

## The critical contact process seen from the right edge

J.T. Cox<sup>1,\*</sup>, R. Durrett<sup>2,\*\*</sup>, R. Schinazi<sup>1,\*</sup>

<sup>1</sup> Department of Mathematics, Syracuse University, Syracuse, NY 13210, USA

<sup>2</sup> Department of Mathematics, Cornell University, Ithaca, NY 14853, USA

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**Summary.** Durrett (1984) proved the existence of an invariant measure for the critical and supercritical contact process seen from the right edge. Galves and Presutti (1987) proved, in the supercritical case, that the invariant measure was unique, and convergence to it held starting in any semi-infinite initial state. We prove the same for the critical contact process. We also prove that the process starting with one particle, conditioned to survive until time  $t$ , converges to the unique invariant measure as  $t \rightarrow \infty$ .

### 1. Introduction

We consider the one dimensional contact process  $\xi_t$ , a Markov process with state space  $\{0, 1\}^Z$ , where  $Z =$  the integers. For  $x \in Z$ ,  $\xi_t(x) = 1$  means that site  $x$  is occupied by a particle at time  $t$ , and  $\xi_t(x) = 0$  means that site  $x$  is vacant at time  $t$ . The contact process evolves according to the following rules:

(i) if  $\xi_t(x) = 1$ , then  $\lim_{s \rightarrow 0} \frac{1}{s} P(\xi_{t+s}(x) = 0 | \xi_t) = 1$ ,

(ii) if  $\xi_t(x) = 0$ , then  $\lim_{s \rightarrow 0} \frac{1}{s} P(\xi_{t+s}(x) = 1 | \xi_t) = \lambda(\xi_t(x-1) + \xi_t(x+1))$ ,

where  $\lambda > 0$  is a parameter. Liggett (1985) and Durrett (1988) are good references for background on the contact process.

For  $\eta \in \{0, 1\}^Z$  let  $\xi_t^\eta$  denote the process with initial state  $\eta$ , and let  $*$  denote the configuration which has a single particle located at the origin. Let

$$|\eta| = \sum_{x \in Z} \eta(x)$$

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be the number of occupied sites and let

$$\tau^* = \inf\{t > 0: |\xi_t^*| = 0\}$$

be the extinction time starting from a single occupied site. There is a critical value  $\lambda_c \in (0, \infty)$  defined by

$$\lambda_c = \inf\{\lambda > 0: P(\tau^* = \infty) > 0\}.$$

Although relatively little is known about the critical case, it has recently been shown (see Bezuidenhout and Grimmett 1989) that the critical contact process dies out, i.e., that  $P(\tau^* = \infty) = 0$  for  $\lambda = \lambda_c$ .

To introduce the process seen from the right edge define

$$\begin{aligned} r(\eta) &= \sup\{x \in \mathbb{Z}: \eta(x) = 1\}, \\ X &= \{\eta \in \{0, 1\}^{\mathbb{Z}}: |\eta| = \infty, r(\eta) < \infty\}, \\ \tilde{X} &= \{\eta \in X: r(\eta) = 0\}, \end{aligned}$$

and let  $S: X \rightarrow \tilde{X}$  be the mapping given by

$$S\eta(x) = \eta(x + r(\eta)).$$

In words,  $r(\eta)$  is the rightmost particle of configuration  $\eta$ , and  $S\eta$  shifts  $\eta$  so that the rightmost particle is at 0. We are interested in the process  $S\xi_t^\eta$ , which is the contact process seen from the right edge.

Durrett (1984) proved, for the critical and supercritical cases, that  $S\xi_t^\eta$  has an invariant measure which concentrates on  $\tilde{X}$ . Schonmann (1987) showed that in the subcritical case  $S\xi_t$  does not admit an invariant measure. In both of these papers the discrete time version of the model, oriented percolation, was considered. Andjel et al. (1989) treat a class of continuous time models which includes the contact process. Galves and Presutti (1987) showed that in the supercritical case the invariant measure is unique, and that for any  $\eta \in \tilde{X}$ ,  $S\eta_t^\eta$  converges in law as  $t \rightarrow \infty$  to this invariant measure. In addition, they proved a central limit theorem type result for the position of the right edge  $r(\eta_t)$ . See Kuczek (1990) for a nice proof of this in the discrete time setting.

