events $F_k, 1\leq k\leq[\frac{1}{8}L]$, are independent and each has probability at least $\exp(-2d(d+1))$, so with probability at least

$$1-\{1-\exp(-2d(d+1))\}^{[L/8]}$$

one of these events will occur and give us a place ρ_0 to start our construction.

In case (a) our moving particle ρ_t starts moving at time 0; in case (b) at time 1. In either case ρ_t starts at the location found in the last paragraph and behaves like the tagged particle r_t until time

$$\tau_1=\inf\{t\colon \rho_t^1\geq\tfrac{3}{2}L\}.$$

To keep the particle from flying out of the box $B_{0,0}$ at time τ_1, we stop the repositioning step at time τ_1 when the first coordinate becomes $\frac{3}{2}L$. (We assume L is even.)

If $\tau_1\leq T$ then at time τ_1 we have achieved our first goal of moving the first coordinate into $[L, 2L]$ and the construction enters its second phase which will now be described. Let $\beta_1(x)=x-e_1$ if $x^1>\frac{3}{2}L$, $\beta_1(x)=x+e_1$ if $x^1<\frac{3}{2}L$. If $2\leq i\leq d$ let $\beta_i(x)=\alpha_i(x)$. During this part of the construction, things will be arranged so that at all times:

($\bar{\text{C}}$0): $\eta_t(r_t)=1$.

($\bar{\text{C}}i$) $1\leq i\leq d$: $\eta_t(\beta_i(r_t))=0$.

We do not move our particle until one of the conditions becomes violated. If it is one of the conditions ($\bar{\text{C}}$1)-($\bar{\text{C}}d$) that fails we will use:

Repositioning Algorithm II. Repeatedly apply the following rules until ($\bar{\text{C}}$1)-($\bar{\text{C}}d$) hold:

($\bar{\text{R}}i$) $1\leq i\leq d$: if $\eta_t(\beta_i(r_t))=1$ then we move to $\beta_i(r_t)$.

If several rules can be applied, use the one with the smallest number.

Since each such move brings the particle closer to $(\frac{3}{2}L, 0, \ldots, 0)$ the algorithm stops after a finite number of steps. Notice that now the first coordinate is treated like the others (except for the fact that we try to keep it near $\frac{3}{2}L$).

To prove (**), we begin by observing that the first coordinate of our particle starts at $\rho_0^1>-\frac{3}{2}L$, and our first goal is to get to $\frac{3}{2}L$, so if $\kappa_d=2\cdot 3\cdot 2d(d+2)$ then Proposition 3.2 and Proposition 3.3 imply that with high probability (i.e., with a probability $\to 1$ as $L\to\infty$)

$$\tau_1 < T = \kappa_d L \quad\text{and}\quad \rho_t^1 > -2L \text{ for } t \le \tau_1.$$

Turning now to the behavior of $\rho_t^i, 2\le i\le d$, we begin by observing that ρ_0^i is in $[-\frac{3}{2}L, \frac{3}{2}L]$, so the arguments for $\rho_t^1, 0\le t\le \tau_1$ (see the remark in Section 3) show that if $\tau_1^0 = \inf\{t: \rho_t^i = 0\}$ then with high probability

$$\tau_i^0 < T \quad\text{and}\quad |\rho_s^i| \le 2L \text{ for } s\le \tau_i^0.$$

For $2\le i\le d$ let

$$\sigma_i^k = \inf\{t > \tau_i^{k-1}: \rho_t^i \neq 0\},$$
$$\tau_i^k = \inf\{t > \sigma_i^{k-1}: \rho_t^i = 0\}$$

and $K_i = \inf\{k: \sigma_i^k > T\}$. Every time $\rho_t^i = 0$, it stays there for at least an exponential amount of time with mean $\frac{1}{2}$ (i.e., until ρ_t is killed by a particle at $\rho_t + e_i$ or $\rho_t - e_i$), so with high probability $K_i \le 4T = 4\kappa_d L$. An easy generalization of Proposition 3.3 shows

$$P(|\rho_t^i| \ge L \text{ for some } t\in[\sigma_i^k, \tau_i^k)) \le C\,\mathrm{e}^{-\gamma L}.$$

Combining the last two results we conclude that with high probability, $|\rho_t^i| \in (-L, L)$ for all $t\in[\tau_i^0, T]$. For the case $i=1$, replacing 0 by $\frac{3}{2}L$ in the definition of σ_i^k and τ_i^k for $i=1$ and setting $\tau_1^0 = \tau_1$, the last argument also shows that with high probability $\rho_t^1 \in (L, 2L)$ for $t\in[\tau_1, T]$.

