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Multicolor Particle Systems with Large Threshold and Range

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In this paper we consider the Greenberg–Hastings and cyclic color models. These models exhibit (at least) three different types of behavior. Depending on the number of colors and the size of two parameters called the threshold and range, the Greenberg–Hastings model either dies out, or has equilibria that consist of "debris" or "fire fronts." The phase diagram for the cyclic color models is more complicated. The main result of this paper, Theorem 1, proves that the debris phase exists for both systems.

KEY WORDS: Particle systems; cellular automata; excitable media; cyclic color models; Greenberg–Hastings model.

1. INTRODUCTION

We consider two families of processes in which the state at time t is $\xi_t: \mathbb{Z}^d \to \{0, 1,..., \kappa-1\}$. Before describing the dynamics we need some preliminary definitions. For $1 \leqslant p < \infty$ and $x \in \mathbb{R}^d$, let $\|x\|_p = (|x_1|^p + \cdots + |x_d|^p)^{1/p}$ and let $\|x\|_\infty = \sup_i |x_i|$. In both systems the neighbors of a point x will be the $y \in B_p(x, N) = \{y: \|y-x\|_p \leqslant N\}$. We will primarily be interested in the cases $p = 1$ and $p = \infty$ but $p = 2$ is also interesting and it is just as easy to formulate the results in general.

Example 1. (Greenberg–Hastings Dynamics.) In this model we think of the sites as being occupied by neurons that can be rested (0), excited (1), or in a sequence of recovery states $(2,..., \kappa-1)$. The system evolves according to the following rules:

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a. If x is in state $i > 0$ then after an exponential amount of time with mean 1, it progresses to state $i + 1$. Here and throughout the paper arithmetic involving states is done modulo κ, e.g., $(\kappa - 1) + 1 = 0$.

b. If x is in state 0 then at the times of a rate one Poisson process x looks at its neighbors. If the number of excited neighbors is at least θ, the threshold, then x becomes excited; otherwise nothing happens.

When $\kappa = 2$ and $\theta = 1$ this model is the threshold contact process in which the birth rates and death rates are 1. See Cox and Durrett[2] for more on this process. Durrett and Gray[4] have investigated the case $\kappa = 2$, $\theta = 2$. When $\kappa = 3$ and $\theta = 1$ this is an epidemic model in which the infection, death, and birth rates are all 1. See Durrett and Neuhauser[6] and references therein. The results in Refs. 4 and 6 primarily concern the case $N = p = 1$.

The name for the processes in Example 1 comes from their deterministic counterparts in which time is discrete and

a. If $\xi_n(x) = i > 0$ then $\xi_{n+1}(x) = i + 1$.

b. If $\xi_n(x) = 0$ and at least θ neighbors are 1 then $\xi_{n+1}(x) = 1$, otherwise $\xi_{n+1}(x) = 0$.

These "cellular automata" with $\theta = 1$ and $\kappa = 3$ were introduced by Greenberg and Hastings.[13] See Durrett and Steif[7] and Fisch, Gravener, and Griffeath[10] for recent results and references. We would like to point out that the last three authors initiated the study of "threshold range scaling" of these models that is the primary focus of this investigation.

The second family of examples is simpler to describe but much harder to analyze.

Example 2. (Cyclic Particle Systems.) In this family of models we think of the states as colors and the system evolves as follows:

If x is in state i then at the times of a rate one Poisson process x looks at its neighbors. If at least θ neighbors are in state $i + 1$ then x progresses to state $i + 1$; otherwise nothing happens.

When $\kappa = 2$ and $\theta = 1$ this system is the threshold voter model which has been studied recently by Cox and Durrett.[2] The systems with $\kappa > 2$ and $\theta = 1$ have been studied by Bramson and Griffeath[1] in one dimension. Their deterministic counterparts have been investigated by Fisch[8] in one dimension and Fisch, Gravner, and Griffeath[9] in $d > 1$. The August 1989 *Scientific American* has a popular account of their results.

The basic question concerning these systems is: When do they have nontrivial stationary distributions—that is, one that assigns no mass to absorbing states? In Example 1 there is only one absorbing state $\xi(x) \equiv 0$ but in Example 2 there are a large number of configurations in which no transitions can occur. To give our first answer we need some definitions. When we say "multicolor system" we mean that the result applies to both examples. The constant N appearing in the definition of the neighborhood set is the range of interaction and we let $v_N = |B_p(0, N)|$ be the number of neighbors.

Theorem 1. Consider a multicolor system with threshold $\theta = av_N$. If $a < 1/2\kappa$ and N is large, then there is a nontrivial stationary distribution.

