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Complete convergence theorem for a competition model

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Summary. In this paper we consider a hierarchical competition model. Durrett and Swindle have given sufficient conditions for the existence of a nontrivial stationary distribution. Here we show that under a slightly stronger condition, the complete convergence theorem holds and hence there is a unique nontrivial stationary distribution.

1. Introduction

In this paper we consider Markov processes in which the state at time t is ζ_t: $Z^d \to \{0, 1, 2\}$. We think of $0 =$ grass, $1 =$ bushes, and $2 =$ trees, and formulate the evolution as follows: *(i)* 1's and 2's each die (i.e., become 0) at rate 1. *(ii)* 1's (resp. 2's) give birth at rate λ_1 (resp. λ_2). *(iii)* If the birth occurs at x, the offspring is sent to a site chosen at random from $\{y: y-x \in \mathcal{N}\}$, $\mathcal{N} =$ the set of neighbors of 0. *(iv)* If $\zeta_t(y) \geqq \zeta_t(x)$ then the birth is suppressed. That is, trees can give birth onto sites occupied by bushes but not conversely. Since 2's can replace 1's or 0's, it should be clear that $\xi_t = \{y: \zeta_t(y) = 2\}$ is a Markov process. In the terminology of Liggett [10] or Durrett [4], it is the contact process with neighborhood set $\mathcal{N}$.

It is not hard to show that there is a constant $c_{\mathcal{N}}$ so that if $\lambda_2 > c_{\mathcal{N}}$ then 1*'s die out*. That is, if $|\xi_0| = \infty$, then $\zeta_t \Rightarrow \mu_2$ the limit starting from all sites $= 2$. Here $\Rightarrow$ denotes weak convergence, which in this setting is just convergence of finite dimensional distributions. Durrett and Swindle [7] showed that the other alternative can occur.

Theorem 1. *Suppose $\mathcal{N} = \{y: \|y\|_\infty \leqq M\}$ where $\|y\|_\infty = \sup|y_i|$. If $\lambda_1 > \lambda_2^2 > 1$ then coexistence occurs for $M \geqq M_1(\lambda_1, \lambda_2)$. That is, there is a translation invariant stationary distribution μ_{12} that concentrates on the configurations with infinitely many* 1*'s and* 2*'s.*

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When coexistence occurs, the process has four stationary distributions. One is trivial μ_0, the pointmass on the $\equiv 0$ configuration. Two are not interesting: μ_i, $i=1,2$, the limit starting from all sites $=i$. (The limits exist since in this case the process reduces to a one type contact process). The main result of this paper shows that under the hypotheses of Theorem 1, μ_{12} is the only interesting stationary distribution and is the limit starting from any initial configuration with infinitely many 1's and 2's. To state that result we need some notation. Let $\eta_t=\{y:\zeta_t(y)=1\}$. Let $\tau_1=\inf\{t:\eta_t=\emptyset\}$ and $\tau_2=\inf\{t:\xi_t=\emptyset\}$.

Theorem 2. *Suppose $\mathcal{N}=\{y:\|y\|_\infty\leqq M\}$. If $\lambda_1>\lambda_2^2>1$ and $M\geqq M_2(\lambda_1,\lambda_2)$ then*

$$\zeta_t\Rightarrow P(\tau_1<\infty,\tau_2<\infty)\,\mu_0+P(\tau_1=\infty,\tau_2<\infty)\,\mu_1$$
$$+P(\tau_1<\infty,\tau_2=\infty)\,\mu_2+P(\tau_1=\infty,\tau_2=\infty)\,\mu_{12}.$$

The last result, called the "complete convergence theorem", implies that all stationary distributions are convex combinations of μ_0, μ_1, μ_2, and μ_{12}, or less formally, there is only one interesting stationary distribution $-\mu_{12}$. The constant M_2 in Theorem 2 is somewhat larger than M_1, which is enormous. With more work we might be able to take $M_2=M_1$, but the interesting problem is to show that the complete convergence theorem holds whenever there is coexistence.

The first step in proving Theorem 2 is to give a definition of μ_{12}. Let $\zeta_t^{A,B}$ be the process that starts with 1's on A, 2's on B, and 0's on $A^c\cap B^c$. Let ζ_t^{12} denote the special case in which $A=B^c$ and B is random with distribution μ_2, let $\eta_t^{12}=\{y:\zeta_t^{12}(y)=1\}$ and $\xi_t^{12}=\{y:\zeta_t^{12}(y)=2\}$. (In general, we will use η and ξ to denote the set of 1's and 2's in the corresponding ζ.) Since the 2's are in equilibrium and the 1's initially occupy all the other sites it should not be surprising that for all finite sets C and D,

$$t\to P(\eta_t^{12}\cap C=\emptyset,\ \xi_t^{12}\cap D=\emptyset) \tag{2.3}$$

is increasing. Since these probabilities determine the finite dimensional distributions, it follows that $\zeta_t^{12}\Rightarrow\zeta_\infty^{12}$. As in the case of the contact process, ζ_∞^{12} is a translation invariant stationary distribution and coexistence occurs if and only if $P(\zeta_\infty^{12}(x)=i)>0$ for $i=1,2$.

The key to the proof of (2.3) (and to the proof of Theorem 2 itself), is duality. Recall that when the contact process is defined on a graphical representation then by working backwards in time we can define dual processes $\hat{\xi}_s^{A,t}$, $0\leqq s\leqq t$ that have the property

$$\{\hat{\xi}_t^{A,t}\cap B\neq\emptyset\}=\{A\cap\xi_t^B\neq\emptyset\}$$

(If you don't recall this, it will be explained in Sect. 2.) In the same way we can define dual processes $\hat{\eta}_s^{A,B,t}$, $0\leqq s\leqq t$ that have the property

$$\{\hat{\eta}_t^{A,B,t}\cap C\neq\emptyset\}=\{A\cap\eta_t^{C,B}\neq\emptyset\}$$

Notice that here B is the set of sites occupied by 2's at time 0. One way of thinking about this relation is that we first go forward in time to determine the sites occupied by 2's, then work backwards to see if $A\times\{t\}$ can be reached by a path that avoids the set of 2's.

With our two dual processes defined, the proof of the complete convergence theorem follows the same general approach as for the ordinary contact process. For that process one begins with Griffeath's observation that it is enough to show that if ξ_t^A and $\tilde{\xi}_t^B$ are independent

$$P(\xi_t^A \neq \emptyset, \tilde{\xi}_t^B \neq \emptyset, \xi_t^A \cap \tilde{\xi}_t^B = \emptyset) \to 0$$

Then one uses the fact that a supercritical contact process, when viewed on suitable length and time scales, dominates oriented percolation with parameter p close to 1. (See Bezuidenhout and Grimmett [2] or Durrett [5] for more details.) Here, we follow the same outline. In Sect. 2 we construct the process on a graphical representation, define the dual processes, and prove (2.3). In Sect. 3 we reduce the proof of Theorem 2 to showing

(3.4) $$\lim_{r\to\infty} \lim_{s\to\infty} P(\eta_s^{A_1,A_2} \neq \emptyset, \hat{\eta}_r^{B_1,A_2,r+s} \neq \emptyset, \eta_s^{A_1,A_2} \cap \hat{\eta}_r^{B_1,A_2,r+s} = \emptyset) = 0.$$

To prove (3.4) we show in Sect. 4 that the forward and dual one processes, when viewed on suitable length and time scales, dominate oriented percolation with parameter p close to 1, and then in Sect. 5 use some results for oriented percolation to show (3.4). The arguments in Sect. 4 and 5 follow the approach of Durrett and Schonmann [6] but use a new trick to avoid the "restart" argument.