At this point we have done what we promised to do. We have given a procedure that moves a particle from I_0 to I_1 with high probability. A little reflection (pun intended) shows we can also move a particle from I_0 to I_{-1}. This shows that (**) is satisfied. To get from (**) to (*) we use induction. Since knowledge of the variables $\{\omega_{i,j}: (i,j)\in\mathscr{L}, |i|\le j\}$ is enough to compute $\{W_n^0, n\ge 0\}$, we will only define those variables. We start with an A with $A\cap I_0\neq\emptyset$, so (**) implies we can define $\omega_{0,0}\in\{0,1\}$ so that $P(\omega_{0,0}=1)=1-\varepsilon$ and $\{\omega_{0,0}=1\}\subset G_{0,0,A}$. Let $n\ge 1$. Suppose now that the $\omega_{i,j}$ have been defined for $j<n$ and we have $\chi_n^0 \supset W_n^0$. Since the good events $G_{k,n,\eta_{nT}}A$ for the boxes $B_{k,n}$ with $k\in\chi_n^0$ have probability $\ge 1-\varepsilon$ and are conditionally independent given η_{nT}^A, we can define independent $\omega_{m,n}\in\{0,1\}$, $(m,n)\in\mathscr{L}$ with $|m|\le n$ that have $P(\omega_{m,n}=1)=1-\varepsilon$, and $\{\omega_{k,n}=1\}\subset G_{k,n,\eta_{nT}}A$, and are independent of the $\omega_{j,k}$ with $k<n$. The last inclusion and the definition of the good event imply $\chi_{n+1}^0 \supset W_{n+1}^0$. The proof of (*) is complete, and as indicated in the introduction, Theorem 1.1 follows. □

5. Proof of Theorem 1.3

In this section $\delta=0$. It suffices to prove the result when $\eta_0 = A\neq\emptyset$ is not random. Let $B\neq\emptyset$ be a finite set and $\tilde\eta_t^B$ be an independent copy of the process with initial state B. By Griffeath's observation (Proposition 1.1) it suffices to show that

$$P(|\eta_t^A \cap \tilde\eta_t^B| \text{ is odd}) \to \tfrac{1}{2} \quad\text{as } t\to\infty. \tag{5.1}$$

To do this, we begin by recalling some facts about the set of wet sites W_n^0 in oriented percolation. Here and throughout the rest of the section, we will use notation introduced in the last section. It is well known (see Durrett, 1980, 1984, 1988) that on $\Omega_\infty = \{W_n^0 \neq \emptyset \text{ for all } n\}$ we almost surely have

$$\frac{1}{n}\sup W_n^0 \to \alpha(p), \qquad \frac{1}{n}\inf W_n^0 \to -\alpha(p) \quad \text{and} \quad |W_n^0|/n \to \alpha(p)\rho(p),$$

where $\alpha(p)$ is a constant, and $\rho(p) = P(\Omega_\infty)$. It is also known that $\alpha(p), \rho(p) \uparrow 1$ as $p \uparrow 1$. Pick p_0 so that $\alpha(p)\rho(p) \geq \frac{2}{3}$ for $p \geq p_0$. Let $\tilde{W}_n^0$ be an independent copy of W_n^0. Since

$$W_n^0, \tilde{W}_n^0 \subset \{-n, -n+2, \ldots, n-2, n\},$$

it follows that

$$|W_n^0 \cap \tilde{W}_n^0| \geq \tfrac{1}{4}n \quad \text{for large } n \text{ a.s. on } \Omega_\infty \cap \tilde{\Omega}_\infty \tag{5.2}$$

where, of course, $\tilde{\Omega}_\infty = \{\tilde{W}_n^0 \neq \emptyset \text{ for all } n\}$.

