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CONDITIONED LIMIT THEOREMS FOR SOME NULL RECURRENT MARKOV PROCESSES[1]

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Let $\{v_k, k \geqq 0\}$ be a discrete time Markov process with state space $E \subset (-\infty, \infty)$ and let S be a proper subset of E. In several applications it is of interest to know the behavior of the system after a large number of steps, given that the process has not entered S. In this paper we show that under some mild restrictions there is a functional limit theorem for the conditioned sequence if there was one for the original sequence. As applications we obtain results for branching processes, random walks, and the M/G/1 queue which complete or extend the work of previous authors. In addition we consider the convergence of conditioned birth and death processes and obtain results which are complete except in the case that 0 is an absorbing boundary.

1. Introduction. Let $\{v_k, k \geqq 0\}$ be a discrete time Markov process with state space $E \subset (-\infty, \infty)$ and let S be a proper subset of E. In several applications (see [8], [12] and [13]) it is of interest to know the behavior of the system after a large number of steps given the process has not entered S. For example, if v_n is a branching process and $S = \{0\}$ a limit theorem for $(v_n \mid v_m \neq 0, 1 \leqq m \leqq n)$ gives information about the size of v_n on the set $\{v_n > 0\}$.

In [2], Seneta and Vere-Jones have given conditions for the convergence of

$$\alpha_{ij}(n) = P(v_n = j \mid v_0 = i, N_S > n) \tag{1}$$

where $N_S = \inf\{m \geqq 1 : v_m \in S\}$. In many cases, however, all the limits in (1) are zero. Applying the results of [2] when v_n is a branching process and $S = \{0\}$ gives that $\alpha_j^* = \lim_{n\to\infty} \alpha_{1j}(n)$ is a probability distribution when $m = E(v_1 \mid v_0 = 1) < 1$ and $\alpha_j^* \equiv 0$ when $m \geqq 1$. To obtain an interesting theorem in the second case we have to look at the limit of $(v_n/c_n \mid v_0 = i, N_S > n)$ where the c_n are constants which $\uparrow \infty$.

In this instance the most desirable type of result is a functional limit theorem, i.e., a result asserting the convergence of the sequence of stochastic processes $\{V_n^+(t), 0 \leqq t \leqq 1\}$ defined by

$$V_n^+(t) = (v_{[nt]}/c_n \mid v_0 = i, N_S > n) \tag{2}$$

where $[x]$ is the largest integer $\leqq x$.

In this paper we will show that under some mild restrictions $\{V_n^+(t), 0 \leqq t \leqq 1\}$

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converges if there is a corresponding functional limit theorem for the unconditioned sequence. As applications we will obtain results which complete the work of Lamperti and Ney (1968), Iglehart (1974) and Kennedy (1974).

To describe our results in detail we have to state the basic assumptions. The first and most natural are: (i) v_k, $k \geqq 0$ is a Markov process with state space $E \subset (-\infty, \infty)$; (ii) there are constants $c_n \uparrow \infty$ with $c_{n+1}/c_n \to 1$ so that if $x_n \to x$ and $x_n c_n \in E$ for all n then

$$V_n^{x_n} = (v_{[n\cdot]}/c_n \mid v_0/c_n = x_n) \Rightarrow (V \mid V(0) = x) = V^x,$$

where V is a Markov process with V^y nondegenerate for some $y > 0$; and (iii) $P\{\inf_{0 \leqq s \leqq t} V^x(s) > 0\} > 0$ for all $t, x > 0$.

Here the symbol $\Rightarrow$ means that the sequence $V_n^{x_n}$ converges weakly as a sequence of random elements of D—the space of right continuous functions on $[0, 1]$ which have left limits (see [20] for a description). Nondegenerate means that $P\{V^x = f\} < 1$ for all $f \in D$.

Let $N = N_{(-\infty,0]}$. It is under assumptions (i)—(iii) that we will derive conditions for the convergence of $(V_n^{x_n} \mid N > n)$ (a) for all $x_n \to x \geqq 0$ and (b) when $x_n c_n \equiv y \in E$.

We will obtain our conditions for the case $x_n \to x > 0$ by solving a more general problem. In Section 2 we give sufficient conditions for the convergence of $P_n(\cdot \mid A_n) = P_n(\cdot \cap A_n)/P_n(A_n)$ when the P_n are probability measures with $\inf_n P_n(A_n) > 0$. Applying these results to sets $A_n = \{f : \inf_{0 \leqq s \leqq t_n} f(s) > 0\}$ with $t_n \to t \in [0, 1]$ we find that if $P_n^{x_n}$ and P^x are the probability measures induced on D by $V_n^{x_n}$ and V^x, then $x_n \to x > 0$, (ii), and $P_n^{x_n}\{N > nt_n\} \to P^x\{T_0 > t\}$ are sufficient for $(V_n^{x_n} \mid N > nt_n) \Rightarrow (V^x \mid T_0 > t)$ when $T_0 = \inf\{s > 0 : V(s) \text{ or } V(s-) \leqq 0\}$. (We will work with T_0 instead of the natural hitting time $T_0' = \inf\{t > 0 : f(t) \leqq 0\}$ since

$$\{f : f(0) > 0, T_0(f) > t\} = \{f : \inf_{0 \leqq s \leqq t} f(s) \geqq 0\}$$

is open.)

In Section 3 we consider the convergence of the conditioned processes when $x_n \to 0$ and, in particular, when $x_n = y/c_n$. In either situation $P_n^{x_n}\{N > n\} \to 0$ (in most cases) so a special analysis is required. Our method for proving convergence will be to show that if

$$V_n^+ = (v_{[n\cdot]}/c_n \mid v_0 = x_n c_n, N > n) \qquad \text{and} \qquad T_\varepsilon^n = \inf\{k : v_k/c_n \geqq \varepsilon\}$$

then

$$\begin{aligned} \lim_{n\to\infty} V_n^+ &= \lim_{\varepsilon\to 0} \lim_{n\to\infty} (v_{[T_\varepsilon^n + n\cdot]}/c_n \mid v_0 = x_n c_n, N > n) \\ &= \lim_{\varepsilon\to 0} \lim_{n\to\infty} (v_{[n\cdot]}/c_n \mid v_0 = V_n^+(T_\varepsilon^n/n), N > n - T_\varepsilon^n) \\ &= \lim_{\varepsilon\to 0} \lim_{n\to\infty} (v_{[n\cdot]}/c_n \mid v_0 = x, N > n). \end{aligned}$$

In Section 3 we will show that these three equalities hold if (in addition to (i)—(iii)) we have

(iv) $P_n^{x_n}\{N > nt_n\} \to P^x\{T_0 > t\}$ whenever $x_n \to x > 0$, $t_n \to t > 0$; and
(v) $P_n^{x_n}\{N > nt_n\} \to 0$ whenever $x_n \to 0$ and $t_n \to t > 0$.

The key to our proof is the following fact (first observed by Lamperti in [25]):

THEOREM 3.2. *If* (ii) *holds then there is a* $\delta \geqq 0$ *so that for all* $c > 0$, $V^{cx} =_d cV^x(\cdot c^{-\delta})$.(*)

This scaling relationship identifies the processes which can occur as limits in (ii) and can be used to deduce many properties of the limit process. In Section 3.1 we use (*) to compute relationships between the numbers $P^x\{T_0 > t\}$. These relationships are used to identify trivial cases and obtain sufficient conditions for (iii), (iv), and (v) to hold.

In Sections 3.2 and 3.3 we use these preliminaries to prove our conditioned limit theorems. To do this we reverse the usual procedure for proving weak convergence. In Section 3.2 we develop sufficient conditions for V_n^+ to be tight. In Section 3.3 we find conditions for the convergence of finite dimensional distributions. The main results of these two sections are:

THEOREM 3.6. V_n^+ *is tight if and only if*

$$\lim_{K\to\infty} \limsup_{n\to\infty} P\{V_n^+(1) > K\} = 0 \quad \text{and} \tag{6a}$$

$$\lim_{t\to 0} \limsup_{n\to\infty} P\{V_n^+(t) > h\} = 0 \quad \text{for each} \quad h > 0\,. \tag{6b}$$

THEOREM 3.10. *Suppose* (i)—(v) *hold and* V_n^+ *is tight. If* $V^+ = \lim_{x\downarrow 0}(V^x \mid T_0 > 1)$ *exists and is* $\not\equiv 0$ *then* $V_n^+ \Rightarrow V^+$ *if and only if*

$$\lim_{h\downarrow 0} \liminf_{n\to\infty} P\{V_n^+(t) > h\} = 1 \quad \text{for all} \quad t > 0\,.$$

From the first result we see that to prove the sequence is tight it is enough to prove that $V_n^+(t) \Rightarrow V^*(t)$ for all $t > 0$ and $V^*(t) \Rightarrow 0$ as $t \to 0$. The second result shows that if we do this and find that $P\{V^*(t) > 0\} = 1$ for all $t > 0$ then $V^*(t) =_d V^+(t)$ (provided that V^+ exists).

In Sections 4.1—4.4 we use the results of Section 3 to prove conditioned limit theorems for random walks, branching processes, birth and death processes, and the M/G/1 queue which contain the corresponding results of [6], [8], [12], and [13] as special cases. It seems likely that the methods can be extended for the non-Markovian examples studied by [7] and [11], but I have not tried this.

A more interesting unsolved problem is to generalize the results of Section 3 to other types of conditioning. There are three types of theorems in the literature to which it seems our methods can be applied. The first and most closely related are the results of Belkin (1970, 1972) and Port and Stone (1971) on random walks conditioned on $\{N_B > n\}$ when B is a bounded subset of the state space. A second type of result concerns conditioning on $\{v_n \in A\}$ or $\{(v_{n-1}, v_n) \in B\}$. Several limit theorems of this type have been obtained for $A = \{x\}$ or $[a_n, b_n]$ (see [15], [17], [18]) and $B = (0, \infty) \times (-\infty, 0)$ (see [19]). A third possibility

can be constructed by taking the intersection of a condition of the second type with $\{N > n - 1\}$ or $\{N_B > n - 1\}$. The condition $\{N_{\{0\}} = n\}$ is an example of this type which has been studied by Kaigh (1976).

2. Conditions for the convergence of $P_n(\cdot \mid A_n)$ when $\inf_n P_n(A_n) > 0$. In this section we shall give several conditions under which the weak convergence of a sequence of probability measures P_n on a metric space (S, ρ) is sufficient for the convergence of the conditional measures $P_n(\cdot \mid A_n) = P_n(\cdot \cap A_n)/P_n(A_n)$ when $\inf_n P_n(A_n) > 0$. The main result is:

THEOREM 1. *Let P_n, $n \geqq 0$, be probability measures and A_n, $n \geqq 0$, be a sequence of events. If* (i) $P_n \Rightarrow P_0$, (ii) *there are sets $G_m \uparrow A_0$ such that for each m, $P_0(\partial G_m) = 0$ and there is a positive integer $k(m)$ so that $A_n \supset G_m$ for all $n \geqq k(m)$, and* (iii) $P_0(A_0) \geqq \limsup_n P_n(A_n) > 0$ *then $P_n(A_n) \to P_0(A_0)$ and $P_n(\cdot \mid A_n) \Rightarrow P_0(\cdot \mid A_0)$.*

PROOF. It suffices to check that $P_n(B \cap A_n) \to P_0(B \cap A_0)$ for all B with $P_0(\partial B) = 0$. From (ii)

$$\liminf_{n\to\infty} P_n(B \cap A_n) \geqq \liminf_{n\to\infty} P_n(B \cap G_m) .$$

Since $P_0(\partial(B \cap G_m)) \leqq P_0(\partial B) + P_0(\partial G_m) = 0$

$$\lim_{n\to\infty} P_n(B \cap G_m) = P_0(B \cap G_m) .$$

Letting $m \to \infty$ now gives $\liminf_{n\to\infty} P_n(B \cap A_n) \geqq P_0(B \cap A_0)$. Since $\partial(B^c) = \partial B$, $P(\partial(B^c)) = 0$ and we have

$$\liminf_{n\to\infty} P_n(B^c \cap A_n) \geqq P_0(B^c \cap A_0) .$$

Using (iii) now gives

$$\begin{aligned} \limsup_{n\to\infty} P_n(B \cap A_n) &\leqq \limsup_{n\to\infty} P_n(A_n) - \liminf_{n\to\infty} P_n(B^c \cap A_n) \\ &\leqq P_0(B \cap A_0) , \end{aligned}$$

which completes the proof.

When applying this theorem we will typically be given P_n, $n \geqq 0$ and A_n, $n \geqq 1$, and we will have to find an appropriate sequence G_m. Condition (iii) suggests that we would like to construct the largest A_0 for which there is a sequence $G_m \uparrow A_0$ which satisfies (ii). To do this observe that if G_m and A_0 satisfy (ii) then

$$G_m \subset \bigcap_{n \geqq k(m)} A_n \qquad \text{and} \qquad P(\partial G_m) = 0$$

so $P[\bigcup_{m=1}^{\infty} (\bigcap_{n\geqq m} A_n)^0] \geqq P(A_0)$. To show that $\bigcup_{m=1}^{\infty} (\bigcap_{n\geqq m} A_n)^0$ (hereafter called LIMNF A_n) is the limit of a sequence G_m which satisfies (ii) we have to introduce some notation.