As the reader can tell from the last sketch, much of the paper is an application of "standard techniques". What then is new here? The main theoretical advance is the introduction of the dual process, which allows us to prove the existence of the stationary distribution μ_{12}. The dual is constructed by running the 2's forward in time and then working backwards to see if the 1's can survive in the spaces left by the 2's. Most of the work in this paper is to adapt existing theory to deal with this evolution in a random environment.

2. Construction and duality

We construct the process from a graphical representation that is like the usual one for the contact process but has indices to keep track of the type of particle. To facilitate letting $M \to \infty$ we will scale space by dividing by M. For each site $x \in Z^d/M = \{z/M : z \in Z^d\}$, we have rate one Poisson processes U_x^1 and U_x^2. At time U_x^i we write a δ_i which kills a type i particle at x. Turning to the births, for each ordered pair of neighbors (x, y), with $\|x-y\|_\infty \leq 1$, we have Poisson processes T_{xy}^1 and T_{xy}^2 with rates $\lambda_1/(2M+1)^d$, and $\lambda_2/(2M+1)^d$. At time T_{xy}^i we draw an arrow from y to x to indicate if $\zeta(y)=i$ and $\zeta(x)<i$ a particle of type i will be born at x.

Even though there are infinitely many Poisson processes and hence no first arrival, an idea of Harris [8] allows us to construct the process starting from any initial configuration. Consider a random graph in which x and y are connected if there is a potential birth from x to y or from y to x before time τ. If τ is small enough a simple argument (compare with a branching process) shows that all the components of our random graph are finite. The evolution of each component can be computed separately. In this way we can construct the process up to time τ, and iterating constructs the process for all time.

Our main tool will be two dual processes that generalize the one for the contact process. The "two-dual" $\{\hat{\xi}_s^{A,t}, 0 \leqq s \leqq t\}$ is obtained by asking the question "Is there a 2 in A at time t?" and working backwards in time. The answer to our question is "Yes if there is a 2 at some site in $\hat{\xi}_s^{A,t}$ at time $t-s$", so a little thought reveals that our dual should evolve as follows. If we encounter a δ_2 at x at time $t-s$, we remove x from $\hat{\xi}_s^{A,t}$. If there is a 2-arrow from y to some point in $\hat{\xi}_s^{A,t}$ at time $t-s$ then we add y to $\hat{\xi}_s^{A,t}$. As the verbal description of our dual suggests, it has the property

$$P(\hat{\xi}_t^{A,t} \cap B \neq \emptyset) = P(\xi_t^B \cap A \neq \emptyset) \tag{2.1}$$

The "one-dual" $\{\hat{\eta}_s^{A,B,t}, 0 \leqq s \leqq t\}$ is obtained by asking the question "Is there a 1 in A at time t when we start with B occupied by 2's at time 0?" To answer this question, we first let the 2-process evolve up to time t, starting from B at time 0. Then we work backwards in time to see if there is a 1 in A. The first step is to discard all the points of A that are occupied by 2's at time t. The set of points that remain, A', is the starting point for the one-dual. As before if we encounter a δ_1 at x at time $t-s$ then we remove x from $\hat{\eta}_s^{A,B,t}$. If we encounter a δ_2 at x at time $t-s$ then we remove x if there was a 2 at x just before time $t-s$. Finally, if there is a 1-arrow from y to some point in $\hat{\eta}_s^{A,B,t}$ and y is not occupied by a 2 then we add y to $\hat{\eta}_s^{A,B,t}$. Again it is an immediate consequence of the definition that

$$P(\hat{\eta}_t^{A,B,t} \cap C \neq \emptyset) = P(\eta_t^{C,B} \cap A \neq \emptyset). \tag{2.2}$$

After seeing the last definition, the reader probably has her own question: "What is it good for?" The first answer to this is that it follows us to define the stationary distribution called μ_{12}, in the introduction. To construct it, let ζ_t^{12} be the process starting from

$$\zeta_0^{12}(x) = \begin{cases} 2 & x \in \xi \\ 1 & x \notin \xi \end{cases}$$

where ξ has distribution μ_2. (Two's start in their stationary distribution, one's are filled in between). First we observe that if we use our convention of letting η stand for the process of one's, and ξ stand for the process of two's the weak limit of ζ_t exists if

$$\lim_{t \to \infty} P(\eta_t^{12} \cap B_1 = \emptyset, \xi_t^{12} \cap B_2 = \emptyset) \text{ exists} \tag{2.3}$$

This is sufficient because all the finite dimensional distributions can be computed from the ones just given by applying the inclusion-exclusion formula.

Proof of (2.3). Note that

$$P(\eta_t^{12} \cap B_1 = \emptyset, \xi_t^{12} \cap B_2 = \emptyset) = P(\xi_t^{12} \cap B_2 = \emptyset) - P(\eta_t^{12} \cap B_1 \neq \emptyset, \xi_t^{12} \cap B_2 = \emptyset)$$

Since the 2's start in equilibrium $P(\xi_t^{12} \cap B_2 = \emptyset)$ is independent of t. It suffices then to show that $\rho_t = P(\eta_t^{12} \cap B_1 \neq \emptyset, \xi_t^{12} \cap B_2 = \emptyset)$ is decreasing in t. Let $\zeta_r^{12,-T}$

be the process that at time $-T$ has all sites occupied by two's, and at time 0 one's are filled in on all vacant sites. Let $\hat{\eta}_s^{B_1, t, 2, -T}$ be the one-dual of $\zeta_r^{12, -T}$, starting at time t with B_1 occupied by one's. Define $\rho_s^t(T) = P(\hat{\eta}_s^{B_1, t, 2, -T} \neq \emptyset, \xi_t^{12, -T} \cap B_2 = \emptyset)$. We begin by observing that $\emptyset$ is an absorbing state for the dual so for $s \leqq t$,

$$\rho_t^t(T) \leqq \rho_s^t(T) = \rho_s^s(T + (t-s))$$

The equality follows by constructing $\zeta_r^{12, -T}$ and $\zeta_r^{12, -T-(t-s)}$ on the same graphical representation with their all 2's configuration at the same place. The second step is to notice

$$\rho_t^t(T) \to \rho_t \quad \text{as} \quad T \to \infty. \tag{2.4}$$

To show this, view $\zeta_r^{12, -T}$ for all T on the same graphical representation. Then using the definition of the upper invariant measure for the contact process, it is easy to see that $\xi_0^{12, -T}$ decreases to a limit that has distribution μ_2. With (2.4) established it is easy to see that ρ_t is decreasing in t because for $t \geqq s$

$$\rho_t - \rho_s = \lim_{T \to \infty} \rho_t^t(T) - \lim_{T \to \infty} \rho_s^s(T + (t-s)) \leqq 0$$

and this gives that the limit in (2.3) exists.