Our first result is that the Galves-Presutti result (concerning the process seen from the right edge) holds true in the critical case. Here and in what follows we suppose  $\lambda = \lambda_c$ .

**Theorem 1.** *There is a unique invariant measure  $\tilde{\mu}$  for  $S\xi_t$ . For every  $\eta \in \tilde{X}$ ,  $S\xi_t^\eta$  converges in law to  $\tilde{\mu}$  as  $t \rightarrow \infty$ .*

The proof of this result relies on some ideas from a construction in Galves and Presutti (1987) and an estimate for the tail of  $\tau^*$  due to Durrett (1988):

$$(1.1) \quad \lim_{t \rightarrow \infty} \sqrt{t} P(\tau^* > t) = +\infty.$$

We remark that if  $\tilde{\mu}_\lambda$  denotes the invariant measure for  $S\xi_t$  with parameter  $\lambda$ , then  $\tilde{\mu}_\lambda$  is continuous on  $[\lambda_c, \infty)$ . To see this, first note that  $S\xi_t$  is a Feller process on  $\tilde{X}$ . Next, it follows from Proposition I.2.14 of Liggett (1986) that if  $\tilde{\mu}_\lambda \Rightarrow \tilde{\nu}$  on  $\{0, 1\}^{\mathbb{Z}}$  as  $\lambda \rightarrow \lambda_0 \geq \lambda_c$  and  $\tilde{\nu}(\tilde{X}) = 1$ , then  $\tilde{\nu}$  is invariant for  $S\xi_t$ . Uniqueness then implies  $\tilde{\nu} = \tilde{\mu}_{\lambda_0}$  at  $\lambda_0$ . To show  $\tilde{\nu}(\tilde{X}) = 1$  we use an inequality from Andjel et al. (1989):

$$\tilde{\mu}_\lambda(A_{i,j}) \leq \frac{\lambda - \alpha(\lambda)}{ip_\lambda(j)}.$$

Here  $A_{i,j} = \left\{ \eta \in \tilde{X} : \sum_{x=-i}^0 \eta(x) < j \right\}$ ,  $\alpha(\lambda) \geq 0$  and

$$p_\lambda(j) = (1 - e^{-1})^{j-1} \exp(\lambda - 2\lambda j).$$

Thus  $\tilde{v}(A_{i,j}) \leq \lambda_0 / ip_{\lambda_0}(j)$ , and letting  $i \rightarrow \infty$  and then  $j \rightarrow \infty$  completes the argument.

Our second result is

**Theorem 2.** *Let  $v_t$  be the distribution of  $S_{\xi_t}^*$  conditioned on  $\{\tau^* > t\}$ . Then  $v_t$  converges in law to  $\tilde{\mu}$  as  $t \rightarrow \infty$ .*

Our work gives some new information about the critical contact process *not* seen from the right edge:

**Corollary.** (i) *For  $L < \infty$ ,*

$$\lim_{t \rightarrow \infty} P(|\xi_t^*| \leq L | \tau^* > t) = 0.$$

(ii) *For  $s < \infty$*

$$\lim_{t \rightarrow \infty} \frac{P(\tau^* > t + s)}{P(\tau^* > t)} = 1.$$

In view of (1.1), the last result should not be surprising.