If H is a subset of S and $\varepsilon > 0$ let $H^\varepsilon = \{y : \{x : \rho(x, y) < \varepsilon\} \subset H\}$. The interior of H, $H^0 = \bigcup_{\varepsilon>0} H^\varepsilon$ so $P(H^\varepsilon) \uparrow P(H^0)$ as $\varepsilon \downarrow 0$. Let $\varepsilon_m \downarrow 0$ and let

$$G_m = (\bigcap_{n\geqq m} A_n)^{\varepsilon_m} .$$

It is immediate from the definition that $G_m \subset A_n$ for $n \geqq m$ and $G_m \uparrow$ LIMNF A_n. The sets G_m may have $P(\partial G_m) > 0$ but this is no problem. If $\varepsilon' < \varepsilon$

$\partial(H^{\varepsilon'}) \subset (H^{\varepsilon})^0$ so $\partial H^{\varepsilon} \cap \partial H^{\varepsilon'} = \emptyset$. From this it follows that $P(\partial H^{\varepsilon}) = 0$ for all but a countable number of ε so we can pick another decreasing $\varepsilon_m' \leqq \varepsilon_m$ for which the associated G_m' have $P(\partial G_m') = 0$.

Using the observations above we can write the result of Theorem 1 in a simpler form.

THEOREM 2. *If* $P_n \Rightarrow P$ *and* $P(\text{LIMNF } A_n) \geqq \limsup_n P_n(A_n) > 0$ *then* $P_n(A_n) \to P(\text{LIMNF } A_n)$ *and* $P(\cdot \mid A_n) \Rightarrow P(\cdot \mid \text{LIMNF } A_n)$.

The reader should note that if $P(\text{LIMNF } A_n) = 1$ then $P_n(\cdot \mid A_n) \Rightarrow P$. To apply Theorem 2 in nontrivial cases it is desirable to reformulate the condition $P(\text{LIMNF } A_n) \geqq \limsup_n P_n(A_n)$ in terms of the sequence A_n and the limit measure P. One way of doing this is to observe that for all $n \geqq m$

$$A_n \subset (\bigcup_{k \geqq m} A_k)^-,$$

so

$$\limsup_n P_n(A_n) \leqq \limsup_n P_n((\bigcup_{k \geqq m} A_k)^-) \leqq P((\bigcup_{k \geqq m} A_k)^-),$$

and letting $m \to \infty$

$$\limsup_n P_n(A_n) \leqq P(\bigcap_{m=1}^{\infty} (\bigcup_{k \geqq m} A_k)^-).$$

If we let $\text{LIMSP } A_n = \bigcap_{m=1}^{\infty} (\bigcup_{k \geqq m} A_k)^-$ and note that $\text{LIMSP } A_n \supset \text{LIMNF } A_n$ we can write Theorem 2 as:

THEOREM 3. *If* $P(\text{LIMSP } A_n - \text{LIMNF } A_n) = 0$, $P(\text{LIMNF } A_n) > 0$ *and* $P(A \,\Delta\, \text{LIMNF } A_n) = 0$ *then* $P_n(A_n) \to P(A)$ *and* $P_n(\cdot \mid A_n) \Rightarrow P(\cdot \mid A)$.

A special case of Theorem 3 which we will need in Sections 3 and 4 is the following:

EXAMPLE. Let $S = D$ and ρ be the Skorokhod metric on D (see [20], page 113). Let $A_n = \{f : \inf_{s \leqq t_n} f(s) > 0\}$ with $t_n \to t > 0$. If $q_n = \sup_{m \geqq n} t_m$ and $r_n = \inf_{m \geqq n} t_m$ then

$$\begin{aligned} \text{LIMSP } A_n &= \bigcap_{n=1}^{\infty} (\{f : \inf_{s \leqq r_n} f(s) > 0\})^- \\ &= \bigcap_{n=1}^{\infty} \{f : \inf_{s < r_n} f(s) \geqq 0\} \\ &= \{f : \inf_{s < t} f(s) \geqq 0\}. \end{aligned}$$

To compute $\text{LIMNF } A_n$ we observe

$$\begin{aligned} \bigcap_{n=m}^{\infty} A_n &= \{f : \inf_{s \leqq q_m} f(s) > 0\} && \text{if } t_n \geqq t \text{ for some } n \geqq m \\ &= \bigcap_{\varepsilon > 0} \{f : \inf_{s < t-\varepsilon} f(s) > 0\} && \text{if } t_n < t \text{ for all } n \geqq m. \end{aligned}$$

Since the interior of the second set is the first, we have

$$\text{LIMNF } A_n = \bigcup_{m=1}^{\infty} \{f : \inf_{s \leqq q_m} f(s) > 0\} = \{f : \inf_{s \leqq t} f(s) > 0\}$$

and

$$\text{LIMSP } A_n - \text{LIMNF } A_n = \{f : \inf_{s \leqq t} f(s) = 0\} \cup \{T_0 = t\}$$

where T_0 is the hitting time defined in the introduction.

Using Theorem 2.3 now gives that we have convergence whenever $P\{T_0 > t\} > 0$ and the two sets in the last equality above have probability zero.

This result is sufficient for most, but not all, of our desired applications. If $P\{f : f \geqq 0\} = 1$ then $P\{f : \inf_{s \leq t} f(s) = 0\} = P\{T_0 \leqq t\}$ and from the computations above we see that Theorem 3 can only be applied in the trivial case $P\{T_0 > t\} = 1$. To obtain our results when $P\{f : f \geqq 0\} = 1$ and $P\{T_0 > t\} \in (0, 1)$ we will use Theorem 2.

3. Conditioning on $T_{(-\infty,0]} > n$.

3.1. *Preliminary results.* In this section we will investigate consequences of assumptions (i) and (ii). Our first result follows immediately from the type of convergence assumed in (ii).

THEOREM 1. *If there is a Markov chain v_n so that $v_{[n\cdot]}/c_n$ converges to V (in the sense specified in* (ii)*) then V has the following weak continuity property*:

$$\text{if} \quad x_n \to x\,, \qquad \text{then} \quad V^{x_n} \Rightarrow V^x\,. \tag{1}$$

This implies, in particular, that V is a strong Markov process.

PROOF. The second fact is a well-known consequence of the first. To prove (1) we observe that if $x_n \to x$ there is a sequence n_k increasing to ∞ so that if $y_n = x_k$ when $n_k \leqq n < n_{k+1}$ then $\lim_{k\to\infty} V^{x_k} = \lim_{n\to\infty} V_n^{y_n} = V^x$ (the lim here means weak convergence).

The processes which can occur as limits in (ii) also have special properties because they result from scaling and contracting time in a single Markov process. The most basic of these is the scaling relationship given in the following theorem.

THEOREM 2. *If assumptions* (i) *and* (ii) *hold, there is a $\delta \geqq 0$ so that*

$$\text{for all} \quad c > 0\,, \qquad V^{cx} =_d cV^x(\cdot c^{-\delta})\,, \tag{2}$$

$$\text{and for all} \quad t > 0\,, \qquad \lim_{n\to\infty} c_{[nt]}/c_n = t^{1/\delta} \quad (\text{here}, \ t^\infty = \lim_{m\to\infty} t^m)\,. \tag{3}$$

NOTE. To simplify notation in what follows we will drop the square bracket from $c_{[nt]}$ and write c_{nt}.

PROOF. Let $\lambda \in (0, 1]$. Let $m_n = m_n(\lambda) = \sup\{m \leqq n : c_m/c_n < \lambda\}$. Since $c_{n-1}/c_n \to 1$ and $c_n \to \infty$, $c_{m_n}/c_n \to \lambda$. If $x_n \to x$ and a subsequence of m_n/n converges to $\rho \in [0, 1]$, it follows from (ii) that

$$(v_{[m_n\cdot]}/c_{m_n} \mid v_0 = x_n c_{m_n}) \Rightarrow V^x$$

and a subsequence of the left-hand side converges to $\lambda^{-1}V^{x\lambda}(\rho\cdot)$ so $V^x =_d \lambda^{-1}V^{x\lambda}(\rho\cdot)$.

Let x_0 be a state with $P\{V^{x_0} \equiv x_0\} < 1$. If m_n/n has two subsequential limits $\rho_1, \rho_2 \in [0, 1]$ with $\rho_1 < \rho_2$ then

$$\lambda^{-1}V^{x_0}(\rho_1\cdot) =_d V^{x_0/\lambda} =_d \lambda^{-1}V^{x_0}(\rho_2\cdot)$$

so if $t > 0$ and n is a positive integer $V^{x_0}(t) =_d V^{x_0}(t(\rho_1/\rho_2)^n)$. Letting $n \to \infty$ and using the right continuity of V^{x_0} at 0 gives $P\{V^{x_0}(t) = x_0\} = 1$ for each t, a contradiction, so $\lim_{n\to\infty} m_n(\lambda)/n$ exists and is positive.

If we let $\rho(\lambda) = \lim_{n\to\infty} m_n(\lambda)/n$ then ρ is a positive nondecreasing function which satisfies $\rho(s)\rho(t) = \rho(st)$. From this it is immediate that $\rho(s) = s^\delta$ for some $\delta \geqq 0$ and (2) holds.

To prove (3) we will consider two cases. First, let $\delta > 0$. If $\lambda_1^\delta < t < \lambda_2^\delta$ then for n sufficiently large $m_n(\lambda_1) < [nt] < m_n(\lambda_2)$ so $\lambda_1 \leqq \liminf_n c_{nt}/c_n \leqq \limsup_n c_{nt}/c_n \leqq \lambda_2$. Since this holds for all λ_1 and λ_2 with $\lambda_1^\delta < t < \lambda_2^\delta$ this means $\lim_{n\to\infty} c_{nt}/c_n = t^{1/\delta}$. If $\delta = 0$ a similar argument shows that if $t < 1$, $\limsup_{n\to\infty} c_{nt}/c_n < \varepsilon$ for all $\varepsilon > 0$ and this completes the proof.

REMARK. A function L is slowly varying if $\lim_{t\to\infty} L(xt)/L(t) = 1$ for all $x > 0$. Using this notation conclusion (3) can be written as $c_n = n^{1/\delta}L(n)$. Since we will write many statements like this in what follows we will use the letter L to denote slowly varying functions. The value of $L(n)$ is rarely important for our arguments and in general will change from line to line. Subscripts and other ornaments will be attached when we want to emphasize that the slowly varying function depends upon the indicated parameters.

If $\delta > 0$ we can let $c = n^{-1/\delta}$ and $x = y/n^{-1/\delta}$ in (2) to obtain

$$V^y =_d n^{-1/\delta} V^{yn^{1/\delta}}(n\cdot) \tag{4}$$

so (1) and (2) characterize the processes which can occur as limits in (ii). If $\delta = 0$, however, (2) becomes $V^{cy} =_d cV^y$ and we can no longer guarantee that there are $c_n \to \infty$ so that $c_n^{-1}V^{c_n x}(n\cdot)$ converges. We have not been able to characterize the limits which can occur when $\delta = 0$. The next few results shows that these processes have some strange properties.

An immediate consequence of Theorem 2 is the fact that for all $c > 0$

$$P^{cx}\{T_0 > t\} = P^x\{T_0 > tc^{-\delta}\}. \tag{5}$$

If $\delta = 0$ this means that $P^y\{T_0 > t\}$ has the same value for all $y > 0$ so using the strong Markov property

$$\begin{aligned} P^y\{T_0 > s + t\} &= E^y[T_0 > s;\, P^{V(s)}\{T_0 > t\}] \\ &= P^y\{T_0 > s\}P^y\{T_0 > t\}. \end{aligned}$$

Since $\varphi(t) = P^y\{T_0 > t\}$ is nonincreasing, nonnegative, and satisfies $\varphi(t + s) = \varphi(s)\varphi(t)$ this means $P^y\{T_0 > t\} = e^{-\lambda t}$ for some $\lambda \geqq 0$ (which is independent of y).

This shows that (iii) is always satisfied if $\delta = 0$. If $\delta > 0$, however, we are not so lucky. In this case taking $c > 1$ in (5) gives only an inequality:

$$P^x\{T_0 > t\} \geqq P^y\{T_0 > t\} \qquad \text{when} \quad x \geqq y > 0, \tag{6}$$

so we are forced to take a new approach.

Let $S_x = \inf\{t\colon P^x\{T_0 > t\} = 0\}$. What we would like to show is that $S_x = \infty$ for each $x > 0$. From (2), we have:

$$\text{if} \quad c > 0, \qquad S_{cy} = c^\delta S_y, \tag{7}$$

so either all the S_x are infinite or none is.

Suppose $S_y < \infty$. For $x > 0$ and $f \in D$ let $T_x(f) = \inf\{t > 0 : f(t) \geqq x\}$. Using the strong Markov property

$$0 = P^y\{T_0 > S_y\} \geqq E^y[T_{y+\varepsilon} < T_0; P^{V(T_{y+\varepsilon})}\{T_0 > S_y - T_{y+\varepsilon}\}] .$$

Since $V^y(T_{y+\varepsilon}) \geqq y + \varepsilon$ and $S_y - T_{y+\varepsilon} < S_y$ it follows from (7) that the integrand is positive so $P^y\{T_{y+\varepsilon} < T_0\} = 0$ for each $\varepsilon > 0$.

Since V is strong Markov process this implies $V^y(t \wedge T_0)$ is nonincreasing. When we note that for each $t > 0$, $0 = P^y\{T_0 > S_y\} \geqq P\{V(t) = y, T_0 > t\}P^y\{T_0 > S_y - t\}$ we have shown:

(8) if $S_y < \infty$, $V^y(t)$ is strictly decreasing for $t < T_0$.

Having arrived at a strange conclusion under the assumption $S_y < \infty$ we might hope to continue and derive a contradiction. The next example (due to W. Vervaat) shows that assumptions (i) and (ii) do not imply (iii).