Sketch of proof. We begin with two disclaimers: (i) In the actual proof some definitions are different for technical reasons but the picture is the same. (ii) The reader who finds this sketch confusing can skip it without loss. The proof of Theorem 1 is done in three steps.

Step 1. We define a deterministic discrete time process $\eta_n: \mathbb{Z}^d \to \{0, 1\}$ by $\eta_{n+1}(x) = 1$ if and only if $|\{y \in B_p(x, M): \eta_n(y) = 1\}| \geq \lambda v_M$. We call this process the threshold contact automaton or t.c.a. The first step is to show that if $\lambda < \frac{1}{2}$ and the t.c.a. is 1 on a large ball then the region occupied by 1's has a linearly growing radius. To be precise, if $\lambda < \frac{1}{2}$ then there are constants R_0, M_0, and Δ so that if $M \geq M_0$ and $\eta_0(x) = 1$ for $x \in B_2(0, rM)$ with $r \geq R_0$ then $\eta_1(x) = 1$ for $x \in B_2(0, (r + \Delta)M)$.

Step 2. Let $H_x = [x_1 L, (x_1 + 1)L) \times \cdots \times [x_d L, (x_d + 1)L)$ for $x \in \mathbb{Z}^d$ and call H_x a *house*. The houses are like the sites in the t.c.a. To make the comparison, pick $\lambda < \frac{1}{2}$ and $\rho < 1/\kappa$ so that $\lambda\rho > a$, let $\sigma \in (\rho, 1/\kappa)$, and set $\zeta_n(x) = 1$ if there are at least σL^d sites in the house in each state at time nL. Suppose the range $N = L(M + 1)$ and pick $K \geq R_0 M$. To use the result of Step 1, a collection of houses H_x, $x \in B_2(zK, K)$ is combined to make a *city* for each $z \in \mathbb{Z}^d$. We call a city *occupied* at time n if all its houses have $\zeta_{nT}(x) = 1$.

Suppose for simplicity that $z = 0$ and $T = 0$. Our choices imply that all the sites in all the cities H_x, $x \in B_2(0, K + \Delta M)$ see an above-threshold number of sites of each color. Now (i) the "single-site chain" that has state space $\{0, 1,..., \kappa - 1\}$ and makes transitions from i to $i + 1$ at rate 1 has a stationary distribution that assigns mass $1/\kappa$ to each state, and (ii) as long as all the sites in a house see all colors above threshold they flip independently so the empirical distribution of the sites will with probability less than $C \exp(-\delta L^d)$ differ by more than εL^d from what we expect.

Step 3. Using Steps 1 and 2, it is straightforward to show that if L

is large then with high probability the occupied ball of houses will expand to $B_2(0, K+\Delta M)$ by time L. Repeating the last construction R_0/Δ times we see that an occupied city will, with high probability, make its neighbors occupied and then a routine comparison with 1-dependent oriented site percolation allows us to construct a nontrivial stationary distribution. □

Remark. The last proof generalizes immediately to hybrids of Examples 1 and 2 in which some updates are automatic (i.e., one progresses to the next state at rate 1) and some are by contact (i.e., only occur if there are enough neighbors of the next color). One can also treat the variant of Example 1 introduced in Ref. 12 in which we view states 1,..., l as excited and transitions $0 \to 1$ occur at rate 1 if more than θ neighbors are excited. In this case the condition becomes $a < l/2\kappa$.

The construction in the proof of Theorem 1 not only allows us to construct a stationary distribution but also allows us to propagate coupling between two stationary distributions and leads to the following uniqueness result. Occupied cities were defined in Step 2 of the sketch of the proof of Theorem 1 and will be defined precisely in Section 2.

Theorem 2A. Consider a multicolor system with threshold $\theta = av_N$. If $a < 1/2\kappa$ and N is large then there is only one extremal stationary distribution in which occupied cities have positive probability.

To turn the last theorem into a uniqueness result one needs process specific arguments to rule out stationary distributions in which occupied cities have zero probability. In this direction we have not been very successful. To state our result for the Greenberg–Hastings model we need to define $h(\xi)$ to be the set of sites that could be 1 at some time $t > 0$ with positive probability.

Theorem 2B. Consider the Greenberg–Hastings model with threshold $\theta = av_N$. (a) If $a < 1/2\kappa$ and N is large then there is a unique stationary distribution concentrated on $\mathscr{H} = \{\xi: h(\xi) = \mathbb{Z}^d\}$. (b) In $d = 1$ if $a < \frac{1}{2}$ then any nontrivial stationary distribution is concentrated on $\mathscr{H}$.

In (b) N does not have to be large and there is no κ. If $\theta > N$ there is nothing to show: The process dies out since an interval of length $> N$ with no 1's will never contain any 1's at later times. In dimensions $d > 1$ we do not know how to rule out the existence of nontrivial stationary distributions concentrated on $\mathscr{H}^c$.