Theorem 1 is proved in Sect. 2, and Theorem 2 is proved in Sect. 3. We complete this section by giving Harris' graphical construction of the contact process. For each  $x, y \in Z$  such that  $|x - y| = 1$  let  $\{S_n^{x,y}, n \geq 1\}$  be the points of a Poisson process with rate  $\lambda$ , and let  $\{U_n^x, n \geq 1\}$  be the points of a Poisson process with rate 1. Assume the Poisson processes are independent of each other, and let  $\mathcal{P}$  be the collection

$$\mathcal{P} = \{ (S_n^{x,y})_{n \geq 1}, (U_n^z)_{n \geq 1}, x, y, z \in Z \}.$$

We call  $\mathcal{P}$  a *percolation substructure*. For  $(x, s)$  and  $(y, t) \in Z \times [0, \infty]$ , with  $s < t$ , say that there is a path from  $(x, s)$  to  $(y, t)$  in  $\mathcal{P}$  if there is a sequence of times  $\{u_i\}$ ,  $u_0 = s < u_1 < \dots < u_n = t$ , and a sequence of sites  $\{z_i\}$ ,  $z_0 = x, z_n = y$ ,  $|z_i - z_{i-1}| = 1$ , such that for  $i = 1, \dots, n$ ,

- (i)  $u_i = S_k^{z_{i-1}, z_i}$  for some  $k$
- (ii) no  $U_k^{z_{i-1}} \in [u_{i-1}, u_i)$ .

For  $\eta \in \{0, 1\}^Z$  and  $s \leq t$  define  $\xi_t^{\eta,s}$  by setting  $\xi_t^{\eta,s}(y) = 1$  if there is a path from  $(x, s)$  to  $(y, t)$  in  $\mathcal{P}$  for some  $x$  such that  $\eta(x) = 1$ , and  $\xi_t^{\eta,s}(y) = 0$  otherwise. Then  $\xi_t^{\eta,0}$  is a version of  $\xi_t$ , and we will use  $\xi_t^\eta$  to denote  $\xi_t^{\eta,0}$ .

## 2. Proof of Theorem 1

We begin by using an idea of Galves and Presutti to construct another version of the contact process on  $\mathcal{P}$ . Define the sequence  $\{T_k, k \geq 0\}$  by  $T_0 = 0$  and

$$T_k = \inf\{t > T_{k-1} : |\xi_t^{*, T_{k-1}}| = 0\}, \quad k \geq 1.$$

For  $\eta \in \tilde{X}$  define the process  $\tilde{\xi}_t^\eta$  by the following prescription: for  $t \in [T_{k-1}, T_k)$  and

$\zeta = \hat{\xi}_{T_{k-1}}^\eta$  let

$$\hat{\xi}_t^\eta = \xi_t^{S_t^{\zeta}, T_{k-1}} + r(\zeta).$$

(For a configuration  $\eta$  and an integer  $x$  define  $\eta + x$  by  $(\eta + x)(y) = \eta(y - x)$ .) In words, at the times  $T_{k-1}$  we shift the configuration  $\zeta = \hat{\xi}_{T_{k-1}}^\eta$  so that its rightmost particle is at the origin, then use  $\mathcal{P}$  to obtain  $S_t^{\zeta}, T_{k-1}$ , and then shift back by  $r(\zeta)$ . Thus  $\hat{\xi}_t^\eta$  is a version of the contact process starting from  $\eta$ , and hence  $S_t^{\hat{\xi}_t^\eta}$  is a version of the contact process seen from the right edge. Note that if  $t \in [T_{k-1}, T_k)$ , and  $|\xi_t^{*, T_{k-1}}| > L$ , then for all  $\eta \in \tilde{X}$ ,  $S_t^{\hat{\xi}_t^\eta}(x) = S_t^{\xi_t^{*, T_{k-1}}}(x)$  for all  $x \in [-L, 0]$ . Finally, for  $i \geq 1$  let  $\tau_i = T_i - T_{i-1}$  and let

$$N(t) = \sum_{i=1}^{\infty} 1_{\{\tau_i \leq t\}}.$$

The  $\{\tau_i\}$  form an i.i.d. sequence, and the distribution of  $\tau_1$  is the same as that of  $\tau^*$ .