EXAMPLE. Let v_n be a Markov chain with state space $\{0, 1, 2 \cdots\}$ which makes transitions according to the following rules:

$$\begin{aligned} & P\{v_{n+1} = 0 \mid v_{-} = 0\} = P\{v_{n+1} = 0 \mid v_n = 1\} = 1 \\ k \geqq 2 \qquad & P\{v_{n+1} = k - 1 \mid v_n = k\} = 1 - (1/k) \\ & P\{v_{n+1} = 0 \mid v_n = k\} = 1/k . \end{aligned}$$

From the definition of v_n it is easy to check that $v_{[n\cdot]}/n$ converges (in the sense specified in (ii)) to a process which has the following form:

$$\begin{aligned} V^x(t) &= x - t \qquad \text{if} \quad t < R_x \\ &= 0 \qquad\quad \text{if} \quad t \geqq R_x \end{aligned}$$

where R_x has $P\{R_x \leqq x\} = 1$ and for $0 < s < x$ $P\{R_x > x - s\} = \lim_{n\to\infty} \prod_{m=[ns]}^{[nx]}(1 - 1/m) = \exp(\log s - \log x) = s/x$.

Up to this point we have only used the scaling relationship for $x > 0$. If we let $x = 0$ in (2) and (5) then we get two more formulas to help us analyze the limit process.

$$V^0 =_d cV^0(\cdot c^{-\delta}) \tag{9}$$

$$P^0\{T_0 > t\} = P^0\{T_0 > tc^{-\delta}\} . \tag{10}$$

If $\delta = 0$, (9) says $V^0 =_d cV^0$ for all $c > 0$ so $V^0 \equiv 0$. Combining this result with the fact that $P^x\{T_0 > t\} = e^{-\lambda t}$ for $x > 0$ gives that for all $h > 0$

$$\lim_{\varepsilon \downarrow 0} P(\sup_{0\leqq t\leqq 1} V^\varepsilon(t) > h \mid T_0 > 1) \leqq e^\lambda \lim_{\varepsilon \downarrow 0} P(\sup_{0\leqq t\leqq 1} V^{\varepsilon'}(t) > h) = 0$$

so $(V^\varepsilon \mid T_0 > 1) \Rightarrow 0$ as $\varepsilon \downarrow 0$. Taking a peek ahead into Section 3.3 we see that this means the only possible limit of V_n^+ is 0 so we will abandon this case and label it trivial.

If $\delta > 0$, (10) shows that $P^0\{T_0 > t\}$ has the same value for all $t > 0$. Since $P^0\{T_0 > 0\} = \lim_{u\downarrow 0} P^0\{T_0 > u\}$ it follows from the Blumenthal 0-1 law ([22],

Theorem 5.17) that

$$P^0\{T_0 > t\} \quad \text{is either} \quad \equiv 0 \quad \text{or} \quad \equiv 1 . \tag{11}$$

Since $\{T_0 > t\}$ is open, using (6) and (1) give

$$P^x\{T_0 > t\} \geqq \lim_{y \downarrow 0} P^y\{T_0 > t\} \geqq P^0\{T_0 > t\} \tag{12}$$

for all $t, x > 0$.

From (12) we see that if $P^0\{T_0 > t\} \equiv 1$ then $P^x\{T_0 > t\} = 1$ for all $t, x > 0$ and so we expect that the conditioning to stay positive will have no effect. For positive levels this is a consequence of the results of Chapter 2: if $x_n \to x > 0$, using Theorem 2.2 gives $(V^{x_n} \mid N > n) \Rightarrow (V^x \mid T_0 > 1) = V^x$.

If $x_n \to 0$ the situation becomes more complicated. If $\liminf_n P_n{}^{x_n}\{N > n\} < 1$ then we cannot apply the results of Chapter 2 (each theorem has $P_n(A_n) \to P(A)$ as a conclusion) and if $\liminf_n P_n{}^{x_n}\{N > n\} = 0$, $V_n{}^+$ may fail to be tight. Conditions for convergence in this case will be given in Section 3.3. The results given there will show that if the limit exists in the sense of (a) then $V_n{}^+ \Rightarrow V^0$, i.e., the conditioning has no effect.

For the rest of the paper we will be mainly concerned with what happens when $P^x\{T_0 > t\} \not\equiv 1$ for some (and hence all) $x > 0$. Since $P^x\{T_0 > t\}$ is decreasing $\lim_{t \to \infty} P^x\{T_0 > t\}$ exists for each $x > 0$. Using the scaling relationship gives that this limit is independent of x. Call it λ. From the Markov property

$$P^x\{T_0 > t + s\} = E^x[T_0 > t; P^{V(t)}\{T_0 > s\}] .$$

Letting $s \to \infty$ gives $\lambda = \lambda P^x\{T_0 > t\}$ so $\lambda = 0$.

If $\delta = 0$, this agrees with our previous calculation. If $\delta > 0$, we can use (5) to conclude

$$\lim_{x \downarrow 0} P^x\{T_0 > t\} = \lim_{u \uparrow \infty} P^1\{T_0 > u\} = 0 \qquad \text{iff} \quad P^x\{T_0 > t\} \not\equiv 1 . \tag{13}$$

The reason for interest in this conclusion is the following result which is an immediate consequence of Theorem 2 of Section 2:

(14) Suppose $\lim_{x \downarrow 0} P^x\{T_0 > t\} = 0$ for all $t > 0$ and (iv) holds.
If for each m, $P\{N > m \mid v_0 = x\}$ is an increasing function of x then
(v) $P^{x_n}\{N > nt_n\} \to 0$ whenever $x_n \to 0$ and $t_n \to t > 0$.

There is a converse to this proved in [41]:

$$\text{if (v) holds then so does (iv)} . \tag{15}$$

Since it is usually more difficult to verify (v) than (iv), (15) is not a useful result for checking that (iv) holds. To obtain the results which we will use to check (iv) in Chapter 4, we will use the results of Chapter 2.

Let $T_0{}^- = \inf\{t > 0 : f(t) < 0\}$. If $P^x\{T_0 = t\} = 0$ and $P^0\{T_0{}^- = 0\} = 1$ then from the strong Markov property $P^x\{f : \inf_{0 \leq s \leq t} f(s) = 0\} = 0$; so using Theorem 2.3 gives $(V_n{}^{x_n} \mid N > nt_n) \Rightarrow (V^x \mid T_0 > t)$ whenever $x_n \to x > 0$ and $t_n \to t > 0$.

From (9)

$$P^0\{T_0^- = 0\} \geqq \lim_{t\to 0} P\{V^0(t) < 0\} = P\{V^0(1) < 0\}, \tag{16}$$

so if $P^0\{V^0(1) < 0\} > 0$ using the Blumenthal 0-1 law gives $P^0\{T_0^- = 0\} = 1$ and the result above can be applied to conclude:

(17) if $P\{V^0(1) < 0\} > 0$ and $P^1\{T_0 = t\} = 0$ for all $t > 0$ then (iv) holds.

On the other hand, if $P\{V^0(1) < 0\} = 0$

$$P\{\inf_{0\leqq s\leqq t} V^0(s) \geqq 0\} \geqq 1 - \sum_{q,\,\text{rational}} P\{V^0(q) < 0\} = 1 \tag{18}$$

so $V^0 \geqq 0$ and Theorem 2.3 cannot be applied. In this case we will use Theorem 2.2 or another trick (see Section 4.4).

3.2. *Conditions for tightness.* According to Theorem 15.2 in [20], a sequence of probability measures on D is tight if and only if the following two conditions hold:

(a) $\lim_{M\to\infty} \limsup_{n\to\infty} P_n\{f : \sup_t |f(t)| > M\} = 0$

(b) $\lim_{d\to 0} \limsup_{n\to\infty} P_n\{f : \omega_f'(d) > \varepsilon\} = 0$

where $\omega_f'(d) = \omega_f'(d; 0, 1)$ is the modulus of continuity defined by

$$\omega_f'(d; a, b) = \inf_{\{t_i\}} [\max_{1\leqq i\leqq r} (\sup_{t_{i-1}\leqq s<t<t_i} |f(s) - f(t)|)]$$

the infimum being taken over all sequences $\{t_i\}$ with $a = t_0 < t_1 < \cdots < t_r = b$ and $\min_i (t_i - t_{i-1}) > d$.

Because of the complexity of the definition of ω_f' the second condition is usually difficult to verify. In this section we will assume (i)—(iv) hold and develop equivalent conditions, which are easier to check in our special case, by examining the behavior of the path before and after hitting $[\varepsilon, \infty)$. Throughout this section we will assume that δ, the exponent in (2) of Section 3.1, is positive.

If $T_\varepsilon(f) > d$ we can let $t_1 = T_\varepsilon(f)$ in the definition of ω_f' and obtain

$$\omega_f'(d) \leqq \varepsilon \vee \omega_f'(d; T_\varepsilon, 1). \tag{1}$$

When $f = V_n^+ = (v_{[n\cdot]}/c_n \mid v_0 = x_n c_n, N > n)$ the last expression is the modulus of continuity of a process which starts from a height $V_n^+(T_\varepsilon \wedge 1)$ and is conditioned to stay positive for $(1 - T_\varepsilon)^+$ time units. Since we have assumed (iv), the results of Section 2 show that $(V_n^{x_n} \mid N > n) \Rightarrow (V^x \mid T_0 > 1)$ when $x_n \to x > 0$ and using the inequality above we can prove the following.

THEOREM 3. *V_n^+ is tight if and only if the following two conditions hold:*

(3a) *for some $\varepsilon > 0$, $\lim_{M\to\infty} \limsup_{n\to\infty} P\{V_n^+(T_\varepsilon) > M\} = 0$;*

(3b) *for all $\varepsilon > 0$, $\lim_{t\to 0} \limsup_{n\to\infty} P\{T_\varepsilon(V_n^+) < t\} = 0$.*

That is, we have tightness if the conditioning does not make the process jump too high or leave zero too fast.

PROOF. The conditions are necessary since they follow from (a) and (b) above. To prove sufficiency define the post-T_ε process

$$X_n^+(\cdot) = (v_{[n(T_\varepsilon+\cdot)]}/c_n \mid T_\varepsilon \leqq 1, N > n) .$$

Since v_n is a Markov chain,

$$X_n^+(\cdot) =_d (v_{[n\cdot]}/c_n \mid v_0 = Y_n, T_0 > L_n)$$

where

$$Y_n = (v_{nT_\varepsilon}/c_n \mid T_\varepsilon \leqq 1, N > n)$$

and

$$L_n = (1 - T_\varepsilon \mid T_\varepsilon \leqq 1, N > n) .$$

From Prohorov's theorem ([20], Theorems 6.1 and 6.2) a sequence of probability measures on D is tight if and only if every subsequence has a further subsequence which converges weakly, so it is enough to show that for any subsequence (a) and (b) hold for some further subsequence.

Let $\varepsilon > 0$. If $P_{n_k}^+\{T_\varepsilon \leqq 1\} \to 0$ as $k \to \infty$ then (a) and (b) hold so it suffices to consider subsequences for which $\liminf_{k\to\infty} P_{n_k}^+\{T_\varepsilon \leqq 1\} > 0$. In this case the tightness of Y_{n_k} follows from (3a). Since $0 \leqq L_n \leqq 1$, (Y_{n_k}, L_{n_k}) is tight and so there is a sequence of integers $m_j = n_{k_j} \uparrow \infty$ so that $(Y_{m_j}, L_{m_j}) \Rightarrow (Y, L)$.

Let h be a bounded continuous function from D to R. If $g_n(x, t) = E(h(V_n^x) \mid T_0 > t)$ then $E(h(X_n^+)) = E(g_n(Y_n, L_n))$. Using (iv) and the results of Section 2 we have that as $x_n \to x > 0$ and $t_n \to t \geqq 0$

$$g_n(x_n, t_n) \to g(x, t) = E(h(V) \mid V(0) = x, T_0 > t)$$

so from the continuous mapping theorem (Theorem 5.5 in [20]) $Eh(X_{m_k}^+) \to Eg(Y, L)$. From this we conclude $X_{m_k}^+ \Rightarrow (V \mid V(0) = Y, T_0 > L)$, a process we will denote by V^*.

Since $X_{m_k}^+ \Rightarrow V^*$ we have that $\limsup_k Eh(X_{m_k}^+) \leqq Eh(V^*)$ whenever h is bounded and upper semicontinuous. Applying this result with $h(f) = 1 \wedge (\sup_t f(t) - (M - 1))^+$ and $h(f) = \omega_f'(d) \wedge 1$ and using the obvious inequalities

$$\sup_t f(t) \leqq \varepsilon \vee \sup_{t \geqq T_\varepsilon} f(t)$$

$$P_n^+\{\omega_f'(d) > \varepsilon\} \leqq P_n^+\{T_\varepsilon < d\} + P_n^+\{\omega_f'(d; T_\varepsilon, 1) > \varepsilon \mid T_\varepsilon \leqq 1\}$$

completes the proof.