The ideas that are used in the proofs of Theorem 2A and 2B can be applied to systems without letting $N \to \infty$. Consider the following:

Example 3. (Threshold Contact Processes.) In this model $\xi_t: \mathbb{Z}^d \to \{0, 1\}$, we think of 0 as vacant and 1 as occupied, and the system evolves according to the following rules:

a. If x is occupied then after an exponential amount of time with mean 1, it becomes vacant.

b. If x is vacant and at least θ neighbors are occupied then x becomes occupied at rate λ.

If $\lambda = 1$ this is just the two-color Greenberg–Hastings model. The proofs of Theorems 2A and 2B can be combined to show the following:

Theorem 2C. Consider the threshold-2 contact process with neighborhood set $B_p(x, N)$. If we exclude the nearest neighbor case $N = p = 1$ then for large λ there is a nontrivial stationary distribution and the complete convergence theorem holds. That is, if ξ^1_∞ is the limit starting from $\xi^1_0(x) \equiv 1$ and $\tau = \inf\{t: \xi_t \equiv 0\}$ then

$$\xi_t \Rightarrow P(\tau < \infty)\, \delta_0 + P(\tau = \infty)\, \xi^1_\infty$$

The existence of a nontrivial stationary distribution is not new but the convergence is. In the case $d = 1$, $N = 1$ the system dies out since an interval of vacant sites of length ≥ 2 cannot become occupied. When $d \geq 2$ and $N = p = 1$, survival is impossible starting from a finite set. If the initial set of 1's is inside a box $B_\infty(x, R)$ then this will be true for all time. This does not mean that the system dies out, however. Durrett and Gray[4] have shown that if $d \geq 2$ and $N = p = 1$ there is a nontrivial stationary distribution for large λ. The proof of Theorem 2C generalizes to some thresholds $\theta > 2$ but if θ is too large, we run into difficulties like those in the proof of Theorem 2B.

Theorem 1 gives a sufficient condition for the existence of a stationary distribution. The next result shows that in one case the condition is (asymptotically) necessary.

Theorem 3. Consider the two-color Greenberg–Hastings model with threshold $\theta = bv_N$. If $b > \frac{1}{4}$ then the system dies out for large N, i.e., the only stationary distribution is δ_0, the point mass on the "all 0" state.

Remark. The proofs of Theorems 1 and 3 generalize easily to show that for the threshold contact process the cutoff is $b_0 = \lambda/2(\lambda + 1)$. That is, if $\theta = bv_N$ then there is a nontrivial stationary distribution for large N if $b < b_0$ but not if $b > b_0$.

A remarkable aspect of Theorem 3 is that it works very well for small N. If we call the largest threshold for which a stationary distribution exists the cutoff then for the B_∞ neighborhood in $d=2$ we have

N	1	2	3
v_N	9	25	49
cutoff	2	6	12

These data were obtained from computer simulation but one cannot just run the system and see if it dies out or not. When $N=3$ and $\theta=14$ the system will run for at least several thousand time units without deviating significantly from a 50–50 mix of 0's and 1's. To see that the system has no nontrivial stationary distribution in this case, we make the sites in a ball vacant and observe that if the radius of the ball is larger than 10 then the vacant region expands. This shows that the state we are observing is not stable but only metastable.

Our final result attempts to quantify the metastability just mentioned. We begin with a simple observation: If $\theta < v_N/\kappa$, N is large, and we start the κ color Greenberg–Hastings model from a product measure in which each color has density $1/\kappa$ then the probability a site in the initial configuration sees all colors above threshold is at least $1-\exp(-\delta N^d)$. Combining this with parts of the proofs of Theorems 1 and 3 leads easily to the following:

Theorem 4. Consider the two-color Greenberg–Hastings model on a torus $(\mathbb{Z} \bmod L)^d$ starting from product measure with density $\frac{1}{2}$. Suppose the threshold $\theta = bv_N$ where $\frac{1}{4}<b<\frac{1}{2}$ and $(\log L)/N^d \to 0$ as $N\to\infty$. If $\tau_N = \inf\{t: \xi_t \equiv 0\}$ then there are constants δ, $\Delta \in (0,\infty)$ so that

$$P(\exp(\delta N^d) < \tau_N < \exp(\Delta N^d)) \to 1 \qquad \text{as} \quad N\to\infty$$

To prove the upper bound we show that by time $\exp(\Delta N^d)$ a large vacant hole will form and grow to wipe out the system. We conjecture that $(\log \tau_N)/N^d \to \gamma$ in probability as $N\to\infty$. To prove this one has to answer the following questions: (a) What is the optimal size and shape for the hole? (b) How does it form? Of course, you also have to show that the process does not die out by some other mechanism. The condition $(\log L)/N^d \to 0$ is important for the conjecture (and to a lesser extent for Theorem 4) since it guarantees that there are no large holes in the initial configuration and the time for a large hole to grow and wipe out the system can be ignored.