Fix  $L < \infty$ ,  $\kappa \in \tilde{X}$ , and define  $C = \{\eta \in \tilde{X} : \eta(x) = \kappa(x), x \in [-L, 0]\}$ . For any  $\eta$  and  $\eta'$  in  $\tilde{X}$

$$|P(S_t^{\hat{\xi}_t^\eta} \in C) - P(S_t^{\hat{\xi}_t^{\eta'}} \in C)| \leq P(\exists x \in [-L, 0], S_t^{\hat{\xi}_t^\eta}(x) \neq S_t^{\hat{\xi}_t^{\eta'}}(x)).$$

Using the coupling property in our construction of  $\hat{\xi}_t^\eta$ , this last expression is no larger than

$$P(|\xi_t^{*, T_{N(t)}}| \leq L).$$

The fact that  $E\tau_1 = \infty$  implies  $t - T_{N(t)}$  converges in probability to  $\infty$ , so the desired result would follow if we could show

$$P(|\xi_s^*| < L | \tau^* > s) \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

The reader can see from Corollary 1 that this is true, but we do not know how to show this directly. Instead, we show convergence to 0 along a subsequence, which is easy to do and good enough to prove Theorem 1. The first step is to observe that if  $P(|\xi_t^*| < L | \tau^* > t) \geq \varepsilon$  too often then  $P(\tau^* > t)$  would go to zero exponentially fast, which would contradict (1.1).

**Lemma 1.** *For any  $L < \infty$  there is a  $\delta = \delta(L) > 0$  such that if  $P(|\xi_t^*| \leq L | \tau^* > t) \geq \varepsilon$  then*

$$\frac{P(\tau^* > t + 1)}{P(\tau^* > t)} \leq 1 - \delta\varepsilon.$$

*Proof.* Since the probability that a particle dies in unit time and is not reinfected by a neighbor is at least  $(1 - e^{-1})e^{-2\lambda_c}$ , it follows that

$$P(\tau^* \leq t + 1 | \tau^* > t, |\xi_t^*| \leq L) \geq [(1 - e^{-1})e^{-2\lambda_c}]^L \equiv \delta.$$

By hypothesis,  $P(|\xi_t^*| \leq L | \tau^* > t) \geq \varepsilon$ , thus

$$P(\tau^* \leq t + 1 | \tau^* > t) \geq P(\tau^* \leq t + 1 | |\xi_t^*| \leq L, \tau^* > t) P(|\xi_t^*| \leq L | \tau^* > t) \geq \delta\varepsilon. \quad \square$$

**Lemma 2.** *For every  $L < \infty$  there exists a sequence of integers  $j_n \nearrow \infty$  such that*

$$\lim_{n \rightarrow \infty} P(|\xi_{j_n}^*| \leq L) = 0.$$

*Proof.* Let  $e(t) = (\log(1 + t))^{-1}$  and let  $f(t) = P(|\xi_{t^*}^*| \leq L | \tau^* > t)$ . Define the deterministic set

$$A = \{s > 0: f(s) > e(s)\}$$

and the random set

$$B = \{r > 0: r - T_{N(r)} \in A\} .$$

Let

$$a(n) = \sup\{k: \exists t_1, \dots, t_k \in A \cap [0, n], t_i - t_{i-1} \geq 1 \text{ for } 2 \leq i \leq k\}$$

and let  $b(n)$  be the number of integers in  $B \cap [0, n]$ .

The first ingredient in the proof is an upper bound on  $a(n)$ . If  $t_1, \dots, t_{a(n)}$  are as in the definition of  $a(n)$ , then

$$P(\tau^* > n) \leq P(\tau^* > t_{a(n)}) \leq \prod_{m=2}^{m=a(n)} P(\tau^* > t_m | \tau^* > t_{m-1}) .$$

Since  $m \rightarrow e(m)$  is decreasing it follows from Lemma 1 that

$$P(\tau^* > n) \leq (1 - \delta e(n))^{a(n)-1} .$$

With this inequality and (1.1) a little algebra shows

$$(2.1) \quad a(n) \leq \frac{K}{e^2(n)}$$

where  $K$  denotes a positive finite constant whose value is unimportant.