Condition (3a) may be difficult to check directly because it involves estimating the value of V_n^+ at a random time. Using the scaling relationship ((2) in Section 3.1) and the Markov property we have for $t < 1$ that

$$P\{V(1) > K \mid V(t) = x\} = P\{V^x(1 - t) > K\} = P\{xV^1((1 - t)x^{-\delta}) > K\} .$$

Since we are assuming $\delta > 0$ it follows from the right continuity of V^1 that as $x \to \infty$ the last expression above converges to 1 uniformly for $t \in [0, 1]$ so

$$\lim_{M\to\infty} P\{V(1) > K \mid V(T_\varepsilon) > M\} = 1 .$$

From the scaling relationship and the right continuity of V^1

$$\lim_{x\uparrow\infty} P^x\{T_0 > 1\} = \lim_{t\downarrow 0} P^1\{T_0 > t\} = 1\,,$$

so the same statement holds for the process V^+. This suggests:

THEOREM 4. *A sufficient condition for* (3a) *is*

$$\lim_{K\to\infty} \limsup_{n\to\infty} P\{V_n^+(1) > K\} = 0\,.$$

REMARK. From (a) it is clear that this is necessary for tightness. An argument similar to that given in the proof below will show that this is necessary for (3a).

PROOF. Using the Markov property, if $\varepsilon < K$

$$P\{V_n^+(1) > K\} = E[T_\varepsilon \leqq 1;\, q_K^n(V_n^+(T_\varepsilon),\, 1 - T_\varepsilon)]$$

where $q_K^n(x, t) = P(V_n(1) > K \mid V_n(1 - t) = x,\, T_0 > 1)$. From (iv) it follows that, if $x_n \to x > 0$ and $t_n \to t \geqq 0$

$$\liminf_{n\to\infty} q_K^n(x_n, t_n) \geqq q_K(x, t)$$

where $q_K(x, t) = P(V(1) > K \mid V(1 - t) = x,\, T_0 > 1)$ so for $u \leqq 1$

$$\begin{aligned}\limsup_{n\to\infty} & P\{V_n^+(1) > K\} \\ &\geqq \limsup_{n\to\infty} E[V_n^+(T_\varepsilon) > 2Ku^{-1/\delta};\, q_K^n(V_n^+(T_\varepsilon),\, 1 - T_\varepsilon)] \\ &\geqq [\inf\{q_K(x, s)\colon x \geqq 2Ku^{-1/\delta},\, 0 \leqq s \leqq 1\}] \limsup_{n\to\infty} P\{V_n^+(T_\varepsilon) > 2Ku^{-1/\delta}\}\,.\end{aligned}$$

From scaling $q_K(x, t) = q_{Kc}(xc, tc^\delta)$ so if $2K/x \leqq 1$, $q_K(x, t) \geqq q_K(2K, t(2K/x)^\delta)$ and from above

$$\begin{aligned}\limsup_{n\to\infty} & P\{V_n^+(1) > K\} \\ &\geqq [\inf_{0\leq s\leq u} q_K(2K, s)] \limsup_{n\to\infty} P\{V_n^+(T_\varepsilon) > 2Ku^{-1/\delta}\}\,.\end{aligned}$$

Now

$$1 \geqq q_K(2K, s) \geqq \frac{P(V_n(s) > K \mid V_n(0) = 2K) - P(T_0 \leqq s \mid V_n(0) = 2K)}{P(T_0 > s \mid V_n(0) = 2K)}\,.$$

Letting $u \to 0$ gives

$$\limsup_{n\to\infty} P\{V_n^+(1) \geqq K\} \geqq \lim_{M\to\infty} \limsup_{n\to\infty} P\{V_n^+(T_\varepsilon) > M\}$$

and letting $K \to \infty$ gives the desired result.

From Theorem 4 if we know that $V_n^+(1)$ converges then (3a) is satisfied. The next theorem gives a sufficient condition for (3b).

THEOREM 5. *Let P_n^* be the probability measures induced on $D[-1, 1]$ by $V_n^+(t \vee 0)$. If* (3a) *holds $\{P_n^*, n \geqq 1\}$ is tight. If, in addition, for every P^* which is the limit of a subsequence $P_{n_k}^*$ we have $P^*\{f : f(0) \neq f(0-)\} = 0$ then $\{P_n^+, n \geqq 1\}$ is tight.*

PROOF. For all $f \in D[-1, 1]$ which are constant on $[-1, 0)$ if $d < 1$ we have

$$\omega_f'(d;\, -1, 1) \leqq \varepsilon \vee \omega_f'(d;\, T_\varepsilon, 1)\,.$$

From this

$$P_n^*\{\omega_f'(d; -1, 1) > \varepsilon\} \leqq P_n^*\{\omega_f'(d; T_\varepsilon, 1) > \varepsilon\},$$

so using the proof of Theorem 3 we see that (3a) is sufficient for tightness in $D[-1, 1]$.

To prove the other result we note that by Prohorov's theorem it is sufficient to show that if $P_{n_k}^* \Rightarrow P^*$ then $P_{n_k}^+ \Rightarrow P^+ = P^*\pi^{-1}$ where π is the natural projection from $D[-1, 1]$ to $D[0, 1]$. If h: $D[0, 1] \to R$ has $P^+(\Delta_h) = 0$ where Δ_h is the set of discontinuities of h then $P^*\{f : f(0) \neq f(0-)\} = 0$ implies that $P^*(\Delta_{h\circ\pi}) = 0$. Using the continuous mapping theorem ([20], Theorem 5.2) now gives $P_{n_k}^*\pi^{-1}h^{-1} \Rightarrow P^*\pi^{-1}h^{-1}$ for all bounded continuous functions h, which completes the proof.

Combining the conclusions of Theorems 3, 4, and 5 gives the following result.

THEOREM 6. *V_n^+ is tight if and only if*

(6a) $\lim_{K\to\infty} \limsup_{n\to\infty} P\{V_n^+(1) > K\} = 0$,

(6b) $\lim_{t\to\infty} \limsup_{n\to\infty} P\{V_n^+(t) > h\} = 0$ *for each* $h > 0$.

From Theorem 6 if we know that the finite dimensional distributions of V_n^+ converge to those of a process V^* with $P\{V^*(0) = 0\} = 1$, then the sequence is tight.

In Theorem 10 below we will give conditions which imply that if V_n^+ is tight then the limit is $\lim_{x\downarrow 0}(V^x \mid T_0 > 1)$ (assuming this exists), so in cases when the convergence of finite dimensional distributions is not known we would like to check that the sequence is tight without computing the limit of the distributions.

One way of doing this (which we will use in Section 4.3) is to use

THEOREM 7. *If for each* $\varepsilon > 0$ $(V_n^+(T_\varepsilon) - \varepsilon)^+ \Rightarrow 0$ *then* V_n^+ *is tight.*

PROOF. Observe that if $V_{n_k}^+(t \vee 0)$ conveys (as a sequence of random elements of $D[-1, 1]$) to a process V^* with $P\{V^*(0) > 2h\} = p > 0$ for some $h > 0$ then $\liminf_k P\{V_{n_k}^+(T_h) - h > h\} \geqq p$ which contradicts the assumption that $(V_n^+(h) - h)^+ \Rightarrow 0$. This shows that the hypotheses of Theorem 5 are satisfied and proves the desired conclusion.

3.3. *Convergence of finite dimensional distributions.* In this section we will assume V_n^+ is tight and derive conditions for V_n^+ to converge. Our method of proof is not the usual one suggested by the title of this section, however. We will prove convergence by showing that all convergent subsequences have the same limit.

The first step is to consider what processes can occur as limits of the V_n^+. From (i)—(iv) and the results of Section 2, if $x_n \to x > 0$ $(V_n^{x_n} \mid N > n) \Rightarrow (V^x \mid T_0 > 1)$. Letting x_n go to zero very slowly we see that if V_n^+ converges for all $x_n \to 0$ then $\lim_{x\downarrow 0}(V^x \mid T_0 > 1)$ exists and is the limit process for any $x_n \to 0$.

Assuming $\lim_{x \downarrow 0}(V^x \mid T_0 > 1)$ exists and writing $(V^0 \mid T_0 > t)$ for $\lim_{x \downarrow 0}(V^x \mid T_0 > t)$ we can give a simple formula for the processes which can occur as limits of subsequences of V_n^+.

THEOREM 8. *If $V_{n_k}^+ \Rightarrow V^*$ then there are random variables $t^* \in [0, 1]$ and $x^* \geqq 0$ with $P\{t^* = 0, x^* > 0\} = 0$ so that*

$$V^*(\cdot) =_d 1_{\{t^* \leqq \cdot\}}(V^{x^*}(\cdot - t^*) \mid T_0 > 1 - t^*). \tag{1}$$

PROOF. From the proof of Theorem 3.3 $V^*(T_\varepsilon(V^*) + t)$ behaves like V starting from $V^*(T_\varepsilon)$ and conditioned to stay positive for $1 - T_\varepsilon(V^*)$ units of time. As ε decreases, $T_\varepsilon(V^*)$ does not increase so as $\varepsilon \downarrow 0$, $T_\varepsilon(V^*)$ converges to a limit t^*. Since V^* is right continuous this means $V^*(T_\varepsilon)$ converges to a limit x^*.

Under the hypothesis of Theorem 8, $(x, t) \to (V^x \mid T_0 > t)$ is a continuous function from $[0, \infty) \times (0, \infty)$ to $D[0, 1]$; so using the continuous mapping theorem we see that $V^*(T_\varepsilon(V^*) + t) \Rightarrow (V^{x^*} \mid T_0 > 1 - t^*)$. Since $0 \leqq V^* < \varepsilon$ on $[0, T_\varepsilon(V^*))$ this shows V^* has the representation given by (1).

To see that $P\{t^* = 0, x^* > 0\} = 0$ observe that since $V_{n_k}^+ \Rightarrow V^*$ in D, $x_{n_k} = V_{n_k}^+(0) \Rightarrow V^*(0)$ so $V^*(0) = 0$.

Having identified the possible limits of subsequences of $(V_n^{x_n} \mid T_0 > 1)$ the next step in solving problem (a) is to determine for which V^* there is a Markov chain v_n so that $(V_n^{x_n} \mid T_0 > 1) \Rightarrow V^*$ for all $x_n \to 0$.

If $\limsup_n P_n^{x_n}\{N > n\} > 0$ for some $x_n \to 0$ then it is easy to show that a subsequence of $V_{n_k}^+$ converges to V^0. In this case if the convergence takes place in the sense of (a) the conditioning will have no effect. So in what follows we will assume that (v) holds.

To characterize the limits which can occur when (v) holds we will investigate the convergence in the case $x_n c_n \equiv a$. In this instance the limit process results from conditioning and scaling a single sequence of random variables so there is a scaling relationship which allows us to compute the distribution of V^* from that of $V^*(1)$.

THEOREM 9. *Let $x_n c_n \equiv a$, $Q^a(\cdot) = P(\cdot \mid v_0 = a)$. If $V_n^+(1) \Rightarrow 0$ then V_n^+ converges to a process which is $\equiv 0$. If $V_n^+(1) \Rightarrow v^*$ with $P\{v^* = 0\} < 1$ and* (v) *holds then $Q^a\{N > n\} = n^{-\beta}L_a(n)$. In the second case the finite dimensional distributions of $\{V_n^+, 0 < s \leqq 1\}$ converge to those of a nonhomogeneous Markov process V^+ which has*

$$P(V^+(t) \in dy) = t^{-\beta}P(t^{1/\delta}v^* \in dy)P^y\{T_0 > 1 - t\} \tag{2}$$

and

$$P(V^+(t) \in dy \mid V^+(s) = x) = \frac{P(V^x(t - s) \in dy, T_0 > t - s)P^y\{T_0 > 1 - t\}}{P^x\{T_0 > 1 - s\}} \tag{3}$$

for $s < t$, $x > 0$.

If $V^+(t) \Rightarrow 0$ as $t \to 0$ then V_n^+ is tight and $V_n^+ \Rightarrow V^+$.

PROOF. The first result is obvious: observe that if V^* is given by (1) and $x > 0$ then $P(V^*(t+s) > 0 \mid V^*(t) = x) = P(V^x(s) > 0 \mid T_0 > 1 - t) = 1$ so V^* does not hit zero after it hits a positive level.

To prove the second statement, note that if $\lambda > 0$

$$(4) \qquad \frac{Q^a\{N > (1+\lambda)n\}}{Q^a\{N > n\}} = \int_{(0,\infty)} Q^a(\dot{V}_n(1) \in dx \mid N > n) P(N > \lambda n \mid v_0 = xc_n),$$

and from the hypothesis as $x_n \to x \geqq 0$, $\varphi_n{}^{\lambda}(x_n) = P(N > \lambda n \mid v_0 = x_n c_n)$ converges to $P^x\{T_0 > \lambda\} = \varphi^{\lambda}(x)$.

$\varphi^{\lambda}(x) > 0$ for $x > 0$ so if $V_n{}^+(1) \Rightarrow V^*$ with $P\{v^* = 0\} < 1$ then from Theorem 5.5 in [20] $Q^a\{N > (1+\lambda)n\}/Q^a\{N > n\}$ converges to a positive limit. If we let $\rho(1+\lambda)$ denote the value of this limit then since $\rho(st) = \rho(s)\rho(t)$, ρ is measurable, and $\rho(s) \leqq 1$ for $s \geqq 1$ we can conclude $\rho(s) = s^{-\beta}$ for some $\beta \geqq 0$.