Theorem 3 shows that the condition in Theorem 1 is sharp for the Greenberg–Hastings model when $\kappa=2$. We believe the result is not sharp for large κ in $d>1$. To explain this we introduce the following:

Example 4. (An Epidemic Model.) In this model $\xi_t\colon \mathbb{Z}^d \to \{0, 1, 2\}$, we think of 0 as healthy, 1 as infected, and 2 as removed (dead or immune). The dynamics are like the three-color Greenberg–Hastings model but the transition $2 \to 0$ is not allowed.

a. If x is healthy and at least θ neighbors are infected then x becomes infected at rate 1.

b. If x is infected then after an exponential amount of time with mean 1, it becomes removed.

c. Removed sites remain removed for all time.

The epidemic cannot have a nontrivial stationary distribution so attention focuses on the question of survival, i.e., if we start with all sites in $[-CN, CN]^d$ infected and all other sites healthy, then is there positive probability that the set of infected sites is always nonempty?

Conjecture. Consider the epidemic model with threshold $\theta = a v_N$ in dimension $d > 1$. There is a constant a_e, that depends on p and d, so that (a) if $a < a_e$ and N is large the epidemic survives and the Greenberg–Hastings model has a nontrivial stationary distribution for any $\kappa < \infty$ and (b) if $a > a_e$ and N is large then the epidemic dies out.

The conjecture in (a) is based on results of Durrett and Neuhauser,[6] who showed for the case $\theta = 1$, $d = 2$, $N = p = 1$ that if the epidemic survives and the rate for $2 \to 0$ transition is increased to a positive level δ then there is a nontrivial stationary distribution. Having κ colors is something like having $\delta = 1/(\kappa - 2)$, and hence our conjecture. If our conjecture is correct Theorem 1 is not sharp for large κ.

The general idea of investigating the Greenberg–Hastings model and cyclic systems with threshold $\theta > 1$ and the particular idea to consider the behavior when $N \to \infty$ and $\theta / v_N \to a$ are due to David Griffeath. He has made an extensive study of these two cellular automata when $p = 1, \infty$ and $N \leqslant 6$ and has mapped out the parameter values that lead to different qualitative behaviors. See Fisch, Gravner, and Griffeath.[10] Based on their investigations, we guess that the phase diagram for Example 1 should for large N look like the picture in Figure 1.

The "dies out" region is easiest to define: There is no stationary distribution and the system converges exponentially rapidly to the all rested state. To explain "debris" we note that it follows from Theorem 2A and the proof of Theorem 1 that if $\theta = a v_N$ with $a < 1/2\kappa$ then the unique equilibrium concentrated on $\mathscr{H}$ converges to product measure with density $1/\kappa$ as $N \to \infty$. We define the debris region to be the set of (a, κ) for which the last conclusion holds and the "fire front" region to be the pairs not covered

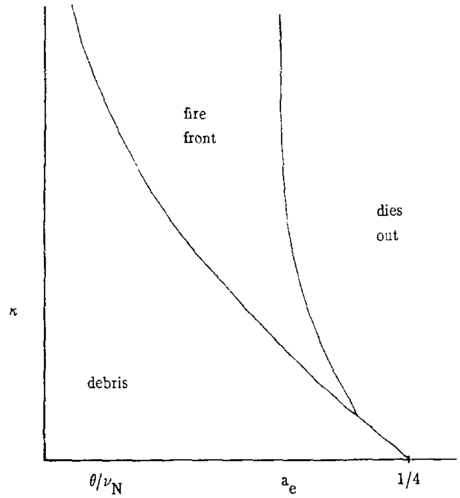
fire
front
dies
out
κ
debris
θ/ν_N
a_e
1/4

Fig. 1.