3. The easy pieces

In this section we will do the easy parts of the proof of Theorem 2. We can suppose without loss of generality that the process starts from a nonrandom initial configuration with 1's on A_1 and 2's on A_2. Let $\tau_1 = \inf\{t: \eta_t^{A_1, A_2} = \emptyset\}$ and $\tau_2 = \inf\{t: \xi_t^{A_2} = \emptyset\}$. Here and in what follows we omit the superscript A_1 from the two process since the initial configuration of 1's does not affect its evolution. Let η_∞^1 (resp. ξ_∞^2) denote the limits when we start with all sites $=1$ (resp. $=2$). For reasons indicated in the discussion of (2.3), it suffices to show that for B_1 and B_2 nonempty finite subsets of Z^d/M the following three results hold.

$$\lim_{t \to \infty} P(\xi_t^{A_2} \cap B_2 \neq \emptyset) = P(\xi_\infty^2 \cap B_2 \neq \emptyset)\, P(\tau_2 = \infty) \tag{3.1}$$

$$\begin{aligned} \lim_{t \to \infty} P(\eta_t^{A_1, A_2} \cap B_1 \neq \emptyset) &= P(\eta_\infty^1 \cap B_1 \neq \emptyset)\, P(\tau_1 = \infty, \tau_2 < \infty) \\ &\quad + P(\eta_\infty^{12} \cap B_1 \neq \emptyset)\, P(\tau_1 = \infty, \tau_2 = \infty) \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\lim_{t \to \infty} P(\eta_t^{A_1, A_2} \cap B_1 \neq \emptyset, \xi_t^{A_2} \cap B_2 \neq \emptyset) \\ &\quad = P(\eta_\infty^{12} \cap B_1 \neq \emptyset, \xi_\infty^{12} \cap B_2 \neq \emptyset)\, P(\tau_1 = \infty, \tau_2 = \infty) \end{aligned} \tag{3.3}$$

(3.1) follows from the complete convergence theorem for the one-type long range contact process. (See [2] or [5].) We claim that to prove (3.2) and (3.3) it suffices to prove

(3.4) $$\lim_{r\to\infty}\lim_{s\to\infty} P(\eta_s^{A_1,A_2}\neq\emptyset,\ \hat\eta_r^{B_1,A_2,r+s}\neq\emptyset,\ \eta_s^{A_1,A_2}\cap\hat\eta_r^{B_1,A_2,r+s}=\emptyset)=0$$

(3.5) $$\lim_{r\to\infty}\lim_{s\to\infty} P(\xi_{s/2}^{A_2}\neq\emptyset,\ \hat\xi_{r+s/2}^{B_2,r+s}\neq\emptyset,\ \xi_{s/2}^{A_2}\cap\hat\xi_{r+s/2}^{B_2,r+s}=\emptyset)=0$$

(3.6) $$\begin{aligned}\lim_{r\to\infty}\lim_{s\to\infty} & P(\eta_s^{A_1,A_2}\neq\emptyset,\ \hat\eta_r^{B_1,A_2,r+s}\neq\emptyset)\\ &=P(\eta_\infty^1\cap B_1\neq\emptyset)\,P(\tau_1=\infty,\tau_2<\infty)\\ &\quad+P(\eta_\infty^{12}\cap B_1\neq\emptyset)\,P(\tau_1=\infty,\tau_2=\infty)\end{aligned}$$

(3.7) $$\begin{aligned}\lim_{r\to\infty}\lim_{s\to\infty} & P(\eta_s^{A_1,A_2}\neq\emptyset,\ \hat\eta_r^{B_1,A_2,r+s}\neq\emptyset,\ \xi_{s/2}^{A_2}\neq\emptyset,\ \hat\xi_{r+s/2}^{B_2,r+s}\neq\emptyset)\\ &=P(\eta_\infty^{12}\cap B_1\neq\emptyset,\ \xi_\infty^{12}\cap B_2\neq\emptyset)\,P(\tau_1=\infty,\tau_2=\infty).\end{aligned}$$

To check our claim, observe that if $r+s=t$ and $r,\ s\geqq 0$ then the probability in (3.2) can be rewritten as

$$\begin{aligned}P(\eta_t^{A_1,A_2}\cap B_1\neq\emptyset)&=P(\eta_s^{A_1,A_2}\cap\hat\eta_r^{B_1,A_2,r+s}\neq\emptyset)\\ &=P(\eta_s^{A_1,A_2}\neq\emptyset,\ \hat\eta_r^{B_1,A_2,r+s}\neq\emptyset)\\ &\quad-P(\eta_s^{A_1,A_2}\neq\emptyset,\ \hat\eta_r^{B_1,A_2,r+s}\neq\emptyset,\ \eta_s^{A_1,A_2}\cap\hat\eta_r^{B_1,A_2,r+s}=\emptyset)\end{aligned}$$

(3.6) implies that the first term has the desired limit. (3.4) says the second term converges to 0. Similarly, the probability in (3.3) is

$$P(\eta_t^{A_1,A_2}\cap B_1\neq\emptyset,\ \xi_t^{A_2}\cap B_2\neq\emptyset)=P(\eta_s^{A_1,A_2}\cap\hat\eta_r^{B_1,A_2,r+s}\neq\emptyset,\ \xi_{s/2}^{A_2}\cap\hat\xi_{r+s/2}^{B_2,r+s}\neq\emptyset)$$

and

$$\begin{aligned}|P(\eta_s^{A_1,A_2}&\cap\hat\eta_r^{B_1,A_2,r+s}\neq\emptyset,\ \xi_{s/2}^{A_2}\cap\hat\xi_{r+s/2}^{B_2,r+s}\neq\emptyset)\\ &-P(\eta_s^{A_1,A_2}\neq\emptyset,\ \hat\eta_r^{B_1,A_2,r+s}\neq\emptyset,\ \xi_{s/2}^{A_2}\neq\emptyset,\ \hat\xi_{r+s/2}^{B_2,r+s}\neq\emptyset)|\\ \leqq P(\eta_s^{A_1,A_2}&\neq\emptyset,\ \hat\eta_r^{B_1,A_2,r+s}\neq\emptyset,\ \eta_s^{A_1,A_2}\cap\hat\eta_r^{B_1,A_2,r+s}=\emptyset)\\ &+P(\xi_{s/2}^{A_2}\neq\emptyset,\ \hat\xi_{r+s/2}^{B_2,r+s}\neq\emptyset,\ \xi_{s/2}^{A_2}\cap\hat\xi_{r+s/2}^{B_2,r+s}=\emptyset)\end{aligned}$$

(3.4) and (3.5) imply that the right hand side converges to 0, so (3.7) gives (3.3).

We will now show that (3.5)–(3.7) hold. The proof of (3.4) is the hard part and will be done in Sects. 4 and 5.