This shows that $Q^a\{N > n\}$ has the indicated form. To prove that the finite dimensional distributions of $V_n{}^+$ converge we will use this fact and the following formula:

If $k \geqq 1$, $0 < t_1 < \cdots < t_k \leqq 1$ and $y_1, \cdots, y_k$ are positive,

$$(5) \qquad P\{V_n{}^+(t_1) \leqq y_1 \cdots V_n{}^+(t_k) \leqq y_k\}$$
$$= \frac{Q^a\{N > nt_1\}}{Q^a\{N > n\}} \int_{(0,y_1]} Q^a\left(\frac{c_{nt_1}}{c_n} V_{nt_1}(1) \in dx \mid N > nt_1\right) \psi_n{}^{t_1}(x)$$

where

$$\psi_n{}^{t_1}(x) = P(V_n(t_2) \leqq y_2 \cdots V_n(t_k) \leqq y_k, \inf_{t_1 \leqq s \leqq 1} V_n(s) > 0 \mid V_n(t_1) = x).$$

From (iv) and Theorem 2.2 if $x_n \to x > 0$ then

$$\psi_n{}^{t_1}(x_n) \to \psi^{t_1}(x) = P(V(t_2) \leqq y_2 \cdots V(t_k) \leqq y_k, \inf_{t_1 \leqq s \leqq 1} V(x) > 0 \mid V(t_1) = x)$$

whenever the y_i are all continuity points of the distributions of the $V(t_i)$, so if we can show $P\{v^* = 0\} = 0$ we can use Theorem 5.5 in [20] to conclude

$$P(V_n{}^+(t_1) \leqq y_1 \cdots V_n{}^+(t_k) \leqq y_k) \to t_1{}^{-\beta} \int_{(0,y_1]} P(t_1{}^{1/\delta} v^* \in dx) \psi_{t_1}(x),$$

which shows the limit process has the indicated form.

Let $G_n(x) = P\{V_n{}^+(1) \leqq x\}$, $G(x) = P\{v^* \leqq x\}$. From (iv), (v), and Theorem 5.5 in [20]

$$\int_{(0,\infty)} G_{n_k}(dx) \varphi_{n_k}^{\lambda}(x) \to \int_{[0,\infty)} G(dx) \varphi^{\lambda}(x).$$

Since $Q^a\{N > (1+\lambda)n\}/Q^a\{N > n\} \to (1+\lambda)^{-\beta}$, using (4) gives

$$(1+\lambda)^{-\beta} = \int_{[0,\infty)} G(dx) \varphi^{\lambda}(x).$$

Now (v) implies $\varphi^{\lambda}(0) = 0$ and we always have $\varphi^{\lambda}(x) \leqq 1$; so this means that $G(0) \leqq 1 - (1+\lambda)^{-\beta}$ for all $\lambda > 0$ or $G(0) = 0$.

To complete the proof of Theorem 9, we observe that the last statement follows from the remark after Theorem 3.6.

Combining the results of Theorems 8 and 9 we observe that if (i)—(v) hold

and V_n^+ converges in the sense specified by problem (a) then the limit is either $\equiv 0$ or > 0 at each $t > 0$ so there are only two possible limits (assuming $\lim_{x\to 0}(V^x \mid T_0 > 1)$ exists).

At this point we are ready to consider conditions for convergence to each of these limits but there is not really much to say. The next result, which summarizes our main conclusions, is an easy consequence of Theorems 8 and 9.

THEOREM 10. *Let v_n be a Markov chain for which* (i)—(iv) *hold. Let $x_n \to 0$ and suppose V_n^+ is tight. $V_n^+ \Rightarrow 0$ if and only if*

$$P\{V_n^+(t) > \varepsilon\} \to 0 \qquad \text{for all} \quad \varepsilon > 0\,, \quad t \geqq 0\,. \tag{6}$$

If $V^+ = \lim_{x\to 0}(V^x \mid T_0 > 1)$ exists and is $\not\equiv 0$ then $V_n^+ \Rightarrow V^+$ if and only if

$$\lim_{\varepsilon \downarrow 0} \liminf_{n\to\infty} P\{V_n^+(t) > \varepsilon\} = 1 \qquad \text{for all} \quad t > 0\,. \tag{7}$$

If we assume in addition that $x_n c_n \equiv a$ and (v) *holds then $V_n^+ \Rightarrow V^+$ if and only if $Q^a\{N > n\} = n^{-\beta} L_a(n)$.*

PROOF. The first result is trivial. The second follows from Theorem 8 since the condition given is equivalent to assuming that for all subsequential limits V^*, $P\{V^*(t) > 0\} = 1$ for all $t > 0$. To prove the third statement we observe that from the proof of Theorem 9 if $V_{n_k}^+ \Rightarrow V^*$ then $V_{n_k}^+(1) \Rightarrow v^*$ with $P\{v^* > 0\} = 1$. Since this holds for all convergent subsequences it follows that $P\{V^*(t) > 0\} = 1$ for all $t > 0$ and the desired conclusion follows from Theorem 8.

4. Examples.

4.1. *Branching processes.* Let z_n, $n \geqq 0$ denote the number of particles in the nth generation of a Galton–Watson process with $z_0 = 1$ and particle production governed by the probability distribution $\{p_i, i = 0, 1, 2, \cdots\}$. (For a detailed definition consult the first few pages of [34] or [35].) Let $f(s) = \sum_{i=0}^{\infty} p_i s^i$ be the generating function of z_1 and for each $n \geqq 2$ let $f_n(s) = f(f_{n-1}(s))$ be the generating function of z_n. Kesten, Ney, and Spitzer ([34], page 19) have shown that

THEOREM 1. *If $Ez_1 = 1$ and $E(z_1 - 1)^2 = 2\lambda \in (0, \infty)$ then*

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1 - f_n(s)} - \frac{1}{1 - s}\right] = \lambda \tag{1}$$

uniformly for $0 \leqq s < 1$.

Setting $s = 0$ in (1) and noting that $P\{z_n > 0\} = 1 - f_n(0)$ we obtain the following formula for $P\{z_n > 0\}$.

THEOREM 2. *As $n \to \infty$,*

$$P\{z_n > 0\} \sim (n\lambda)^{-1}\,. \tag{2}$$

Another immediate consequence of Theorem 1 is the following conditioned limit theorem.

THEOREM 3.

$$\lim_{n\to\infty} P\{z_n/n\lambda > x \mid z_n > 0\} = e^{-x}. \tag{3}$$

PROOF.

$$\begin{aligned} E(e^{-\alpha z_n/n\lambda} \mid z_n > 0) &= E(e^{-\alpha z_n/n\lambda}; z_n > 0)/E(1; z_n > 0) \\ &= (f_n(e^{-\alpha/n\lambda}) - f_n(0))/(1 - f_n(0)) \\ &= 1 - (1 - f_n(e^{-\alpha/n\lambda}))/(1 - f_n(0)). \end{aligned}$$

From (1), $\lim_{n\to\infty} [n(1 - f_n(e^{-\alpha/n\lambda}))]^{-1} = \lambda + \lim_{n\to\infty} [n(1 - e^{-\alpha/n\lambda})]^{-1}$ and from (2), $\lim_{n\to\infty} n(1 - f_n(0)) = 1/\lambda$ so

$$\lim_{n\to\infty} E(e^{-\alpha z_n/n\lambda} \mid z_n > 0) = 1 - 1/(1 + (1/\alpha)) = 1/(\alpha + 1),$$

which completes the proof.

Using the last two results we can compute the limit of $(z_n/n\lambda \mid z_0 = y_n \lambda n)$. Since the $y_n \lambda n$ ancestors act independently, we have from Theorem 2 that if $y_n \to y \geqq 0$ then the number of ancestors which have offspring alive at time n tends to have a Poisson distribution with mean y. Using Theorem 3 now gives that if $y_n \to y \geqq 0$

$$\lim_{n\to\infty} E(e^{-\alpha z_n/n\lambda} \mid z_0 = y_n \lambda n) = e^{-y} \sum_{k=0}^{\infty} \frac{y^k}{k!} (1 + \alpha)^{-k} = \exp(-y\alpha/(1 + \alpha)).$$

Using the Markov property it is easy to compute that the finite dimensional distributions of $Z_n^{y_n} = (z_{[n\cdot]}/n\lambda \mid z_0 = y_n \lambda n)$ converge (a result due to Lamperti [36], Theorem 2.5). In [37], Lindvaal has shown that the sequence is tight so we have the following.

THEOREM 4. *If $y_n \to y \geqq 0$ then $Z_n^{y_n} \Rightarrow (Z \mid Z(0) = y)$ where Z is a nonnegative diffusion with transition probabilities satisfying*

$$\int e^{-\alpha y} P(Z(t + s) \in dy \mid Z(s) = x) = \exp(-x\alpha/(1 + \alpha t))$$

for all nonnegative x, s, and t.

Observe that 0 is an absorbing state so

$$P^x\{T_0 > t\} = P^x\{Z(t) > 0\} = 1 - e^{-x/t} > 0 \tag{4}$$

and we have that (iii) holds. From the remarks after Theorem 3, $P_n^{x_n}\{N > nt_n\} \to 1 - e^{-x/t}$ when $x_n \to x \geqq 0$ and $t_n \to t > 0$ so (iv) and (v) hold.

At this point we have completed our preparation and can apply Theorem 3.9 to conclude:

THEOREM 5. $Z_n^+ = (z_{[n\cdot]}/\lambda n \mid z_0 = 1, z_n > 0) \Rightarrow (Z^+ \mid Z^+(0) = 0)$ *where Z^+ is a Markov process with*

$$P(Z^+(t) \in dx) = t^{-2} e^{-x/t}[1 - e^{-x/(1-t)}]$$

and

$$\begin{aligned} &P(Z^+(t) \in dy \mid Z^+(s) = x) \\ &\quad = x(t - s)^{-2} e^{-(x+y)/(t-s)} \sum_{k=1}^{\infty} \frac{(xy/(t - s)^2)^{k-1}}{k!\, k - 1!} \cdot \frac{1 - e^{-y/(1-t)}}{1 - e^{-x/(1-s)}}. \end{aligned}$$

PROOF. From Theorem 3.9 we have that the finite dimensional distributions of Z_n^+ converge. To obtain the formulas given above from those in Section 3.3 observe that from the discussion following Theorem 3,

$$P(Z(t+s) \in dy,\ Z(t+s) > 0 \mid Z(s) = x) = \sum_{k=1}^{\infty} e^{-x/t} \frac{(x/t)^k}{k!} \left(\frac{t^{-1}(y/t)^{k-1}}{k-1!} e^{-y/t} \right).$$

To prove that the sequence is tight we have to check that for the distributions given above $Z^+(t) \Rightarrow 0$ as $t \to 0$. To do this we observe that if $y > 0$ and $t \to 0$ then

$$P(Z^+(t) > y) \leqq \int_y^{\infty} t^{-2} e^{-x/t}\, dx = t^{-1} e^{-y/t} \to 0 .$$

4.2. *Random walks.* If $X_1, X_2, \cdots$ is a sequence of independent and identically distributed random variables, $S_n = S_{n-1} + X_n$, $n \geqq 1$ defines a random walk. Necessary and sufficient conditions for the convergence of $(S_n - b_n)/a_n$ are known (cf. [29], Chapter 7). In this section we will use some of these results to show that if S_n/a_n converges in distribution to G then (i)—(iv) hold and the results of Chapter 3 can be applied to prove the appropriate conditioned limit theorems.

THEOREM 1. *For the nondegenerate distribution G to be the limit of some sequence of normalized sums $(S_n - b_n)/a_n$ it is necessary and sufficient that it be stable, that is, if $X, X_1, \cdots, X_k$ are independent and have distribution G then there are constants $a_k' > 0$ and b_k' such that*

$$X_1 + \cdots + X_k =_d a_k' X + b_k' .$$

THEOREM 2. *$\varphi(\theta) = Ee^{i\theta X}$ is the characteristic function of a stable law if and only if*

$$\log \varphi(\theta) = i\lambda\theta - c|\theta|^{\alpha}[1 + b\omega_{\alpha}(\theta)/|\theta|] \qquad \text{for} \quad \theta \neq 0 \tag{1}$$

where $0 < \alpha \leqq 2$, $-1 \leqq b \leqq 1$, $c \geqq 0$ and

$$\omega_{\alpha}(\theta) = \tan(\pi\alpha/2) \qquad \text{if} \quad \alpha \neq 1$$
$$= (2/\pi)\log|\theta| \qquad \text{if} \quad \alpha = 1 .$$

α is called the index of the stable law, b is a shape parameter, λ gives the drift, and c is a scaling constant.

DEFINITION. A distribution F is in the domain of attraction of a (nondegenerate) distribution G if there are constants $a_n > 0$, b_n so that $F^{n*}(a_n x + b_n) \Rightarrow G(x)$. (Here F^{n*} is the n-fold convolution of F.)

THEOREM 3. *The distribution F belongs to the domain of attraction of a normal law ($\alpha = 2$) if and only if as $n \to \infty$*

$$n^2 \int_{|x|>n} F(dx) / \int_{|x|<n} x^2 F(dx) \to 0 .$$

F belongs to the domain of attraction of a stable law of index $0 < \alpha < 2$ if and only if

$$[1 - F(x)]/[1 - F(x) + F(-x)] \to p \in [0, 1] \qquad \text{as} \quad x \to \infty$$

and

$$1 - F(x) + F(-x) = x^{-\alpha} L(x) .$$

From the proof of this result in [29], pages 175–180 we can conclude the scaling constants a_n are of the form $n^{1/\alpha}L(n)$ and satisfy

$$n[1 - F(a_n x) + F(-a_n x)] \to \begin{cases} \tilde{c}x^{-\alpha} & \text{if } \alpha < 2 \\ 0 & \text{if } \alpha = 2 . \end{cases}$$

The centering constants can be chosen to be

$$\begin{cases} nEX_1 & \text{if } 1 < \alpha \leqq 2 \\ nE(-a_n \vee X_1 \wedge a_n) & \text{if } \alpha = 1 \quad \text{(see [24], page 315)} \\ 0 & \text{if } 0 < \alpha < 1 . \end{cases}$$

From discussion above we have that if $S_0 = 0$ and $(S_n - b_n)/a_n \Rightarrow Y$ then the finite dimensional distributions of $V_n(t) = (S_{[nt]} - b_{[nt]})/a_n$ converge. Skorokhod has shown (Theorem 2.7 in [32]) that there is also weak convergence.