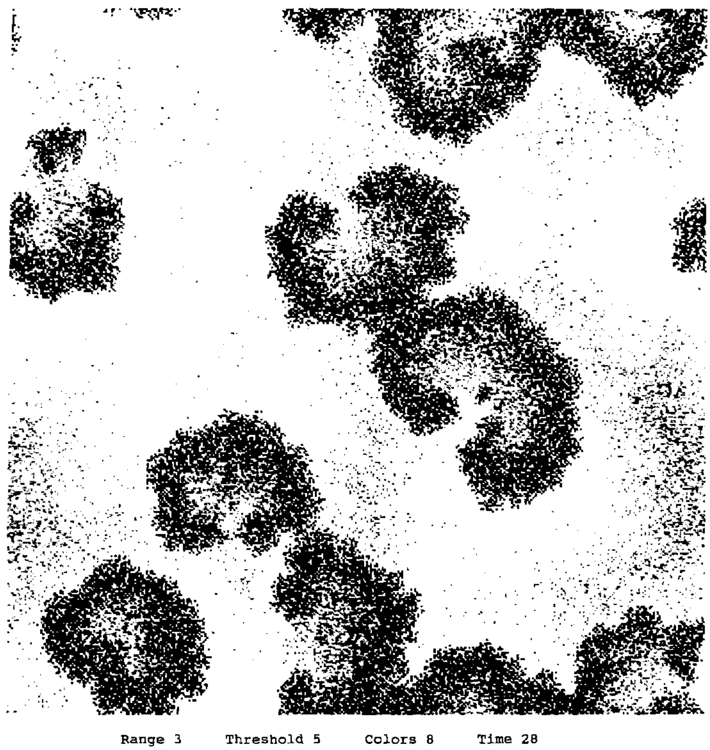
Range 3 Threshold 5 Colors 8 Time 28

Fig. 2.

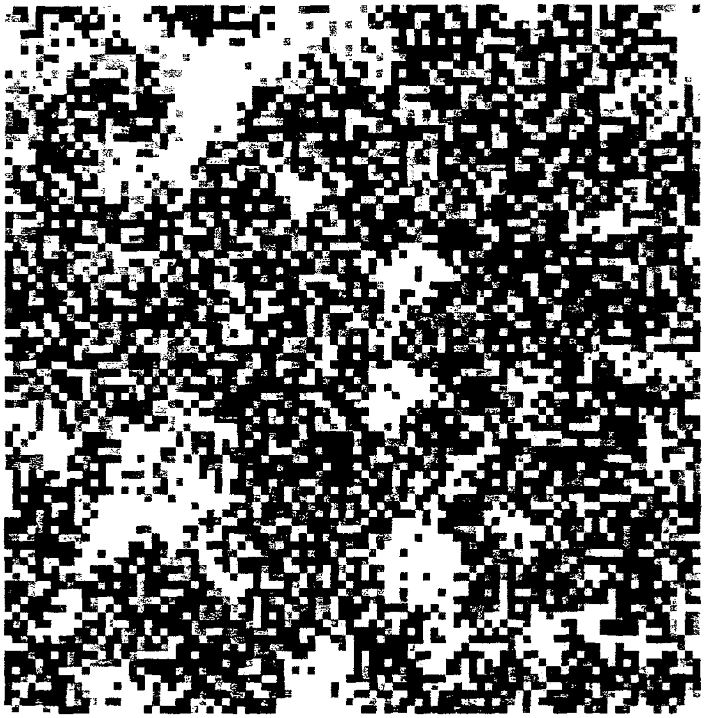

Fig. 3.

by the first two. To explain the name, see Figure 2, which gives a picture of the equilibrium for the threshold 5, eight-color model, with neighborhood set $B_\infty(0, 3)$. Here and in Figure 3, excited sites are black, rested sites are white, and recovering sites are shades of gray.

In accordance with our conjecture we have drawn the boundary of the dying out region to be asymptotic to a_e as $\kappa \to \infty$. There is a portion of the curve where the "dying out" and "debris" phases touch because we conjecture that this occurs when κ is small. As for the boundary of the "debris" region, we believe that it is $1/2\kappa$. In support of this see Figure 3, which gives the equilibrium state for the five-color model with threshold 3 in $d=2$ with the $p=\infty$, $N=2$ neighborhood. We have chosen this particular parameter value because it shows how accurate the $v_N/2\kappa$ formula is when $N=2$. Notice that $v_2=25$ and $25/(2\cdot 5)=2.5$. We have not included a picture of the five-color threshold 2 model because it would look like product measure with density $\frac{1}{5}$.

2. PROOF OF THEOREM 1

The proof is a three-step process. In the first step we introduce the threshold contact automaton and show that if its threshold is λv_N with $\lambda < \frac{1}{2}$ then a large ball grows at a linear rate. The second step is to show that after renormalization a multicolor system with threshold $a v_N$ dominates a t.c.a. with $\lambda < \frac{1}{2}$. In the third and final step, a second renormalization makes the multicolor system dominate supercritical 1-dependent oriented site percolation and standard techniques produce the desired stationary distribution.

Step 1

We define a deterministic discrete time process $\eta_n: \mathbb{Z}^d \to \{0, 1\}$ by $\eta_{n+1}(x) = 1$ if and only if $|\{y \in B_p(x, M): \eta_n(y) = 1\}| \geqslant \lambda v_M$. We call this process the threshold contact automaton or t.c.a. The first step is to show that if the t.c.a. is 1 on a large ball then the region occupied by 1's contains a ball with linearly growing radius.