(5.2) shows that two independent oriented percolations have the property we desire. To get from this to a proof of Theorem 1.3, we note that (∗) in the introduction implies we can pick L large enough so that $B \subset I_0$, $A \cap I_0 \neq \emptyset$, and η_t^A and $\tilde{\eta}_t^B$ dominate independent oriented site percolation processes, W_n^0, $\tilde{W}_n^0$, with $p \geq p_0$. Let $T = \kappa_d L$ be the time scale for the block construction. If $t > T$ we can write $t = nT + r$ where $n \geq 0$ is an integer and $T < r \leq 2T$. Let

$$J_m = 2mLe_1 + [-2L+1, 2L-1]^d \quad \text{and} \quad D_{m,t} = J_m \times [nT, t).$$

Let $U_n = W_n^0 \cap \tilde{W}_n^0$. We say that m is *isolated at time* t if no arrows touch the boundary of J_m during $[nT, t)$ in the graphical representation of either process. Let V_t be the set of $m \in U_n$ that are isolated at time t. If m is isolated, the evolution of the processes η^A and $\tilde{\eta}^B$ in J_m is unaffected by what happens outside.

Since the events which determine the fate (isolated at time t or not) of different m in U_n are independent, it follows easily from (5.2) that we have:

Lemma 5.1. *There is a $c > 0$ so that $|V_t| \geq ct$ for large t a.s. on $\Omega_\infty \cap \tilde{\Omega}_\infty$.*

Proof. Let $b = \frac{1}{5}P$(no arrows touch the boundary of J_m during $[0, 2T]$ in the graphical representation of either process). Since the events $\{m$ is isolated are time $(n+2)T\}$ are independent, conditioning on the value of U_n and computing fourth moments of $V_{(n+2)T}$ shows

$$\sum_{n=1}^{\infty} P(U_n \geq \tfrac{1}{4}n, V_{(n+2)T} \leq bn) \leq \sum_{n=1}^{\infty} \frac{C}{n^2} < \infty.$$

The desired result with $c = b/(3T)$ now follows from (5.2), the Borel-Cantelli lemma, and the fact that

$$V_t \geq V_{(n+2)T} \quad \text{when } t \in ((n+1)T, (n+2)T]. \qquad \square$$

Let $\mathscr{G}_t$ be the σ-field that is generated by η_{nT}^A, $\tilde{\eta}_{nT}^B$, V_t, and all the Poisson points in the graphical representation that are in $\mathbb{Z}^d \times [nT, t)$ but not in $\bigcup\{D_{m,t}: m \in V_t\}$. It is easy to see that

$$P(|\eta_t^A \cap \tilde{\eta}_t^B| \text{ is odd} \mid \mathscr{G}_t) = P\left(\sum_{m \in V_t} g_m = h \bmod 2 \,\middle|\, \mathscr{G}_t \right), \tag{5.3}$$

where

$$g_m = |\eta_t^A \cap \tilde{\eta}_t^B \cap J_m| \bmod 2,$$

$$h = 1 - \{|\eta_t^A \cap \tilde{\eta}_t^B \cap H| \bmod 2\}$$

and

$$H = \mathbb{Z}^d - \bigcup_{m \in V_t} J_m.$$

The intersection of η_t^A or $\tilde{\eta}_t^B$ with a J_m, $m \in V_t$, is a finite state Markov chain with transition probability independent of m, run for an amount of time $\in (T, 2T]$, so there is a $\beta > 0$ with

$$P(g_m = 1 \mid \mathscr{G}_t) \in [\beta, 1-\beta] \quad \text{for } m \in V_t. \tag{5.4}$$

Now V_t and h are measurable w.r.t. $\mathscr{G}_t$, and g_m, $m \in V_t$, are conditionally independent given $\mathscr{G}_t$, so it follows from Lemma 5.1 and Lemma 2.3 that

$$P\left(\sum_{m \in V_t} g_m = h \bmod 2 \,\middle|\, \mathscr{G}_t \right) \to \tfrac{1}{2} \quad \text{as } t \to \infty \text{ a.s. on } \Omega_\infty \cap \tilde{\Omega}_\infty. \tag{5.5}$$

Taking expected values in (5.3) and using the bounded convergence theorem gives

$$\lim_{t \to \infty} P(|\eta_t^A \cap \tilde{\eta}_t^B| \text{ is odd}, \Omega_\infty \cap \tilde{\Omega}_\infty) = \tfrac{1}{2} P(\Omega_\infty)^2.$$

Recall Ω_∞ and $\tilde{\Omega}_\infty$ are independent. As $L \to \infty$, the parameter in the percolation process $p = 1 - \varepsilon(L) \to 1$, so $P(\Omega_\infty)$, $P(\tilde{\Omega}_\infty) \to 1$. From this it follows easily that

$$\lim_{t \to \infty} P(|\eta_t^A \cap \tilde{\eta}_t^B| \text{ is odd}) = \tfrac{1}{2} \tag{5.1}$$

holds and Theorem 1.3 follows. □

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