THEOREM 4. *If S_n is a random walk and $(S_n - b_n)/a_n \Rightarrow Y$ (nondegenerate) then $V_n \Rightarrow V$, a process with stationary independent increments which has $V^0(1) =_d Y$.*

If $\lim_{n\to\infty} b_n/a_n = \mu$ (finite), the centering is unnecessary and S_n/a_n satisfies (i)—(ii). Observe that in this case the scaling exponent $\delta = \alpha$.

The next step is to check that (iii) holds. To do this we observe that if $P^y\{T_0 > t\} = 0$ for some positive y then from (8) of Section 3.1, $\{V^y(t), t < T_0\}$ is decreasing. Since V has independent increments this means $\{V^y(t), t \geqq 0\}$ is decreasing.

Conditions for stable processes to have this property are well known. Using results from [28] we see that if $P^y\{T_0 > t\} = 0$ then $0 < \alpha < 1$, $b = -1$, and $\lambda < 0$ in (1). To complete the proof we will use the scaling relationship to show that none of these processes can occur as limits in (ii).

Let $\varphi_t(\theta) = E\exp(i\theta V^0(t))$. Since V^0 has stationary independent increments $\varphi_t(\theta) = \varphi_1(\theta)^t$. From scaling $V^0(t) =_d t^{1/\alpha}V^0(1)$ so $\varphi_t(\theta) = \varphi_1(t^{1/\alpha}\theta)$. Using $t \log \varphi_1(\theta) = \log \varphi_1(t^{1/\alpha}\theta)$ in (1) gives:

(2) For limits of S_n/a_n, $\lambda = 0$ if $\alpha \neq 1$ and $b = 0$ if $\alpha = 1$.

Since these conditions are incompatible with the ones given above we have shown that (iii) holds.

To prove that (iv) holds we start by observing that stable laws have continuous distributions ([29], page 183) so $P^x\{T_0 = t\} \leqq P\{V^x(t) = 0\} = 0$. If $P\{V^0(1) < 0\} > 0$ then result (17) of Section 3.1 can be applied to give (iv). If $P\{V^0(1) \geqq 0\} = 1$ then $P^x\{T_0 > t\} \equiv 1$ for all $x < 0$ and (iv) follows from (ii) since $\{T_0 > t\}$ is open.

Using (14) of Section 3.1 we see that (v) is satisfied if $P\{V^0(1) < 0\} > 0$ but not otherwise. Having established that (i)—(v) hold when V is not increasing, the next step is to give conditions for the sequence V_n^+ to be tight.

THEOREM 5. *If X_1 has a distribution F so that $F^{n*}(c_n \cdot) \Rightarrow G$, a stable law with $G(0) < 1$, then V_n^+ is tight for $x_n \equiv 0$.*

REMARK. If $G(0) = 1$, V is decreasing so $(V^\varepsilon \mid T_0 > 1) \Rightarrow 0$ as $\varepsilon \downarrow 0$. From the remark after Theorem 3.9, we see that 0 is the only possible limit in this case.

PROOF. The proof will be given in three lemmas, each of which assumes the hypotheses of Theorem 5 and uses the notation of Theorem 3.9.

LEMMA 1. *If* $G(0) = \beta < 1$ *then* $Q^0\{N > n\} = n^{-\beta}L(n)$.

PROOF. Since stable laws have continuous distributions $\lim_{k\to\infty} Q^0\{S_k > 0\} = 1 - \beta$. By a formula due to Spitzer ([33], page 330) if S_k is a random walk then

$$\sum_{n=0}^{\infty} Q^0\{N > n\}t^n = \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k} P\{S_k > 0\}\right).$$

Writing $\theta(t)$ for the generating function of $Q^0\{N > n\}$ and factoring the right-hand side gives

$$\theta(t) = (1 - t)^{\beta-1} \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k} (P\{S_k > 0\} - (1 - \beta))\right).$$

Now $L(1/(1-t)) = \exp(-\sum_{k=1}^{\infty}(t^k/k)a_k)$ is slowly varying whenever $\lim_{k\to\infty} a_k = 0$ (for a proof see [15], page 1159) so applying a Tauberian theorem ([24], page 447) gives

$$\sum_{m=1}^{n} P\{N > m\} = n^{1-\beta}L(n).$$

Since $P\{N > m\}$ is a decreasing function of m, applying a generalization of Landau's theorem ([24], page 446) gives

$$\lim_{n\to\infty} P\{N > n\} \Big/ \frac{1}{n} \sum_{k=1}^{n} P\{N > k\} = 1 - \beta$$

so if $\beta < 1$, $P\{N > n\} = n^{-\beta}L(n)$.

LEMMA 2. *Condition* (3a) *of Theorem* 3.3 *is satisfied whenever the limit process has* $P\{V^0(1) > 0\} > 0$. *If* $\alpha = 2$, *we have in addition that* $(V_n^+(T_\varepsilon) - \varepsilon)^+ \Rightarrow 0$ *so tightness follows from Theorem* 3.7.

PROOF. Let $X_i = S_i - S_{i-1}$. Let $I_n^y = \inf\{i \leqq n : X_i/c_n > y\}$, with $I_n^y = \infty$ if the set is empty.

$$P\{N > n, I_n^y < \infty\} \leqq \sum_{i=1}^{n} P\{N > i - 1 \mid I_n^y = i\}P\{I_n^y = i\}.$$

Given $I_n^y = i$, $X_1, \cdots, X_{i-1}$ are independent and have common distribution function $H_y(x) = (F(x)/F(yc_n)) \wedge 1$. Now $H_y(x) \geqq F(x)$ for all x, so if $U_1, U_2, \cdots, U_{i-1}$ are independent random variables each with a uniform distribution on (0, 1) then

$$\begin{aligned}((X_1, \cdots, X_{i-1}) \mid I_n^y = i) &=_d (H_y^{-1}(U_1), \cdots, H_y^{-1}(U_{i-1})) \\ &\leqq (F^{-1}(U_1), \cdots, F^{-1}(U_{i-1})) =_d (X_1, \cdots, X_{i-1}),\end{aligned}$$

where the equalities are between distributions and the inequality holds almost surely. From this it is clear that $P\{N > i - 1 \mid I_n^y = i\} \leqq P\{N > i - 1\}$. Using this in the first inequality we get

$$P\{I_n^y < \infty \mid N > n\} \leqq \sum_{i=1}^{n} \frac{P\{N > i - 1\}}{P\{N > n\}} P\{I_n^y = i\}.$$

Now $P\{N > n\} = n^{-\beta}L(n)$ and $P\{I_n^y = i\} \leqq P\{X_i > yc_n\}$ so

$$P\{I_n^y < \infty \mid N > n\} \leqq \frac{(1 + \sum_{i=1}^{n-1} i^{-\beta}L(i))}{n(n^{-\beta}L(n))}\, n(1 - F(yc_n)) .$$

$u(x) = [x]^{-\beta}L([x])$ is regularly varying with exponent > -1, so from Karamata's theorem

$$\frac{\sum_{i=1}^{n-1} i^{-\beta}L(i)}{n(n^{-\beta}L(n))} = \frac{\int_1^n u(x)\,dx}{nu(n)} \to 1/(1 - \beta) .$$

From Theorem 3 if $0 < \alpha < 2$

$$\lim_{x\to\infty} \frac{1 - F(x)}{1 - F(x) + F(-x)} = p \in [0, 1]$$

and $\lim_{n\to\infty} n[1 - F(c_n y) + F(-c_n y)] = \bar{c}y^{-\alpha}$ so in this case $\limsup_{n\to\infty} P\{I_n^y < \infty \mid N > n\} \leqq p\bar{c}y^{-\alpha}/(1 - \beta)$. From this we get

$$\lim_{y\to\infty} \limsup_{n\to\infty} P\{V_n^+(T_\varepsilon) > y + \varepsilon\}$$
$$\leqq \lim_{y\to\infty} \limsup_{n\to\infty} P\{I_n^y < \infty \mid N > n\} = 0$$

so (3a) is satisfied for $0 < \alpha < 2$.

To prove the result for $\alpha = 2$ we observe that from above

$$\limsup_{n\to\infty} P\{I_n^y < \infty \mid N > n\} \leqq 2 \limsup_{n\to\infty} n(1 - F(yc_n))$$

so using Theorem 3 gives $(V_n^+(T_\varepsilon) - \varepsilon)^+ \Rightarrow 0$ and applying Theorem 3.7 gives that the sequence is tight when $\alpha = 2$.

To complete the tightness proof when $0 < \alpha < 2$ we use Theorem 3.5 and the following.

LEMMA 3. $\lim_{u\downarrow 0} \limsup_{n\to\infty} P\{V_n^+(u) > y\} = 0$.

PROOF. If $k_n = n - [nst]$ then

$$P\{V_n(st) > y, N > n\}$$
$$= \int_{(y,\infty)} P\left(\frac{c_{nt}}{c_n} V_{nt}\left(\frac{snt}{[nt]}\right) \in dx, N > snt\right) P(N > k_n \mid v_0 = xc_n) .$$

If $m_n = nt - [nst]$ we have

$$P\left(\frac{c_{nt}}{c_n} V_{nt}^+\left(\frac{snt}{[nt]}\right) \in dx\right) = \frac{P\left(\frac{c_{nt}}{c_n} V_{nt}\left(\frac{snt}{[nt]}\right) \in dx, N > snt\right) P(N > m_n \mid v_0 = xc_n)}{P\{N > nt\}} .$$

Using the last two equations gives

$$P\{V_n^+(st) > y\} = \frac{P\{N > nt\}}{P\{N > n\}} \int_{(y,\infty)} P\left(\frac{c_{nt}}{c_n} V_{nt}^+\left(\frac{snt}{[nt]}\right) \in dx\right) \frac{P(N > k_n \mid v_0 = xc_n)}{P(N > m_n \mid v_0 = xc_n)}$$
$$\leqq \frac{P\{N > nt\}}{P\{N > n\}} P\left\{\frac{c_{nt}}{c_n} V_{nt}^+\left(\frac{snt}{[nt]}\right) > y\right\} .$$

From Lemma 2 and Theorem 3.5, V_n^+ is tight in $D[-1, 1]$ so for any subsequence there is a further subsequence with $V_{n_k}^+ \Rightarrow V^*$ in $D[-1, 1]$. Now if u_m is a sequence of numbers which $\downarrow 0$ we can pick $s_m > u_m$ so that $P\{V^*(s_m) \neq V^*(s_m -)\} = 0$ for all m and $u_m/s_m \to t$. From this it follows that

$$\begin{aligned}\lim_{m\to\infty} \limsup_{n\to\infty} P\{V_n^+(u_m) > y\} &\leqq t^{-\beta} \lim_{m\to\infty} P\{t^{1/\alpha} V^*(s_m) > y\} \\ &= t^{-\beta} P\{V^*(0) > yt^{-1/\alpha}\} .\end{aligned}$$

Since $P\{V^*(0) > z\} \leqq \limsup_{n\to\infty} P\{I_n^z < \infty \mid N > n\}$, using an inequality from the proof of Lemma 2 gives

$$t^{-\beta} P\{V^*(0) > yt^{-1/\alpha}\} \leqq pcy^{-\alpha} t^{1-\beta}$$

and we can complete the proof by letting $t \downarrow 0$.

At this point we have given conditions for V_n^+ to be tight and $Q^0\{N > n\}$ to be regularly varying so from Theorem 3.10 to prove the conditional limit theorem in the case $G(0) \in (0, 1)$ it only remains to show $\lim_{x\downarrow 0}(V^x \mid T_0 > 1)$ exists.

THEOREM 6. *If V is a stable process which can occur as a limit in* (ii) *then* $\lim_{x\downarrow 0}(V^x \mid T_0 > 1)$ *exists.*

PROOF. If V is decreasing or if $P^x\{T_0 > 1\} \equiv 1$ then the result is trivial, so for what follows we will assume $P^x\{T_0 > 1\} \not\equiv 1$ and hence $P^x\{T_0 > 1\} \downarrow 0$ as $x \downarrow 0$.

Let $R_0^\varepsilon = 0$ and for $k \geqq 0$

$$R_{k+1}^\varepsilon = \inf\{t > R_k^\varepsilon : V^0(t) - V^0(R_k^\varepsilon) \leqq -\varepsilon\} .$$

Since V^0 has independent increments $R_{k+1}^\varepsilon - R_k^\varepsilon$, $k \geqq 0$ are independent and identically distributed. Since $P\{R_1^\varepsilon \leqq t\} = P^\varepsilon\{T_0 \leqq t\} \to 1$ as $t \to \infty$, each $R_k^\varepsilon < \infty$ P^0 almost surely.

Let $K_\varepsilon = \inf\{k \geqq 1 : R_k^\varepsilon - R_{k-1}^\varepsilon > 1\}$. From (iii), $P\{R_1^\varepsilon > 1\} = P^\varepsilon\{T_0 > 1\} > 0$ so K_ε and $S_\varepsilon = R_{K_\varepsilon - 1}$ are finite P^0 almost surely. Let $U^\varepsilon(t) = \varepsilon + [V^0(S^\varepsilon + t) - V^0(S^\varepsilon)]$. Since V^0 has independent increments it follows from the construction that $U^\varepsilon =_d (V^\varepsilon \mid T_0 > 1)$ (see Lemma 2 of Section 4.3 for a detailed proof of a similar result). To show that $(V^\varepsilon \mid T_0 > 1)$ converges weakly as $\varepsilon \downarrow 0$ we will show S^ε and U^ε converge P^0 almost surely.