Lemma 1. Suppose $\lambda < \frac{1}{2}$. There are constants R_0, Δ, and M_0, so that if $M \geqslant M_0$ and $\eta_0(x) = 1$ for $x \in B_2(0, rM)$ with $r \geqslant R_0$ then $\eta_1(x) = 1$ for $x \in B_2(0, (r + \Delta) M)$.

Proof. In one dimension we can take $R_0 = 1$ and $\Delta = 1 - 2\lambda$. Turning to dimensions $d > 1$, let $Q = \{x \in R^d: \|x\|_p \leqslant 1\}$ and let q be its volume. To prove the result it is convenient to scale space by $1/M$ and translate so that x/M sits at the origin. Any $d-1$ dimensional hyperplane through the origin divides Q into two pieces with volume $q/2$. For $i = 1, 2, 3$ let $\lambda < \lambda_3 < \lambda_2 < \lambda_1 < \frac{1}{2}$. By continuity, there is a $\Delta > 0$ so that if a hyperplane passes within a distance Δ of the origin then it divides Q into two pieces each of which has volume at least $q\lambda_1$. Another application of continuity shows that if R_0 is large and $D = B_2(y, r)$ with $r \geqslant R_0$ and $B_2(y, r) \cap B_2(0, \Delta) \neq \varnothing$ then the volume of $D \cap Q$ is at least $q\lambda_2$.

The last step is to argue that if M is large then the lattice behaves like the "continuum limit" considered above. Pick $\varepsilon > 0$ so that if $D = B_2(y, r)$ is as above then $B_2(y, r-\varepsilon) \cap (1-\varepsilon) Q$ is always larger than $q\lambda_3$. Then pick M_0 so that $1/M_0 < \varepsilon$ and if $M \geqslant M_0$ then $|B_p(0, M)|/qM^d < \lambda_3/\lambda$. Let $\mathscr{X} = (Z^d/M) \cap D \cap Q$. The first part of the choice of M_0 implies that if $M \geqslant M_0$ then

$$B_2(y, r-\varepsilon) \cap Q(1-\varepsilon) \subset \bigcup_{x \in \mathscr{X}} x + \left[\frac{-1}{2M}, \frac{1}{2M}\right]^d$$

so

$$M^{-d}\,|\mathscr{X}| \geqslant q\lambda_3 \geqslant \lambda\,|B_p(0, M)|$$

by the second part of the choice of M_0 and the proof of Lemma 1 is complete. □

Step 2

The next step is to show that if the threshold is av_N with $a < 1/2\kappa$ then after renormalization the multicolor system dominates a t.c.a. with $\lambda < \frac{1}{2}$. Pick $\lambda < \frac{1}{2}$ and $\rho < 1/\kappa$ so that $\lambda\rho > a$, use Lemma 1 to pick R_0, Δ, M_0 and then pick $M_1 \geqslant M_0$ so that

$$\lambda\rho\,|B_p(0, M_1)|\,L^d > a\,|B_p(0, L(M_1+1))| \text{ holds for large } L$$

Let $\sigma \in (\rho, 1/\kappa)$ and suppose that the range of interaction in the multicolor model is $N = L(M_1 + 1)$. Here L is an integer that will be chosen below. For $x \in Z^d$ let

$$H_x = [x_1 L, (x_1+1)L) \times \cdots \times [x_d L, (x_d+1)L)$$

We will call H_x a *house* and set $\zeta_n(x) = 1$ if there are at least σL^d sites in the house in each state at time nL.

Lemma 2. Let $\varepsilon > 0$. If L is large and $\zeta_0(x) = 1$ on $B_2(0, rM_1)$ with $r \in [R_0, 2R_0]$ then with probability at least $1-\varepsilon$, $\zeta_1(x) = 1$ on $B_2(0, (r+\Delta)\,M_1)$.

Proof. Let τ be the first time some house H_x with $x \in B_2(0, rM_1)$ has less than ρL^d sites of some color. The choice of R_0 and M_0 implies that for $r \geqslant R_0$ and $M_1 \geqslant M_0$

$$|B_p(x, M_1) \cap B_2(0, rM_1)| \geqslant \lambda\,|B_p(x, M_1)|$$

for all $x \in B_2(0, (r+\Delta)\,M_1)$. Until time τ, each house in $B_2(0, rM_1)$ has at least ρL^d sites in each state, so each site in each house in $B_2(0, (r+\Delta)\,M_1)$ sees at least $\lambda\,|B_p(0, M_1)|\,\rho L^d$ sites of each color and by the choice of M_1 this number is at least $a\,|B_p(0, N)|$ if L is large. The last observation implies that until time τ the sites in each house H_x with $x \in B_2(0, (r+\Delta)\,M_1)$ behave like a system of independent flips in which state i changes to $i+1$ at rate 1. Independent flips converge to a product measure in which each color has density $1/\kappa$ so a simple large deviations estimates will allow us to conclude that (a) $P(\tau \leqslant L) \to 0$ as $L \to \infty$, and