Proof of (3.5). This is an immediate consequence of the complete convergence theorem for the one-type process

$$
\begin{aligned}
&P(\xi_{s/2}^{A_2} \neq \emptyset, \xi_{r+s/2}^{B_2,r+s} \neq \emptyset, \xi_{s/2}^{A_2} \cap \xi_{r+s/2}^{B_2,r+s} = \emptyset) \\
&\quad = P(\xi_{s/2}^{A_2} \neq \emptyset, \xi_{r+s/2}^{B_2,r+s} \neq \emptyset) - P(\xi_{s/2}^{A_2} \cap \xi_{r+s/2}^{B_2,r+s} \neq \emptyset) \\
&\quad = P(\xi_{r+s/2}^2 \cap B_2 \neq \emptyset)\, P(\xi_{s/2}^{A_2} \neq \emptyset) - P(\xi_{r+s}^{A_2} \cap B_2 \neq \emptyset) \\
&\quad \to P(\xi_\infty^2 \cap B_2 \neq \emptyset)\, P(\tau_2 = \infty) - P(\tau_2 = \infty)\, P(\xi_\infty^2 \cap B_2 \neq \emptyset) = 0
\end{aligned}
$$

as $s \to \infty$.

To prove (3.6) and (3.7), we need three preliminary results.

(3.8) Lemma. *Given $r > 0$ and $\varepsilon > 0$, there is a q that depends on $|B_1|$ but does not depend on A_2 so that the dual $\hat{\eta}_r^{B_1,A_2,r}$, will with probability at least $1-\varepsilon$ not inspect any particle outside $B_1 + [-q, q]^d$.*

Proof. The number of particles we start off with is $|B_1|$. Each time either a one-arrow or a two-arrow points towards the present configuration, we must inspect the two-dual starting from the endpoint of the arrow to see, if it emanates from a two at time 0. Thus, at most W_r particles are to be inspected, where W_r is a branching process with birth rate $\lambda_1 + \lambda_2$ and death rate 0. (To get an upper bound we ignore deaths.) Chebyshev's inequality implies $P(W_r > q) \leqq EW_r/q \leqq \varepsilon$ for q large enough. Starting from B_1, these q particles are all contained in $B_1 + [-q, q]^d$, since arrows have length (in the supremum norm) less than or equal to 1.

(3.9) Lemma. *For q and A_2 fixed, there exists an S such that for $s \geqq S$*

$$P(\xi_s^{A_2}(x) \neq \xi_s^{2,s/2}(x) \quad \textit{for some } x \in B_1 + [-q, q]^d, \tau_2 > s/2) \leqq \varepsilon$$

where $\xi_t^{2,s/2}$ is the process that starts at time $s/2$ with all sites occupied by two's, and we write $\xi_s(x) = 2$ if $x \in \xi_s$, $\xi_s(x) = 0$ otherwise.

Proof. The complete convergence theorem for the contact process implies

$$\xi_s^{A_2} \Rightarrow \delta_\emptyset\, P(\tau_2 < \infty) + \xi_\infty^2\, P(\tau_2 = \infty).$$

Since we can construct the two-process starting from A_2 at time 0 and $\xi_t^{2,s/2}$ on the same graphical representation with $\xi_t^{A_2} \subseteq \xi_t^{2,s/2}$ for all times $t \geqq s/2$, and disjoint pieces of the graphical representation are independent, we get

$$
\begin{aligned}
&P(\xi_s^{A_2}(x) \neq \xi_s^{2,s/2}(x), \tau_2 > s/2) = P(\xi_s^{A_2}(x) = 0, \xi_s^{2,s/2}(x) = 2, \tau_2 > s/2) \\
&\quad = P(\xi_s^{2,s/2}(x) = 2, \tau_2 > s/2) - P(\xi_s^{A_2}(x) = 2) \\
&\quad = P(\xi_s^{2,s/2}(x) = 2)\, P(\tau_2 > s/2) - P(\xi_s^{A_2}(x) = 2) \\
&\quad \to P(\xi_\infty^2(x) = 2)\, P(\tau_2 = \infty) - P(\xi_\infty^2(x) = 2)\, P(\tau_2 = \infty) = 0
\end{aligned}
$$

as s goes to infinity by the complete convergence theorem for the one type process. Thus for s large enough

$$\begin{aligned} &P(\xi_s^{A_2}(x) \neq \xi_s^{2,s/2}(x) \text{ for some } x \in B_1 + [-q,q]^d, \tau_2 > s/2) \\ &\leqq \sum_{x \in B_1 + [-q,q]^d} P(\xi_s^{A_2}(x) \neq \xi_s^{2,s/2}(x), \tau_2 > s/2) \leqq \varepsilon. \end{aligned}$$

The last preliminary we need is the following trivial observation.

(3.10) *For all s large enough,* $P(s/2 \leqq \tau_1 < \infty) \leqq \varepsilon$.

Proof of (3.6). This is just a question of combining the above observations. First

$$|P(\eta_s^{A_1,A_2} \neq \emptyset, \hat{\eta}_r^{B_1,A_2,r+s} \neq \emptyset) - P(\tau_1 > s/2, \hat{\eta}_r^{B_1,A_2,r+s} \neq \emptyset)| \leqq \varepsilon$$

by (3.10), and

$$\begin{aligned} (3.11) \qquad P(\tau_1 > s/2, \hat{\eta}_r^{B_1,A_2,r+s} \neq \emptyset) &= P(\tau_1 > s/2, \tau_2 \leqq s/2, \hat{\eta}_r^{B_1,(\emptyset,s/2),r+s} \neq \emptyset) \\ &\quad + P(\tau_1 > s/2, \tau_2 > s/2, \hat{\eta}_r^{B_1,A_2,r+s} \neq \emptyset) \end{aligned}$$

where as the reader can probably guess, $\hat{\eta}_r^{B_1,(\emptyset,s/2),r+s}$ is the one-dual starting with B_1 occupied at time $r+s$ when there are no 2's at time $s/2$. The first term on the right is easy to deal with

$$P(\tau_1 > s/2, \tau_2 \leqq s/2, \hat{\eta}_r^{B_1,(\emptyset,s/2),r+s} \neq \emptyset) = P(\tau_1 > s/2, \tau_2 \leqq s/2)\, P(\hat{\eta}_r^{B_1,(\emptyset,s/2),r+s} \neq \emptyset)$$

since disjoint pieces of the graphical representation are independent, and

$$P(\hat{\eta}_r^{B_1,(\emptyset,s/2),r+s} \neq \emptyset) = P(\hat{\eta}_r^{B_1,\emptyset,r} \neq \emptyset) = P(\eta_r^1 \cap B_1 \neq \emptyset)$$

where η_r^1 is the set of 1's at time r when initially all sites are occupied by 1's. So letting $s \to \infty$ and then $r \to \infty$

$$P(\tau_1 > s/2, \tau_2 \leqq s/2, \hat{\eta}_r^{B_1,(\emptyset,s/2),r+s} \neq \emptyset) \to P(\tau_1 = \infty, \tau_2 < \infty)\, P(\eta_\infty^1 \cap B_1 \neq \emptyset)$$