Let $m(t) = \inf_{0\leqq s\leqq t} V^0(s)$. Let $S = \inf\{t : m(t) = m(t+1)\}$. Since we have assumed V^0 is not decreasing $P\{V^0(t) = m(t)\} < 1$ for some $t > 0$. From the scaling relationship it follows that $P\{V^0(t) = m(t)\} < 1$ is independent of t and $V^0(t) - m(t) =_d t^{1/\alpha}[V^0(1) - m(1)]$. From this it follows easily that $P^0\{S < \infty\} = 1$.

LEMMA 1. $\liminf_{\varepsilon\downarrow 0} S^\varepsilon \geqq S$, P^0 *almost surely.*

PROOF. Suppose $S^{\varepsilon_m} \to t < \infty$. By choosing a subsequence we can guarantee that either $S^{\varepsilon_m} \geqq t$ for all m or $S^{\varepsilon_m} < t$ for all m. If $S^{\varepsilon_m} \downarrow t$, it follows that $m(t) = m(t+1)$ so $S \leqq t$.

To prove $S \leqq t$ in the second case observe that if $h > 0$ and $t - S^{\varepsilon m} < h$

$$-\varepsilon_m \leqq \inf_{0\leqq s\leqq 1} V^0(S^{\varepsilon m} + s) - V^0(S^{\varepsilon m}) \leqq [\inf_{0\leqq s\leqq 1-h} V^0(t + s)] - V^0(S^{\varepsilon m})$$
$$\to [\inf_{0\leqq s\leqq 1-h} V^0(t + s)] - V^0(t-)$$

so $m(t) = m(t + 1 - h)$ for all $h > 0$.

To conclude $m(t) = m(t + 1)$ it suffices to show $V^0(t + 1) = V^0((t + 1)-)$. To do this we observe $\max_{1\leqq m\leqq n}(S^{\varepsilon m} + 1)$ is an increasing sequence of stopping times which are less than $t + 1$ so the desired conclusion follows from the "quasi left continuity" of V (see [22], page 45 and Exercise I.9.14).

LEMMA 2. $\limsup_{\varepsilon\downarrow 0} S^\varepsilon \leqq S$, P^0 *almost surely.*

PROOF. Let $X = V^0(S)$. The first step is to show $V^0(S) = V^0(S-)$ P^0 almost surely. To do this we observe:

(a) if Q is a stopping time and $P^0\{T_0^- = 0\} = 1$ then $P\{\inf_{Q\leqq s\leqq Q+1} f(s) = f(Q)\} = 0$ so $P\{S = Q\} = 0$.

(b) If we let $Q_{a,b} = \inf\{t > 0: V(t) - V(t-) \in (a, b)\}$ we will have $P(S = Q_{a,b}) = 0$ for all rational a, b so $P(V^0(S) \neq V^0(S-)) = 0$.

Now R^ε_{k+1} is the first time $m(t) - m(R_k^\varepsilon) \leqq -\varepsilon$; so we have for all ε there is a K_ε' so that $V^0(R^\varepsilon_{K_\varepsilon'}) \in [X - \varepsilon, X]$. Since $K_\varepsilon \leqq K_\varepsilon'$ this shows $\limsup_{\varepsilon\downarrow 0} S^\varepsilon \leqq S$ P^0 almost surely.

Having shown $S^\varepsilon \to S$, to show $U^\varepsilon \to U = V^0(S + t)$, it suffices to show $V^0(S) = V^0(S-)$ and $V^0(S + 1) = V^0((S + 1)-)$ hold a.s. The first equality follows from the proof above, the second from the independence of increments.

REMARK. Although this completes the proof of the conditioned limit theorem in the case $G(0) \in (0, 1)$, our solution is still somewhat incomplete because we have not given the distribution of the limit. If V is Brownian motion the formulas can be found in [26]. If V is a stable process, however, the distribution of the limit is known only in one special case (see Section 4.5).

4.3. *Birth and death processes.* We will call an integer valued Markov process $\{U(t), t \geqq 0\}$ a birth and death process if starting from state j, U remains there for a random length of time having an exponential distribution with mean $(\lambda_j + u_j)^{-1}$; and upon leaving j, U moves to states $j - 1$ and $j + 1$ with probabilities $u_j(\lambda_j + u_j)^{-1}$ and $\lambda_j(\lambda_j + u_j)^{-1}$ respectively.

It is easy to see that if a birth and death process satisfies (ii) then the limit is a strong Markov process with continuous paths, or a diffusion. In [41], Stone has identified which diffusions can occur as limits in (ii) and given necessary and sufficient conditions for the convergence of birth and death processes to these limits.

As the reader can imagine these conditions are different when the state space of the limit process is $(-\infty, \infty)$ and $[0, \infty)$ and in the latter case also depend upon the nature of the boundary at 0. To keep things simple we will give the

results first in the case where the state space is $(-\infty, \infty)$ and the diffusion is regular, and then consider the other possibilities.

DEFINITION. Let $\tau_x = \inf\{t \geqq 0: V(t) = x\}$. A diffusion V with state space $(-\infty, \infty)$ is regular if $P^x\{\tau_y < \infty\} > 0$ for all x, y.

THEOREM 1 ([41], pages 51–58). *A necessary and sufficient condition that there exists a strictly increasing sequence c_n, such that as $n \to \infty$ $U(n\cdot)/c_n$ converges (in the sense of* (ii)) *to a regular diffusion on $(-\infty, \infty)$, is that the sequence defined by $\pi_n = \pi_{n-1}\lambda_{n-1}/u_n$, $\pi_1 = 1$ satisfy $(\lambda_n\pi_n)^{-1} = n^{\alpha_1-1}L_1(n)$ and $\pi_n = n^{\alpha_2-1}L_2(n)$ where the $\alpha_i > 0$ and the L_i have $\lim_{y\to\infty} L_i(xy)/L_i(y) = 1$ for all $x > 0$ and $\lim_{x\to\infty} L_i(-x)/L_i(x) = d_i \in (0, \infty)$.*

In this case $c_n = n^{1/(\alpha_1+\alpha_2)}L(n)$ and the limit process is a diffusion with scale J and speed measure m given by

$$\begin{aligned} J(x) &= Ax^{\alpha_1} && x \geqq 0 \\ &= -d_1A|x|^{\alpha_1} && x < 0 \\ m(x) &= Bx^{\alpha_2} && x \geqq 0 \\ &= -d_2B|x|^{\alpha_2} && x < 0 \end{aligned}$$

where A and B are positive constants.

NOTE. To work with this theorem we will have to use some facts about the speed and scale measures of diffusions. A complete discussion of this topic is given in [38], but very little of the information given there is needed to prove our conditioned limit theorems.

To show that (iii) holds we observe that if $P^y\{T_0 > t\} = 0$ then from (8) of Section 3.1, $V^y(t \wedge T_0)$ is decreasing for each $t > 0$. Since V has continuous paths and the strong Markov property this implies $P^y\{\tau_z < \infty\} = 0$ for $z > y$, which contradicts the assumed regularity.

To prove (iv) we will use (17) of Section 3.1. Since V is regular, $V^0 \not\equiv 0$ and it follows from the scaling relationship that $P\{V^0(1) < 0\} > 0$. To establish that $P^x\{T_0 = t\} = 0$ we recall that Itô and McKean (see Section 4.11 of [38]) have shown that the transition functions of a diffusion have densities with respect to the speed measure so

$$P^x\{T_0 = t\} \leqq P\{V^x(t) = 0\} = 0\,.$$

Since V is regular $P^x\{T_0 > t\} \not\equiv 1$ and from (13) it follows that $\lim_{x\downarrow 0} P^x\{T_0 > t\} = 0$ for all $t > 0$. Since $P\{N > m \mid v_0 = x\}$ is an increasing function of x and (iv) holds using (14) gives that (v) holds.

Having established (i)—(v) we will now prove the conditioned limit theorem by checking the hypothesis of Theorem 3.10. The first two steps are easy. Since $(V_n^+(T_\varepsilon) - \varepsilon)^+ \leqq 1/c_n \to 0$ it is immediate from Theorem 3.7 that V_n^+ is tight for $x_n \to 0$. To get the asymptotic formula for $Q^0\{N > n\}$ we observe that from [40], page 253 we have $Q^0\{N > n\} = n^{-\beta}L(n)$ where $\beta = \alpha_1/(\alpha_1 + \alpha_2)$.

To complete the proof we have to show:

THEOREM 2. *If V is a diffusion which can occur as a limit in Theorem* 1 *then* $\lim_{x \downarrow 0}(V^x \mid T_0 > 1)$ *exists.*

PROOF. Suppose V is defined on a probability space with σ-fields $\mathscr{F}_t = \sigma\{V(s): s \leqq t\}$ and shift operators $\{\theta_t; t \geqq 0\}$. Let $S_\varepsilon = \inf\{s > 0: V(s) = \varepsilon, V(u) > 0 \text{ for } s < u \leqq s + 1\}$ and let $Z_\varepsilon(t) = V(S_\varepsilon + t)$.

LEMMA 1. *For $\varepsilon > 0$ and all x $S^\varepsilon < \infty$ P^x almost surely. As $\varepsilon \downarrow 0$, $S_\varepsilon \downarrow S_0$ and $Z_\varepsilon \to Z_0$ P^0 almost surely.*

PROOF. For $\varepsilon > 0$ let $R_\varepsilon^0 = -1$ and $R_\varepsilon^{k+1} = \inf\{t \geqq R_\varepsilon^k + 1: V(t) = \varepsilon\}$. If $y \neq \varepsilon$ then from [39], page 53:

$$P^y\{R_\varepsilon^1 < \infty\} = \lim_{M\to\infty} P^y\{\tau_\varepsilon < \tau_{(y-\varepsilon)M}\} = \lim_{M\to\infty} \frac{J(y) - J((y-\varepsilon)M)}{J(\varepsilon) - J((y-\varepsilon)M)} = 1$$

so using the strong Markov property and induction gives that $P^x\{R_\varepsilon^k < \infty\} = 1$ for all x and k. Now if V has no zero in $[R_\varepsilon^k, R_\varepsilon^k + 1]$ then $S_\varepsilon \leqq R_\varepsilon^k$ so

$$P^x\{S_\varepsilon \leqq R_\varepsilon^k \mid S_\varepsilon > R_\varepsilon^{k-1}\} \geqq P^\varepsilon\{T_0 > 1\} > 0$$

and hence

$$P^x\{S_\varepsilon < \infty\} = 1\,.$$

For $0 \leqq \delta < \varepsilon$, $S_\delta \leqq \sup\{t < S_\varepsilon: V(t) = \delta\}$ so $S_\varepsilon \downarrow$ as $\varepsilon \downarrow$. To see that $S_\varepsilon \downarrow S_0$ note that

$$\rho = \inf\{t - (S_0 + 1): t > S_0, V(t) = 0\} > 0$$

so $S_{V(S_0+\lambda\rho)} - S_0 \leqq \lambda\rho$ for all $0 < \lambda < 1$ and the result follows by letting $\lambda \downarrow 0$. Since V has continuous paths and $Z_\varepsilon(t) = V(S_\varepsilon + t)$, $S_\varepsilon \downarrow S_0$ implies $Z_\varepsilon \to Z_0$.

Having proven Lemma 1 to complete the proof of Theorem 2 it suffices to show:

LEMMA 2. *For $\varepsilon > 0$, Z_ε and $(V^\varepsilon \mid T_0 > 1)$ have the same distribution.*

PROOF. Let F be a Borel subset of D. Clearly,

$$P\{Z_\varepsilon \in F\} = P\{Z_\varepsilon \in F, S_\varepsilon = \tau_\varepsilon\} + P\{Z_\varepsilon \in F, S_\varepsilon > \tau_\varepsilon\}\,. \tag{1}$$

(Note: when P is written without a superscript the indicated probability is independent of the initial distribution.) Since τ_ε is a stopping time and V a strong Markov process,

$$P\{Z_\varepsilon \in F, S_\varepsilon = \tau_\varepsilon\} = E[P(Z_\varepsilon \in F, S_\varepsilon = \tau_\varepsilon \mid \mathscr{F}_{\tau_\varepsilon})] = P\{V^\varepsilon \in F, T_0 > 1\}\,. \tag{2}$$

If $S_\varepsilon > \tau_\varepsilon$ then $V(s) = 0$ for some $s \in (\tau_\varepsilon, \tau_\varepsilon + 1]$. Letting $\tau_\varepsilon' = \inf\{s: s \in (\tau_\varepsilon, \tau_\varepsilon + 1], V(s) = 0\}$ where $\tau_\varepsilon' = \infty$ if the last set is empty, we have

$$P\{Z_\varepsilon \in F, S_\varepsilon > \tau_\varepsilon\} = P\{Z_\varepsilon \in F, \tau_\varepsilon' < \infty\} = E[\tau_\varepsilon' < \infty; E(1_{\{Z_\varepsilon \in F\}} \mid \mathscr{F}_{\tau_\varepsilon'})]\,. \tag{3}$$

On the set $\{\tau_\varepsilon' < \infty\}$, $1_{\{Z_\varepsilon \in F\}}$ can be written as $\varphi(\theta_{\tau_\varepsilon'})$ so from (3) and the strong Markov property we get

$$P\{Z_\varepsilon \in F, S_\varepsilon > \tau_\varepsilon\} = (E^0\varphi)P\{\tau_\varepsilon' < \infty\} = P\{Z_\varepsilon \in F\}(1 - P^\varepsilon\{T_0 > 1\})\,. \tag{4}$$

Combining (1), (2), and (4) gives

$$P\{Z_\varepsilon \in F\} = P\{V^\varepsilon \in F, T_0 > 1\} + P\{Z_\varepsilon \in F\}(1 - P^\varepsilon\{T_0 > 1\})$$

so

$$P\{Z_\varepsilon \in F\} = P(V^\varepsilon \in F \mid T_0 > 1),$$

which proves Lemma 2.