(b) if $x \in B_2(0, (r+\Delta) M_1)$ and L is large then with high probability there are at least σL^d sites of each color in H_x at time L. The large-deviations estimate is the following:

Lemma 3. Let $X_1, \ldots, X_n$ be i.i.d. with $P(X_i = 1) = p$, $P(X_i = 0) = 1 - p$. Then

$$P(X_1 + \cdots + X_n \leqslant n(p - \varepsilon)) \leqslant \exp(-\varepsilon^2 n/2)$$

Remark. This result and its proof are standard but for us it is useful that the right-hand side does not depend on p.

Proof. If $\alpha > 0$ then

$$P(X_1 + \cdots + X_n \leqslant n(p-\varepsilon)) \exp(-\alpha n(p-\varepsilon)) \leqslant (pe^{-\alpha} + (1-p))^n$$

so rearranging and using $\log(1+x) \leqslant x$ we have

$$\begin{aligned} &\frac{1}{n} \log P(X_1 + \cdots + X_n \leqslant n(p-\varepsilon)) \\ &\qquad \leqslant \alpha(p-\varepsilon) + \log(1 + p(e^{-\alpha} - 1)) \\ &\qquad \leqslant \alpha(p-\varepsilon) + p(e^{-\alpha} - 1) = -\alpha\varepsilon + p(e^{-\alpha} - 1 + \alpha) \end{aligned}$$

Now $e^{-\alpha} - 1 + \alpha = \alpha^2/2 - \alpha^3/3! + \cdots \leqslant \alpha^2/2$ for $0 \leqslant \alpha \leqslant 1$, so taking $\alpha = \varepsilon$ and using $p \leqslant 1$ gives

$$P(X_1 + \cdots + X_n \leqslant n(p-\varepsilon)) \leqslant \exp(-\varepsilon^2 n/2)$$

which completes the proof of Lemma 3. □

To approach the proof of (a) and (b), consider a Markov chain Z_t with state space $\{0, 1, \ldots, \kappa - 1\}$ in which state i changes to $i+1$ at rate 1. As in the Introduction, we will call this the "single-site" Markov chain. Let $u_{i,j}(t) = P_i(Z_t = j)$ and observe that $u_{i,j}(t) = u_{0,j-i}(t)$. Let v_i be the number of sites in state i in H_x at time 0. Then the expected number of sites in state j at time t is $w_j(t) = \sum_i v_i u_{i,j}(t)$.

To prove (a) apply Lemma 3 with $n = v_i \geqslant \sigma L^d$ to the sites that start in state i to see that with probability at least $1 - \exp(-\varepsilon^2 \sigma L^d/2)$, at least $v_i(u_{i,j}(t) - \varepsilon)$ of the sites that start in state i will be in state j at time t. Taking $\varepsilon = (\sigma - \rho)$ and summing over i gives

$$\sum_i v_i(u_{i,j}(t) - \varepsilon) \geqslant \sigma L^d \left(\sum_i u_{i,j}(t) \right) - L^d \varepsilon \geqslant \rho L^d$$

since $\sum_i u_{i,j}(t) = \sum_i u_{0,j-i}(t) = 1$. So with probability at least $1 - \kappa \exp(-\varepsilon^2 \sigma L^d/2)$, at least ρL^d sites in H_x will be in state j at time t.

The last bound is for a fixed time but it is easy to extend it to $[0, L]$. Let $\delta = \varepsilon^2\sigma/2$, let $J = \exp(\delta L^d/2)$, and $t_k = k/J$ for $1 \leqslant k \leqslant JL$. The probability that the number of sites in state i is less than ρL^d at some time t_k is at most

$$JL\kappa \exp(-\varepsilon^2 \sigma L^d/2) = \kappa L \exp(-\delta L^d/2)$$

The probability that two sites flip in some interval (t_{k-1}, t_k) is at most

$$JL \binom{L^d}{2} (\exp(-\delta L^d/2))^2 \leqslant L^{2d+1} \exp(-\delta L^d/2)$$

When we never have two flips in any interval, the state at each $t \in (t_{k-1}, t_k)$ agrees with the state at one of the two endpoints. Combining the estimates we have

$$P(\tau \leqslant L) \leqslant (4R_0 M_0 + 1)^d (\kappa L + L^{2d+1}) \exp(-\delta L^d/2)$$

since there are at most $(4R_0 M_0 + 1)^d$ houses in $B_2(0, rM_0)$, proving (a).