The next ingredient is

$$(2.2) \quad EN(n) \leq \frac{2n}{E(\tau_1 \wedge n)} .$$

To see this notice that for any  $\varepsilon > 0$  the sequence  $\tau'_i = \tau_i \wedge (n + \varepsilon)$  has the same number of renewals in  $[0, n]$  as does the  $\tau_i$  sequence. Letting  $T'_n = \tau'_1 + \dots + \tau'_n$  and using Wald's equation at the stopping time  $N(n) + 1$  gives

$$E(N(n) + 1)E(\tau_1 \wedge (n + \varepsilon)) = ET'_{N(n)+1} \leq ET'_{N(n)} + n + \varepsilon \leq 2n + \varepsilon .$$

Rearranging and letting  $\varepsilon \rightarrow 0$  gives (2.2). Since (1.1) implies  $E(\tau_1 \wedge n)/\sqrt{n} \rightarrow \infty$ ,

$$(2.3) \quad EN(n) \leq \sqrt{n}$$

for large  $n$ .

A little thought reveals that  $b(n) \leq (N(n) + 1)a(n)$  a.s., so (2.1) and (2.3) imply

$$(2.4) \quad Eb(n) \leq K \frac{\sqrt{n}}{e^2(n)}$$

for large  $n$ . On the other hand,

$$Eb(n) \geq \frac{n}{2} \min\{P(j \in B): \frac{n}{2} \leq j \leq n\} .$$

By combining this fact with (2.4) we can find integers  $j_n$  such that  $j_n \in [\frac{n}{2}, n]$  and

$$(2.5) \quad P(j_n \in B) \leq \frac{2}{n} Eb(n) \rightarrow 0 .$$

It remains to show that for this sequence  $j_n$ ,  $P(|\hat{\xi}_{j_n}^*, T_{N(j_n)}| \leq L) \rightarrow 0$ . To do this, note that

$$(2.6) \quad P(|\hat{\xi}_{j_n}^*, T_{N(j_n)}| \leq L) \leq P(j_n \in B) + P(j_n \notin B, |\xi_{j_n}^*, T_{N(j_n)}| \leq L).$$

The second term on the right side above equals

$$(2.7) \quad \int_0^{j_n} \sum_{m \geq 0} P(j_n \notin B, N(j_n) = m, |\xi_{j_n}^{*,s}| \leq L, T_m \in ds).$$

Given that  $N(j_n) = m$  and  $T_m = s$ ,  $j_n \notin B$  is just  $j_n - s \notin A$ , and the distribution of  $\xi_{j_n}^{*,s}$  is the same as that of  $\xi_{j_n-s}^*$  conditioned on  $\{\tau^* > j_n - s\}$ . Thus,

$$\begin{aligned} &P(|\xi_{j_n}^{*,s}| \leq L, j_n \notin B | T_m = s, N(j_n) = m) \\ &= 1_{\{j_n-s \notin A\}} P(|\xi_{j_n-s}^*| \leq L | \tau^* > j_n - s) \leq e(j_n - s) \end{aligned}$$

by the definition of  $A$ . This implies that (2.7) is no larger than

$$\int_0^{j_n} \sum_{m \geq 0} e(j_n - s) P(N(j_n) = m, T_m \in ds).$$

Fix  $0 < k < j_n$  and break the above integral into two integrals, one over  $[0, j_n - k]$  and the other over  $[j_n - k, j_n]$ . Since  $e(t)$  is decreasing, we easily bound these two integrals by

$$e(k) + P(j_n - T_{N(j_n)} \leq k)$$

for any positive  $k$ . But  $E(\tau_1) = \infty$ , so  $t - T_{N(t)}$  converges in probability to infinity. Thus for any  $k < \infty$