This completes our development for the "regular" case. The next step is to determine in what other cases we can get a nontrivial conditioned limit theorem.

To do this we observe that from (16) and (18) or Section 3.1 either $P^0\{T_0^- = 0\} = 1$ or $V^0 \geqq 0$ so if $P^0\{T_0^- = 0\} < 1$ there is no loss of generality in assuming the state space is $[0, \infty)$. In Section 3.1 we argued that if 0 was inaccessible from positive levels then the limit theorem is trivial so we will assume $P^x\{T_0 > t\} \not\equiv 1$. In this case (13) of 3.1 implies $\lim_{x \downarrow 0} P^x\{T_0 > t\} = 0$ so (12) of 3.1 gives $P^0\{T_0 = 0\} = 1$. Since $P^0\{T_0^+ = 0\} = 1$ if and only if $P^0\{V^0(1) > 0\} > 0$ there are only boundary possibilities to consider:

(a) reflecting: $P^0\{T_0^+ = 0\} = P^0\{T_0 = 0\} = 1$;
(b) absorbing: $V^0 \equiv 0$.

Conditions for convergence in these cases can be obtained from [41]:

THEOREM 3. *Let* $\{U(t), t \geqq 0\}$ *be a birth and death process with state space* $\{0, 1, 2, \cdots\}$.

If 0 *is a reflecting boundary for* V *then* $U(n\cdot)/c_n \Rightarrow V$ *and* (iv) *holds if and only if the sequence* π *defined in Theorem* 1 *has* $(\lambda_n \pi_n)^{-1} = n^{\alpha_1 - 1} L_1(n)$ *and* $\pi_n = n^{\alpha_2 - 1} L(n)$ *where* α_1 *and* $\alpha_1 + \alpha_2$ *are positive and the* L_i *have* $\lim_{\theta \to \infty} L_i(\theta x)/L_i(\theta) = 1$ *for all* $x > 0$.

If 0 *is an absorbing boundary for* V *and* $\lambda_0 = 0$ *in* U *then* $U(n\cdot)/c_n \Rightarrow V$ *and* (iv) *holds if and only if in addition to the conditions stated above we have*

$$\lim_{x \to \infty} \limsup_{n \to \infty} \int_0^x \frac{u(zc_n)}{u(c_n)}\, v_n(dz) = 0$$

where $v_n(x) = (v(xc_n) - v(c_n))/(v(2c_n) - v(c_n))$, $v(i) = \sum_{j=1}^{i} \pi_j$ *and* $u(i) = \sum_{j=1}^{i} (\lambda_j \pi_j)^{-1}$.

In each case $c_n = n^{1/(\alpha_1 + \alpha_2)} L(n)$ *and there are positive constants* A *and* B *so that the limit process is a diffusion with scale* $J(x) = Ax^{\alpha_1}$ $x \geqq 0$ *and a speed measure* m *concentrated on* $(0, \infty)$ *given by*

$$\begin{aligned} m(x) &= Bx^{\alpha_2} && \text{if } \alpha_2 \neq 0 \\ &= B \log x && \text{if } \alpha_2 = 0 . \end{aligned}$$

If $\alpha_2 > 0$, 0 *is a reflecting boundary. In the other cases* 0 *is absorbing.*

Since Theorem 3 gives conditions for (ii) and (iv) to hold and the arguments above for (iii) and (v) still apply, we have that (i)—(v) hold. From Theorem 3.7, V_n^+ is tight for $x_n \to 0$.

If 0 is a reflecting boundary it is easy to use Theorem 3.10 to show V_n^+ converges: a similar argument works to show $\lim_{\varepsilon \downarrow 0}(V^\varepsilon \mid T_0 > 1)$ exists (we only have to change the proof that $P^x\{S_\varepsilon < \infty\} = 1$) and it follows from [40], page 253 that $Q^0\{N > n\} = n^{-\alpha_1/(\alpha_1+\alpha_2)}L(n)$.

If 0 is an absorbing boundary, however, both of these arguments fail. We leave it to the interested reader to decide whether the conditioned limit theorem will hold in general in this case.

4.4. *The* M/G/1 *queue.* In the M/G/1 queue customers arrive at the jump times of a Poisson process $A(t)$, $t \geqq 0$ with rate λ and have service times which are independent positive random variables with the same distribution.

If ξ_i denotes the amount of service required by the ith customer to arrive after time 0 then $S(t) = \sum_{i=1}^{A(t)} \xi_i$ is the amount of work that has arrived at the facility at time t. If the initial backlog of work is x and the server is not idle at any moment before t then $L(t) = x + S(t) - t$ is the amount of work not completed at time t. If the server has been idle then we have to add to this number the amount of time he has been idle, so the amount of work that remains in general is given by $V(t) = L(t) + (\min_{0 \leq s \leq t} L(s))^-$.

It is easy to use Donsker's theorem to obtain conditions for V to satisfy (ii).

THEOREM 1. *Suppose* $\lambda E\xi_1 = 1$ *and* $E(\xi_1 - 1/\lambda)^2 = \sigma^2 \in (0, \infty)$. *If* $x_n \to x \geqq 0$ *then* $(V(n\cdot)/\sigma n^{\frac{1}{2}} \mid V(0) = x_n\sigma_n^{\frac{1}{2}})$ *converges to* $(\bar{B} \mid \bar{B}(0) = x)$ *where* $\bar{B}$ *is the reflecting Brownian motion.*

PROOF. $S(t)$ is the sum of a Poisson number of independent random variables with mean $E\xi_1$, so from [20] Theorem 17.2 $(S(n\cdot) - \lambda(E\xi_1)nt)/\sigma n^{\frac{1}{2}}$ converges to a Brownian motion B. From this it follows that if $x_n \to x \geqq 0$ $(L(n\cdot)/\sigma n^{\frac{1}{2}} \mid L(0) = x_n\sigma n^{\frac{1}{2}})$ converges to $(B \mid B(0) = x)$, and the desired conclusion now follows from the continuous mapping theorem.

Since the limit in Theorem 1 is reflecting Brownian motion (iii) holds. To see that (iv) and (v) are satisfied we observe that if $x_n \to x \geqq 0$ and $t_n \to t > 0$

$$\begin{aligned} P(\inf_{0 \leq s \leq t_n} V(ns) > 0 \mid V(0) = x_n\sigma n^{\frac{1}{2}}) &= P(\inf_{0 \leq s \leq t_n} L(ns) > 0 \mid L(0) = x_n\sigma n^{\frac{1}{2}}) \\ &\to P(\inf_{0 \leq s \leq t} B(s) > 0 \mid B(0) = x) \\ &= P(\inf_{0 \leq s \leq t} \bar{B}(s) > 0 \mid \bar{B}(0) = x)\,. \end{aligned}$$

Having verified (i)—(v) the next step is to compute the asymptotic formula for the probability of the conditioning event. To do this we will study the Laplace transform of

$$T_0 = \inf\{t > 0 \colon U(t) = 0\}\,.$$

Let $\varphi_x(\alpha) = E(e^{-\alpha T_0} \mid U(0) = x)$ for $x \geqq 0$. Since the arrivals form a Poisson process we have

$$\varphi_{x+y}(\alpha) = \varphi_x(\alpha)\varphi_y(\alpha) \tag{1}$$

and also that

$$\varphi_x(\alpha) = e^{-\alpha x} \int_{[0,\infty)} \varphi_y(\alpha) P(S(x) \in dy)\,. \tag{2}$$

From (1) it follows that there is a number $\eta(\alpha)$ so that $\varphi_x(\alpha) = e^{-x\eta(\alpha)}$. Using this fact in (2) gives

$$e^{-x\eta(\alpha)} = e^{-\alpha x}E(e^{-\eta(\alpha)S(x)}) . \tag{3}$$

Now if $\theta(\beta) = E(e^{-\beta\xi_1})$ then $E(e^{-\beta S(x)}) = e^{-\lambda x(1-\theta(\beta))}$ so (3) may be written as

$$-x\eta(\alpha) = -\alpha x - \lambda x(1 - \theta(\eta(\alpha)))$$

or

$$\eta(\alpha) = \alpha + \lambda - \lambda\theta(\eta(\alpha)) . \tag{4}$$

If H is the distribution of ξ_1, Takács ([46], pages 47–49) has shown that equation (4) has a unique positive solution given by

$$\eta(\alpha) = \alpha + \lambda\left[1 - \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{j!} e^{-(\lambda+\alpha)x}x^{j-1}H^{j*}(dx)\right] \tag{5}$$

where H^{j*} denotes the j-fold convolution of H.

Writing $\gamma(\alpha)$ for the sum in (5) we have

$$\varphi_x(\alpha) = e^{-x\eta(\alpha)} = \exp[-x(\alpha + \lambda(1 - \gamma(\alpha)))] .$$

Brody ([43], page 78) has shown that if $E\xi_1^2 = \mu_2 < \infty$ then

$$1 - \gamma(\alpha) \sim (2/\mu_2)^{\frac{1}{2}}\alpha^{\frac{1}{2}} \qquad \text{as} \quad \alpha \downarrow 0 ,$$

so

$$1 - \varphi_x(\alpha) \sim x\lambda(2/\mu_2)^{\frac{1}{2}}\alpha^{\frac{1}{2}} \qquad \text{as} \quad \alpha \downarrow 0 .$$

Using a result of Dynkin ([44], page 179) now shows that

$$P(T_0 > t \mid L(0) = x) \sim x\lambda(2/\pi\mu_2)^{\frac{1}{2}}t^{-\frac{1}{2}} \qquad \text{as} \quad t \uparrow \infty .$$

At this point we are ready to use Theorem 3.10 to prove the conditioned limit theorem. From results in 4.1 or 4.3 we have that $\lim_{x\downarrow 0}(V^x \mid T_0 > 1)$ exists so it remains to show that the sequence V_n^+ is tight. To do this we will imitate the proof given in Section 4.2.

Let $J_h^n = \inf\{j \geqq 1 : \xi_j > h\sigma n^{\frac{1}{2}}\}$.

$$\begin{aligned} Q^a\{T_0 > n, J_h^n < \infty\} &\leqq \sum_{k=1}^{\infty} \int_0^1 Q^a\left(T_0 > ns, J_h^n = k, \frac{A^{-1}(k)}{n} \in ds\right) \\ &= \sum_{k=1}^{\infty} \int_0^1 Q^a(T_0 > ns \mid A^{-1}(k) = ns, J_h^n = k) \\ &\qquad \times P(J_h^n = k \mid A^{-1}(k) = ns)P\left(\frac{A^{-1}(k)}{n} \in ds\right) \\ &\leqq \sum_{k=1}^{\infty} \int_0^1 Q^a(T_0 > ns \mid A^{-1}(k) = ns) \\ &\qquad \times P\{\xi_1 \leqq h\sigma n^{\frac{1}{2}}\}^{k-1}P\{\xi_1 > h\sigma n^{\frac{1}{2}}\} \frac{(ns)^{k-1}e^{-ns/\lambda}}{k - 1!\,\lambda^k} ds . \end{aligned}$$

Since $Q^a(T_0 > ns \mid A^{-1}(k) = ns) \leqq Q^a(T_0 > ns \mid A(ns) = k - 1)$ the last expression above is

$$\begin{aligned} &\leqq \lambda^{-1}P\{\xi_1 > h\sigma n^{\frac{1}{2}}\} \sum_{k=1}^{\infty} \int_0^1 Q^a(T_0 > ns \mid A(ns) = k - 1)e^{-ns/\lambda} \frac{(ns/\lambda)^{k-1}}{k - 1!} ds \\ &= \lambda^{-1}P\{\xi_1 > h\sigma n^{\frac{1}{2}}\} \int_0^1 Q^a\{T_0 > ns\} ds . \end{aligned}$$

Dividing by $Q^a\{T_0 > n\}$ gives

$$Q^a(J_h^n < \infty \mid T_0 > n) \leqq \lambda^{-1} n P\{\xi_1 > h\sigma n^{\frac{1}{2}}\} \frac{\int_0^1 Q^a\{T_0 > ns\}\, ds}{nQ^a\{T_0 > n\}}.$$

Since ξ_1 has finite variance $nP\{\xi_1 > h\sigma n^{\frac{1}{2}}\} \to 0$ as $n \to \infty$. Now $Q^a\{T_0 > n\} = n^{-\frac{1}{2}}L(n)$ so using Karamata's theorem gives

$$\frac{\int_0^1 Q^a\{T_0 > ns\}\, ds}{nQ^a\{T_0 > n\}} \to 2$$

and $Q^a\{J^n < \infty \mid T_0 > n\} \to 0$. Since h was arbitrary it follows from this that $(V_n^+(T_\varepsilon) - \varepsilon)^+ \Rightarrow 0$ for all $\varepsilon > 0$ and so Theorem 3.7 implies that the sequence V_n^+ is tight.

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