As for the second term on the right in (3.11), (3.8) and (3.9) imply that for $s \geqq S$

$$|P(\tau_1 > s/2, \tau_2 > s/2, \hat{\eta}_r^{B_1,A_2,r+s} \neq \emptyset) - P(\tau_1 > s/2, \tau_2 > s/2, \hat{\eta}_r^{B_1,(2,s/2),r+s} \neq \emptyset)| \leqq 3\varepsilon$$

The last step decouples what happens after time $s/2$ from what happened before and we can write

$$\begin{aligned} &P(\tau_1 > s/2, \tau_2 > s/2, \hat{\eta}_r^{B_1,(2,s/2),r+s} \neq \emptyset) \\ &= P(\tau_1 > s/2, \tau_2 > s/2)\, P(\hat{\eta}_r^{B_1,(2,s/2),r+s} \neq \emptyset) \\ &= P(\tau_1 > s/2, \tau_2 > s/2)\, P(\eta_r^{1\,2,-s/2} \cap B_1 \neq \emptyset) \end{aligned}$$

where $\eta_r^{12,-s/2}$ is the set of 1's in the process that starts with all 2's at time $-s/2$ and has 1's filled in at the vacant sites at time 0. Letting $s \to \infty$ and recalling the proof of (2.3) we see that the last quantity

$$\to P(\tau_1 = \infty, \tau_2 = \infty)\, P(\eta_r^{12} \cap B_1 \neq \emptyset)$$

Letting $r \to \infty$ now gives (3.6).

Proof of (3.7). (3.8), (3.9) and (3.10) apply as before to show that for large s

$$\begin{aligned}|P(&\eta_s^{A_1,A_2} \neq \emptyset, \hat{\eta}_r^{B_1,A_2,r+s} \neq \emptyset, \xi_{s/2}^{A_2} \neq \emptyset, \xi_{r+s/2}^{B_2,r+s} \neq \emptyset)\\ &- P(\tau_1 > s/2, \hat{\eta}_r^{B_1,(2,s/2),r+s} \neq \emptyset, \tau_2 > s/2, \xi_{r+s/2}^{B_2,r+s} \neq \emptyset)| \leqq 4\varepsilon\end{aligned}$$

The last step decouples what happens after time $s/2$ from what happened before and we can write

$$\begin{aligned}P(\tau_1 > s/2, &\hat{\eta}_r^{B_1,(2,s/2),r+s} \neq \emptyset, \tau_2 > s/2, \xi_{r+s/2}^{B_2,r+s} \neq \emptyset)\\ &= P(\hat{\eta}_r^{B_1,(2,s/2),r+s} \neq \emptyset, \xi_{r+s/2}^{B_2,r+s} \neq \emptyset)\, P(\tau_1 > s/2, \tau_2 > s/2)\\ &= P(\eta_r^{12,-s/2} \cap B_1 \neq \emptyset, \xi_r^{12,-s/2} \cap B_2 \neq \emptyset)\, P(\tau_1 > s/2, \tau_2 > s/2)\end{aligned}$$

where in the last step we have again rewritten the event in terms of the process $\zeta_r^{12,-s/2}$ that starts with all 2's at time $-s/2$ and has 1's filled in on the vacant sites at time 0. Letting $s \to \infty$, recalling the proof of (2.3), and then letting $r \to \infty$, we get the desired limit

$$P(\eta_\infty^{12} \cap B_1 \neq \emptyset, \xi_\infty^{12} \cap B_2 \neq \emptyset)\, P(\tau_1 = \infty, \tau_2 = \infty).$$

4. The block construction

To prove (3.4) we will show that *(i)* the forward and dual one processes, when viewed on suitable length and time scales, dominate supercritical oriented site percolation and then *(ii)* use some results for percolation to show (3.4). In this section we carry out *(i)*. We begin by reviewing the block construction of Durrett and Swindle [7] for the forward one-process and generalize it to the one-dual. As in that paper we first define mean field (i.e., $M = \infty$) versions of the processes under consideration, show that when viewed on suitable length and time scales they dominate supercritical oriented site percolation, and then use "continuity" to show that the last conclusion holds for the processes of interest.

To define the mean field version X_t of the process of two's ξ_t we recall that the contact process can be thought of as a branching random walk in which two particles that occupy the same site coalesce to one. If we start with a finite number of particles then the probability of a collision before some fixed time goes to 0 as $M \to \infty$ so we define X_t to be the branching random walk in which

(i) Particles die at rate 1.
(ii) Particles give birth at rate λ_2.

(iii) The offspring of a particle at x is sent to a y chosen uniformly from $\{y: \|x-y\|_\infty \leqq 1\}$.

To define the mean field version Y_t of the process of one's η_t we begin by observing that Bramson, Durrett, and Swindle [1] have shown that the upper invariant measure μ_2 for ξ approaches a product measure with density $(\lambda_2-1)/\lambda_2$ as $M \to \infty$. We will therefore define Y_t to be the branching random walk in which

(i) Particles die at rate $1+\lambda_2(\lambda_2-1)/\lambda_2=\lambda_2$. (1's die when they meet δ_1's or a 2 branches onto their site.)
(ii) Particles give birth at rate λ_1/λ_2. (New 1's are born at rate λ_1, but the site to which they are sent is occupied by a 2 with probability $(\lambda_2-1)/\lambda_2$.)
(iii) The offspring of a particle at x is sent to a y chosen uniformly from $\{y: \|x-y\|_\infty \leqq 1\}$.

In the mean field version Z_t for the one-dual process, we will again assume that the two's at each fixed time are distributed as a product measure with density $(\lambda_2-1)/\lambda_2$. The first step is to thin Z_0 by flipping coins with probability $(\lambda_2-1)/\lambda_2$ of heads to remove the sites in Z_0 that are occupied by 2's. We claim that after the initial thinning, Z_t is the same branching process as Y_t. To check *(ii)* and *(iii)* we observe that births occur at rate λ_1 but are suppressed if they land on live two's, so births occur at rate λ_1/λ_2, and in the limit $M \to \infty$ the new particle is displaced from the parent site by an amount uniformly distributed over $[-1,1]^d$. *(i)* is much more subtle. Particles in Z_t die when meeting δ_1's, but also when meeting δ_2's with live two's on the other side. An occupied site clearly encounters δ_1's at rate 1 but somehat surprisingly encounters δ_2's at rate λ_2. To see the second claim note that if (x,r) is not occupied by a 2 then working backwards $S=\inf\{s$: there is a δ_2 at $(x,r-s)\}$ must occur before $T=\inf\{t$: there is a 2-arrow from an occupied site attacking x at time $r-t\}$ and

$$P(S=s<T \mid S<T)=e^{-s}e^{-(\lambda_2-1)s}/(1/\lambda_2)=\lambda_2\, e^{-\lambda_2 s}$$

Since the probability we will find a two on the other side of a δ_2 is δ_2 is $(\lambda_2-1)/\lambda_2$, the total death rate is thus $1+(\lambda_2-1)=\lambda_2$.