$$\lim_{n \rightarrow \infty} P(|\xi_{j_n}^*, T_{N(j_n)}| \leq L) \leq e(k).$$

Letting  $k \rightarrow \infty$  completes the proof.  $\square$

*Proof of Theorem 1.* Fix  $L < \infty$ , let  $j_n$  be the sequence constructed in Lemma 2, let  $\kappa \in \tilde{X}$ , and define  $C = \{\eta \in \tilde{X} : \eta(x) = \kappa(x), x \in [-L, 0]\}$ . For any  $\eta$  and  $\eta'$  in  $\tilde{X}$

$$|P(S_{\hat{\xi}_{j_n}^*}^{\eta} \in C) - P(S_{\hat{\xi}_{j_n}^*}^{\eta'} \in C)| \leq P(\exists x \in [-L, 0], S_{\hat{\xi}_{j_n}^*}^{\eta}(x) \neq S_{\hat{\xi}_{j_n}^*}^{\eta'}(x)).$$

Using the coupling property in our construction of  $\hat{\xi}_t$ , this last expression is no larger than

$$P(|\xi_{j_n}^*, T_{N(j_n)}| \leq L).$$

From Lemma 2 it now follows that

$$(2.8) \quad \lim_{n \rightarrow \infty} \sup_{\eta, \eta'} |P(S_{\hat{\xi}_{j_n}^*}^{\eta} \in C) - P(S_{\hat{\xi}_{j_n}^*}^{\eta'} \in C)| = 0.$$

Suppose now that  $\zeta$  and  $\zeta'$  are in  $\tilde{X}$  and  $t > j_n$ . By the Markov property,

$$P(S_{\hat{\xi}_t}^{\zeta} \in C) = \int P(S_{\hat{\xi}_{t-j_n}}^{\zeta} \in d\eta) P(S_{\hat{\xi}_{j_n}}^{\eta} \in C).$$

Since a similar decomposition holds for  $P(S_{\hat{\xi}_t}^{\zeta'} \in C)$ , (2.8) implies

$$(2.9) \quad \lim_{t \rightarrow \infty} \sup_{\zeta, \zeta' \in \tilde{X}} |P(S_{\hat{\xi}_t}^{\zeta} \in C) - P(S_{\hat{\xi}_t}^{\zeta'} \in C)| = 0.$$

Finally, we use the fact that there is an invariant measure  $\tilde{\mu}$  for  $S_{\hat{\xi}_t}$ . Invariance of  $\tilde{\mu}$  and (2.9) imply

$$|P(S_{\hat{\xi}_t}^{\zeta} \in C) - \tilde{\mu}(C)| \rightarrow 0$$

as  $t \rightarrow \infty$ . Since this is true for any cylinder event  $C$  we have proved Theorem 1.  $\square$

### 3. Proof of Theorem 2

Recall that  $\tilde{\mu}$  is the unique invariant measure for  $S\xi_t^*$  that concentrates on  $\tilde{X}$ , and  $\nu_t$  is the law of  $\xi_t^*$  conditioned on  $\{\tau^* > t\}$ . Fix  $\varepsilon > 0$ ,  $L < \infty$ , and  $\kappa \in \tilde{X}$  and let  $C$  be the cylinder event  $C = \{\eta \in \tilde{X} : \eta(x) = \kappa(x) \text{ for all } x \in [-L, 0]\}$ . We will construct an initial configuration  $\eta \in \tilde{X}$  and a version of the contact process  $\eta_t^\eta$  such that

$$(3.1) \quad |P(S\eta_t^\eta \in C) - \nu_t(C)| \leq 2\varepsilon.$$

Since by Theorem 1  $S\eta_t^\eta$  converges in law to  $\tilde{\mu}$  as  $t \rightarrow \infty$ , this is enough to prove Theorem 2.