Oriented site percolation. Before we proceed to the description of the block construction, we will say a few words about oriented site percolation. For more details, see Durrett [3]. Let

$$\mathscr{L}=\{(m,n)\in Z^2 : m+n \text{ even}\}$$

and draw arrows from (m,n) to $(m-1,n+1)$ and to $(m+1,n+1)$. The site percolation system is constructed from independent random variables $\{\omega(m,n):(m,n)\in\mathscr{L}, n\geqq 0\}$ taking values 1 with probability p and 0 with probability $1-p$. $\omega(m,n)$ indicates whether the site (m,n) is open $(\omega(m,n)=1)$ or closed $(\omega(m,n)=0)$. We write $(m_1,n_1)\to(m_2,n_2)$ if it is possible to get from (m_1,n_1) to (m_2,n_2) following the arrows and only passing through open sites, with (m_1,n_1) and (m_2,n_2) open as well. The cluster containing $(0,0)$ is $C=\{(m,n)\in\mathscr{L}:(0,0)\to(m,n)\}$. We will call the sites in C wet and use $\psi_n^0=\{m:(m,n)\in C\}$ to denote the set of wet sites at time n. Finally, $\Omega_\infty = \{|C| = \infty\} = \{\psi_n^0 \neq \emptyset \text{ for all } n\}$ is the event "percolation occurs" and $p_c = \inf\{p: P_p(\Omega_\infty) > 0\}$ is the "critical probability".

The block construction for the forward process. We will now describe in some detail the block construction for the one-process as done in Durrett and Swindle [7]. We do this not only to save the reader a trip to the library, but also because the proof for the one-dual is almost the same. Let $e_1 = (1, 0, \dots, 0)$ and $T = L^2$. Let $B = (-2L, 2L]^d \times [0, T]$, and for $(m, n) \in \mathscr{L}$, let $\varphi(m, n) = (2mLe_1, nT)$, $B_{m,n} = \varphi(m, n) + B$. We will say that $B_{m,n}$ is good and set $\omega(m, n) = 1$ if certain good events happen in the graphical representation in that box. Note that the boxes $B_{m,n}$ are disjoint. We do this so that the $\omega(m, n)$ will be independent. Let $I = [-L, L]^d$, $I' = [-L+1, L-1]^d$, and for $(m, n) \in \mathscr{L}$ let $I_m = 2mLe_1 + I$, and $I'_m = 2mLe_1 + I'$. We will say (m, n) is occupied if I_m contains N one's at time nT. The good events will be designed so that if (m, n) is occupied and $B_{m,n}$ is good then $(m+1, n+1)$ and $(m-1, n+1)$ will be occuppied.

We begin by proving the corresponding statement for the branching process Y_t with I replaced by I'. The shrinkage by one unit at the edges of I is to leave room for the limit $M \to \infty$. Here and in what follows, X_t^A and Y_t^A will denote the mean field processes starting with A occupied at time 0. The first step is to introduce a truncation designed to make the good events independent. Let μ be chosen to satisfy $\mu > (\lambda_2 - 1)/\lambda_2$, and

$$1 + (1 + \lambda_2 \mu)\mu < \lambda_1(1 - \mu).$$

To see this is possible and to prepare for our use of this condition in the next paragraph, note that $(1 + \lambda_2 \mu) > \lambda_2$ and is an equality when $\mu = (\lambda_2 - 1)/\lambda_2$. Having picked μ the next step is to note

(4.1) *l can be chosen large enough to make the probability that $X_t^{\{0\}}$ survives until time l or reaches a point outside $[-l+1, l-1]^d$ less than μ.*

In view of (4.1) and the choice of μ we will declare a site y to be occupied by a two at time t if the two-dual starting from y at time t does not die out before it reaches the boundary of the space-time box $(y, t) + [-l, l]^d \times [0, -l]$. To make events inside two disjoint boxes independent, let $\bar{Y}_t$ be the modification of Y_t in which

(i) Particles die at rate $1 + \lambda_2 \mu$.
(ii) Particles give birth at rate $(1 - \mu)\lambda_2$.
(iii) The offspring of a particle at x is sent to a y chosen uniformly from $\{y: \|x - y\|_\infty \leqq 1\}$.
(iv) Particles that land outside $(-2L + l + 1, 2L - l - 1)^d$ are killed.

We shrink $(-2L, 2L]^d$ by l so that when we follow two-duals backwards we will stay in $B_{m,n}$. We shrink by 1 more to allow for the limit $M \to \infty$. Let $\tilde{Y}_t$ be $\bar{Y}_t$ modified so that in the first l units of time no particles are added, and every particle that is hit by a δ_1 or a two-arrow is killed. When we are within l time units of the bottom of the box we cannot follow the two-duals backwards for l units of time so we take the pessimistic view that all sites we investigate are occupied by 2's. $E|\tilde{Y}_t|$ is reduced by a factor of $\exp(-(1 + \lambda_2) l)$ in the first l units of time but this deficit can be made up by $\bar{Y}_t$. Let $T' = T - l$. A simple calculation shows

(4.2) *L can be chosen large enough to make* $\inf_{x\in[-L,L]^d} E|\bar{Y}_T^{\{x\}}\cap I_1'|\geqq 4\exp((1+\lambda_2)\,l)$.

Fix ε to make $1-\varepsilon>p_c$, where p_c is the critical value for independent oriented site percolation.

(4.3) *N can be chosen large enough to make* $P(|\tilde{Y}_T^A\cap I_1'|\geqq N)>1-\varepsilon/3$ *for all* $A\subseteq[-L,L]^d$ *with* $|A|\geqq N$.

Sketch of proof. If N is large then with high probability there will be at least $N\exp(-(1+\lambda_2)\,l)/2$ particles alive at time l. In view of (4.2) the expected number of particles in I_1' is at least $2N$. The desired result now follows by computing second moments and using Chebyshev's inequality. See Sect. 3.1 of Durrett and Swindle [7] for more details.

Finally we have to show that the last result holds for the real one-process. Let $\bar{\eta}_t$ be a modification of η_t in which we assume that all sites outside $(-2L,2L]^d$ are always occupied by 2's.

(4.4) *M can be chosen large enough to make* $P(|\bar{\eta}_T^{A,B}\cap I_1|\geqq N, |\bar{\eta}_T^{A,B}\cap I_{-1}|\geqq N)>1-\varepsilon$ *for all A and B disjoint with* $A\subseteq[-L,L]^d$ *and* $N\leqq|A|\leqq 2N$.

Sketch of proof. Without loss of generality we can suppose $B=A^c$. The trick here is to construct the real process from the mean field version. We define a function π_M that maps a uniform distribution on $[-1,1]^d$ to a uniform on $[-1,1]^d\cap Z^d/M$ and has $\|\pi_M(x)-x\|_\infty\leqq 1/M$ for all $x\in[-1,1]^d$. If U_i is the ith displacement in the mean field dual, we let $\pi_M(U_i)$ be the ith displacement in the real process. We have a huge number of particles to keep track of – the one-dual and all the two-duals of particles that branch onto it. However, the number of particles does not depend on M, so it is easy to show that if M is large then with high probability there is no collision (branch onto an occupied site) and all the particles in the real one-dual and the associated two-duals are within one unit of their analogues in the mean-field version.

(4.4) shows that if (m,n) is occupied then with probability at least $1-\varepsilon>p_c$, $(m-1,n+1)$ and $(m+1,n+1)$ will both be occupied. A simple induction argument now shows that the set of occupied $(m,n)\in\mathscr{L}$ dominates the set of wet sites in oriented percolation with parameter $1-\varepsilon$. The details are spelled out in Sect. 3.2 of Durrett and Swindle [7].

The block construction for the one-dual process. We will show that with small changes, the block construction just described can be extended to the one-dual process. We begin, as before, by considering the mean field version Z_t of the one-dual, which after the initial coin flips decimate Z_0 has the same distribution as Y_t. To make the events inside disjoint boxes independent we let $\bar{Z}_t$ be the modification of Z_t in which

(i) Particles die at rate $1+(1+\lambda_2\,\mu)\,\mu$.
(ii) Particles give birth at rate $(1-\mu)\,\lambda_1$.
(iii) The offspring of a particle at x is sent to a y chosen uniformly from $\{y\colon\|x-y\|_\infty\leqq 1\}$.
(iv) Particles that land outside $(-2L+l+1, 2L-l-1)^d$ are killed.

Here μ is the number chosen earlier to satisfy $\mu>(\lambda_2-1)/\lambda_2$ and $1+(1+\lambda_2\,\mu)\,\mu<\lambda_1(1-\mu)$. To explain the birth rate in *(i)*, recall our discussion of the mean field dual Z_t and observe that now 2-arrows from occupied sites occur at rate

$\lambda_2 \mu$. Repeating our previous argument shows that sites occupied by 1's encounter δ_2's at rate $1+\lambda_2 \mu$ and with probability μ we find a 2 on the other side.

Again when we are within l time units of the bottom of the box, we cannot follow the two-duals backwards for l units of time, so we take the pessimistic view that all the sites we investigate are occupied by 2's. This leads us to define $\tilde{Z}_t$, which is $\bar{Z}_t$ modified so that at times $T' \leqq t \leqq T$ no particles are added, and every particle that is hit by a δ_1 or a δ_2 is killed.

The proof of (4.2) generalizes easily to show

(4.5) *L can be chosen large enough to make* $\inf_{x \in [-L, L]^d} E|\tilde{Z}_{T'}^{\{x\}} \cap I'_1| \geqq 4 \exp(2l)$.

To explain the choice of the constant in (4.5), note that if we start the one-dual with N particles, then (4.5) shows that the expected number of particles in I'_1 at time T' is at least $4N \exp(2l)$. The factor $\exp(2l)$ is there because the probability that a site will not be hit by a δ_1 or δ_2 in l units of time is $\exp(-2l)$. Computing second moments and using Chebyshev's inequality leads as before to

(4.6) *N can be chosen large enough to make* $P(|\tilde{Z}_T^A \cap I'_1| \geqq N) > 1 - \varepsilon/3$ *for all* $A \subseteq [-L, L]^d$ *with* $|A| \geqq N$.

Finally we have to show that the analogue of (4.6) holds for the real one-dual. Let $\tilde{\eta}_t^{B,*,T}$ be the modification of the one-dual in which we decide that a site we investigate is occupied by a 2 if the two-dual escapes from the space-time box $B_{0,0}$ and we do not add a site unless there is a δ_2 there before the site is first attacked by a 2-arrow from an occupied site. The continuity argument used to prove (4.4) generalizes easily to show

(4.7) *M can be chosen large enough to make* $P(|\tilde{\eta}_T^{B,*,T} \cap I_1| \geqq N, |\tilde{\eta}_T^{B,*,T} \cap I_{-1}| \geqq N) > 1 - \varepsilon$ *for all* $B \subseteq [-L, L]^d$ *with* $N \leqq |B| \leqq 2N$.

5. Proof of (3.4)

In the last section we showed that the two processes of interest dominate oriented site percolation with p close to 1. The first step in using this to prove (3.4) is to give some results that say when p is close to 1, oriented site percolation, ψ_0^n, survives with high probability and is thick. Proofs of all the facts cited can be found in Durrett [3]. Let ψ_n^Z (resp. ψ_n^x) denote the state at time n when oriented percolation starts with all sites occupied (resp. x occupied).

(5.1) *If* $l_n = \inf \psi_n^0$ *and* $r_n = \sup \psi_n^0$ *then* $\psi_n^0 = \psi_n^Z \cap [l_n, r_n]$ *on* $\{\psi_n^0 \neq \emptyset\}$.

(5.2) *If* $p > p_c$ *then there is a constant* $\alpha(p) > 0$ *so that*

$$r_n/n \to \alpha(p), \qquad l_n/n \to -\alpha(p) \text{ a.s. } \quad \text{on} \quad \Omega_\infty.$$

(5.3) $\{1_{(x \in \psi_{2n}^Z)}, x \in 2Z\} \stackrel{d}{=} \{1_{(\psi_{2n}^x \neq \emptyset)}, x \in 2Z\}$

(5.4) *Let* $\phi(x) = 1_{(\psi_n^x \neq \emptyset \text{ for all } n)}$ *for* $x \in 2Z$. ϕ *is stationary and ergodic with*

$$P_p(\phi(x) = 1) = P_p(\Omega_\infty) \to 1 \quad \text{as} \quad p \to 1.$$

Let $\delta=\varepsilon/100$ and pick ρ so that $P_\rho(\Omega_\infty)>1-\delta/d$. Having fixed ρ choose parameters L, T, and N for the block construction so that the forward and dual one processes dominate oriented percolation with parameter ρ. We call the box $4zL+[-L,L]^d$ a *source* for the block construction if it contains at least N particles. The first phase of our construction is to wait long enough so that with high probability either our processes have died out or have enough particles to make a source for the block construction. The first step is to decide how many particles are "enough".

If there are K particles, then there are at least $K/(4LM)^d$ boxes $4zL+(-2L,2L]^d$, $z\in Z^d$, that contain at least one particle. Let $\bar\zeta_t^A$ be our process starting with 1's on A, 2's everywhere else and modified to always have 2's outside $(-2L,2L]^d$. Let $\bar\eta_t^A$ be the set of sites occupied by 1's. It is easy to see that

$$\inf_{A\subseteq(-2L,2L]^d} P(|\bar\eta_1^A\cap[-L,L]^d|\geqq N)>0 \tag{5.5}$$

and hence K can be chosen large enough so that if we start with K particles at time 0 then with probability $>1-\delta$ there will be a source at time 1.

The next step in setting up our construction is to show that if we wait long enough then our processes will with high probability either be $\emptyset$ or have K particles. We start with the forward one-process. Here and in what follows A_1, A_2, and B_1 are fixed.

(5.6) Lemma. *If δ and K are given there is a σ so that*

$$P(0<|\eta_s^{A_1,A_2}|<K)\leqq\delta \qquad \textit{for } s\geqq\sigma.$$

Proof. Let $\tau_1(C,D)=\inf\{t:\eta_t^{C,D}=\emptyset\}$. It is easy to see that there is a constant $\gamma>0$ so that $P(\gamma_1(C,D)<\infty)\geqq\gamma^K$ whenever $|C|\leqq K$. (Consider the event that all the particles die before they give birth.) Now

$$P(s<\tau_1(A_1,A_2)<\infty)\geqq P(0<|\eta_s^{A_1,A_2}|<K)\,\gamma^K$$

and the left-hand side $\to 0$ as $s\to\infty$, so the desired result follows.