Our construction begins with a collection  $\{\mathcal{P}_i, i \geq 0\}$  of i.i.d. percolation substructures,

$$\mathcal{P}_i = \{(S_{n,i}^{x,y})_{n \geq 1}, (U_{n,i}^z)_{n \geq 1}, x, y, z \in Z\}.$$

Let  $x_i$  be a decreasing sequence of integers with  $x_0 = 0$ , and let  $k_i$  be any sequence of positive integers such that

$$(3.2) \quad (x_{i-1} - k_{i-1}) - (x_i + k_i) > L.$$

Let  $\xi_t^i$  denote the contact process defined using the percolation substructure  $\mathcal{P}_i$ , starting with a single particle at  $x_i$ . Note that the  $\xi_t^i$  are independent, and each  $S\xi_t^i$  is a version of  $S\xi_t^*$ . Using  $l(\xi)$  to denote the leftmost position of a particle in  $\xi$ , define

$$G_i = \{r(\xi_t^i) < x_i + k_i \text{ and } l(\xi_t^i) > x_i - k_i \text{ for all } t > 0\}$$

and

$$G = \bigcap_{i=0}^{\infty} G_i.$$

On  $G_i$  the process  $\xi_t^i$  lives entirely in the space-time region  $[x_i - k_i, x_i + k_i] \times [0, \infty]$ . Since the critical process dies out we can make  $P(G_i)$  close to one by choosing  $k_i$  large. Thus a simple Borel-Cantelli argument shows that there are sequences  $x_i$  and  $k_i$  satisfying (3.2) such that

$$P(G) > 1 - \varepsilon.$$

Having fixed the  $x_i$  and  $k_i$  we now define a percolation substructure  $\mathcal{P}$  in terms of the  $\mathcal{P}_i$ . For  $x > x_0 - k_0$  let

$$S_n^{x,y} = S_{n,0}^{x,y}, \quad U_n^x = U_{n,0}^x$$

and for  $i \geq 1$  and  $x \in (x_i - k_i, x_{i-1} - k_{i-1}]$  let

$$S_n^{x,y} = S_{n,i}^{x,y}, \quad U_n^x = U_{n,i}^x.$$

We take  $\mathcal{P}$  to be the collection of these Poisson processes. Let  $\eta_t$  denote the contact process constructed using  $\mathcal{P}$  and let  $\eta$  be the configuration with particles located precisely at the points  $x_i$ .

In order to prove (3.1) we compare  $\eta_t^\eta$  with the  $\xi_t^i$ . The idea is that the particles  $x_i$  are so far apart that  $\eta_t^\eta$  is essentially the same as the union of the  $\xi_t^i$ , so  $S\eta_t^\eta$  should be the same (locally) as  $S\xi_t^i$  where  $i$  is the smallest  $i$  such that  $\xi_t^i$  is still alive at time  $t$ .

Thus, the law of  $S\eta_t^\eta$  should be the same as the law of  $S\xi_t^i$  conditioned on  $\xi_t^i$  being alive at time  $t$ , i.e.  $v_t$ . To make this precise define

$$K_t = \inf\{i \geq 0: |\xi_t^i| > 0\}$$

and

$$\zeta_t = S\xi_t^i \quad \text{on} \quad \{K_t = i\}.$$

Observe that on  $G$ , for all  $t$  and  $x$ ,  $\eta_t(x) = 1$  if and only if  $\xi_t^i(x) = 1$  for exactly one  $i$ . Thus, on  $G$ ,  $S\eta_t^\eta(x) = \zeta_t(x)$  for  $-L \leq x \leq 0$ , so

$$(3.3) \quad P(S\eta_t^\eta \in C, G) = P(\zeta_t \in C, G).$$

Now

$$(3.4) \quad P(\zeta_t \in C) = \sum_{i \geq 0} P(K_t = i, S\xi_t^i \in C).$$

Since the  $\xi_t^i$  are constructed using the independent  $\mathcal{P}_i$ ,

$$P(S\xi_t^i \in C | K_t = i) = P(S\xi_t^i \in C | |\xi_t^i| > 0) = v_t(C).$$

This fact and (3.4) imply  $P(\zeta_t \in C) = v_t(C)$ . Using this with (3.3) we obtain

$$|P(S\eta_t^\eta \in C) - v_t(C)| \leq 2P(G^c) \leq 2\varepsilon.$$

This proves (3.1), so we are done.

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