The last argument generalizes easily to show

(5.7) Lemma. *If δ and K are given there is an $S\geqq\sigma$ so that*

$$P(0<|\hat\eta_S^{B_1,\emptyset,t}|<K)\leqq\delta$$
$$P(0<|\hat\eta_S^{B_1,\mu_2,t}|<K)\leqq\delta.$$

We do not worry about other initial configurations of 1's because the proof of (3.6) shows that if t is large then the dual will see a collection of 2's that looks like $\emptyset$ or μ_2. The third and final step in the setup is to localize the last two results. Let $D_Q=\{y:\|y\|_\infty\leqq Q\}$.

(5.8) Lemma. *If S is fixed then we pick $Q=(4R+2)L$ large enough so that*

$$P(\eta_S^{A_1,A_2}\neq\emptyset,|\eta_S^{A_1,A_2}\cap D_Q|<K)\leqq 2\delta$$
$$P(\hat\eta_S^{B_1,\emptyset,t}\neq\emptyset,|\hat\eta_S^{B_1,\emptyset,t}\cap D_Q|<K)\leqq 2\delta$$
$$P(\hat\eta_S^{B_1,\mu_2,t}\neq\emptyset,|\hat\eta_S^{B_1,\mu_2,t}\cap D_Q|<K)\leqq 2\delta.$$

At this point we have shown that if we wait $S+1$ units of time then our processes will with probability at least $1-3\delta$ either *(i)* have died out or *(ii)* have N particles in $4zL+[-L,L]^d$ for some $z\in Z^d$ with $\|z\|_\infty \leqq R$. (5.1)–(5.4) guarantee that the construction will succeed with high probability and that most of the boxes that can be reached from the source will contain N particles. This will enable us to conclude that when the forward and dual one processes do not die out they both have N particles in a large number of boxes. To turn this into an actual intersection of the two processes, we observe that if, as in the proof of (5.5), we let $\bar{\zeta}_t^A$ be the process starting with 1's on A, 2's on A^c, and modified to always have 2's outside $(-2L,2L]^d$ then

(5.9) Lemma. *There is a $\kappa>0$ so that $P(\bar{\eta}_t^A \cap B \neq \emptyset) \geqq \kappa$ for all $t\in[2T,6T]$ and $A, B\subset[-L,L]^d$ with $|A|, |B| \geqq N$.*

Having identified a positive probability of success, our next step is to pick J large enough so that

(5.10) *If $X_1, \dots X_J$ are independent with $P(X_i>0)\geqq\kappa$ then*

$$P(X_1+\dots+X_J>0)>1-\delta$$

(5.11) *If $\phi(x)=1_{(\psi_n^x \neq \emptyset\ for\ all\ n)}$ then*

$$P_\rho\left(\sum_{x=-J}^{J} \phi(2x)<(1-2\delta/d)(2J+1)\right)<\delta/d$$

To prepare for the later use of (5.11) the reader should recall $\delta=\varepsilon/100\leqq 0.01$ so $1-2\delta\geqq 0.98$.

Having finally made all our choices, it is time to prove that if $r\geqq r_0$ and $s\geqq s_0(r)$ (i.e., s_0 may depend on r) then

(5.12) $$P(\eta_s^{A_1,A_2}\neq\emptyset, \hat{\eta}_s^{B_1,A_2,r+s}\neq\emptyset, \eta_s^{A_1,A_2}\cap\hat{\eta}_s^{B_1,A_2,r+s}=\emptyset)<\varepsilon$$

We carry out the argument first for $d=1$. By (5.3) we can pick n_0 so that

(5.13) $$P(l_n\leqq -J-R, r_n\geqq J+R)\geqq 1-2\delta \quad \text{for } n\geqq n_0$$

Let $r_0=S+1+n_0T$. As we remarked after (5.8) at time $S+1$ the one-dual will, with probability $>1-3\delta$, either *(i)* be empty or *(ii)* have N points in $4mL+[-L,L]$ for some integer m with $|m|\leqq R$. Using (5.13), (5.1), (5.3), and (5.11), we see that in case *(ii)*, at time r_0 the dual will with probability $>1-3\delta$ have N points in $4jL+[-L,L]$ for 98% of the $j\in\{-2J,\dots,2J\}$.

Fix $r\geqq r_0$. The reasoning in the last paragraph applies to the forward one process but this time we cannot just set $s_0=S+1+n_0T$. We need to pick $s_0(r)$, which depends on r, large enough to guarantee that for times $t\leqq r$, the one-dual will with high probability see a collection of 2's that looks like $\emptyset$ or μ_2. The proof of (3.6) shows that this is possible.

Given $r\geqq r_0$ and $s\geqq s_0(r)$, pick n_1 even so that $r-(S+1+n_1T)\in[T,3T)$ and pick n_2 even so that $s-(S+1+n_2T)\in[T,3T)$. We have shown that when the forward one process does not die out, then with probability at least $1-6\delta$ there is a set G_1 of integers $j\in\{-J,\dots,j\}$ with $|G_1|\geqq 0.98(2J+1)$ so that at time $S+1+n_2T$ there are N ones in $2jL+[-L,L]$. Likewise, with probability at least $1-7\delta$ there is a set G_2 of integers $j\in\{-J,\dots,J\}$ with $|G_2|\geqq 0.98(2J+1)$

so that at time $S+1+n_1 T$ there are N ones in $2mL+[-L, L]$. (We have an extra probability δ that the one-dual sees a collection of 2's that does not look like $\emptyset$ or μ_2.) Now $|G_1 \cap G_2| \geqq 0.96(2J+1) > J$ so (5.9) and (5.10) guarantee that (5.12) (i.e., (3.4)) holds (recall $\delta = \varepsilon/100$).

The last argument does not work in $d=2$. The block construction makes the processes grow in a strip $Z \times (4kL+[-2L, 2L])$, so if the sources for the forward and dual one processes occur in boxes $4zL+[-L, L]^2$ and $4z'L+[-L, L]^2$ with $z_2 \neq z'_2$ the proof given above does not guarantee an intersection. The remedy is simple though. We pick $n_1, n_2 \geqq 2n_0$ to be multiples of 4 (replacing 3 by 5 in their definition and 6 by 10 in (5.9)). We let the processes grow in the first direction for the first $n_i/2$ steps and then employ a modification of the construction that interchanges the roles of the two coordinates to get growth in the second direction. A straightforward generalization of the arguments above shows that if the forward one process does not die out then with probability $1-6\delta$ at time $S+1+n_2 T$ there are N ones in $4zL+[-L, L]$ for 98% of the $z \in \{-J, \ldots, J\}^2$. (Here we use the fact that $P_\rho(\Omega_\infty) > 1-\delta/d$ and the errors in (5.11) have the form δ/d.) A similar statement holds for the dual one-process and the desired conclusion follows in the same way. The extension to $d>2$ is straightforward. Further details are left to the reader.

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