The proof of (b) is similar but simpler. Let $x \in B_2(0, (r + \Delta) M_0)$. When $\tau > L$ each site in H_x always sees at least $a\, |B_p(0, N)|$ sites of each color and flips from i to $i+1$ at rate 1. If L is large then all the quantities $u_{i,j}(L)$ introduced in the proof of (a) are at least $(\sigma + 1/\kappa)/2$ so using Lemma 3 again as in the proof of (a) shows that the fraction of sites in H_x in state i at time L is at least σL^d with probability at least $1 - \kappa \exp(-\delta L^d/2)$, which is more than enough to give (b) and completes the proof of Lemma 2. □

Step 3

Our final step is to show that if $\varepsilon > 0$ then after a second renormalization, the multicolor system dominates a 1-dependent oriented percolation process on $\mathscr{L} = \{(x, n) \in \mathbb{Z}^2 : x + n \text{ is even}\}$ with density at least $1 - \varepsilon$. The first step is to define the percolation process. Given random variables $\omega(x, n)$, $(x, n) \in \mathscr{L}$ that indicate whether the sites are open (1) or closed (0), we say that (y, n) can be reached from (x, m) and write $(x, m) \to (y, n)$, if there is a sequence of points $x_m, ..., x_n$ so that $|x_k - x_{k-1}| = 1$ for $m < k \leqslant n$ and $\omega(x_k, k) = 1$ for $m \leqslant k < n$. Up to this point the $\omega(x, n)$ could be arbitrary random variables. The phrase "1-dependent with density at least $1 - \varepsilon$" means that if (x_i, n_i), $i \in I$ is a sequence of points with $(|x_i - x_j| + |n_i - n_j|)/2 > 1$ for all i then

$$(*) \qquad P(\omega(x_i, n_i) = 0 \text{ for all } i) \leqslant \varepsilon^I$$

To make the comparison with oriented percolation, it is convenient to construct the process from a "graphical representation." For states $i \in \{0, 1,..., \kappa - 1\}$ and $x \in \mathbb{Z}^d$ let $\{T_n^{i,x}, n \geq 1\}$ be independent Poisson processes with rate 1. At time $t = T_n^{i,x}$, $\zeta_t(x)$ jumps from i to $i+1$ if it was in state i and the conditions of its neighbors dictate that a transition should occur. Let $K = R_0 M_1$, and let $T = R_0/\Delta$. By decreasing Δ, we can suppose T an integer without loss of generality. If $(x, n) \in \mathscr{L}_{\mathscr{L}}$ we say (x, n) is occupied if all the houses H_z, $z \in B_2(xK, K)$ are occupied at time nTL. Here the radius K was chosen so that Lemma 2 applies. Applying that result T times we see that if all the houses in $B_2(xK, K)$ are occupied at time nTL then with high probability all the houses in $B_2(xK, 2K)$ are occupied at time $(n+1)\,TL$ and $B_2(xK, 2K) \supset B_2((x \pm 1)K, K)$. Finally, if we condition on the state of the multicolor system at time nTL then the "good" event that makes city x grow and populate its neighbors is measurable with respect to the Poisson points in $A_{x,n} = B_2(xKL, 2KL) \times [nTL, (n+1)\,TL]$, and has probability at least $1 - \varepsilon$ if L is large. The percolation process will be 1-dependent since the space–time cylinders $A_{x,n}$ have the property that $A_{x,m} \cap A_{y,n} = \emptyset$ unless $m = n$ and $|x - y| \leq 2$. Recall $\mathscr{L} = \{(x, n) \in \mathbb{Z}^2$: $x + n$ is even$\}$.

To formalize the comparison with oriented percolation, we need to define the random variables $\omega(x, n)$ for $(x, n) \in \mathscr{L}$. To do this, we consider two cases. If (x, n) is occupied, we set $\omega(x, n) = 1$ if the "good" event occurs in $A_{x,n}$, and $\omega(x, n) = 0$ otherwise. If (x, n) is vacant we define $\omega(x, n)$ by flipping a coin with probability $1 - \varepsilon$ of heads (1) and ε of tails (0). Let $V_n = \{x\colon (x, n)$ is occupied$\}$, let $W_0 = V_0$, and let $W_n = \{y\colon (x, 0) \to (y, n)$ for some $x \in W_0\}$. It follows easily from the definitions and induction that $V_n \supset W_n$ and $(*)$ holds.

To produce a nontrivial stationary distribution, we start the multicolor system from a product measure in which each color has density $1/\kappa$, run the system to time S, take the Cesaro average of the distribution at times $0 \leq s \leq S$, and extract a convergent subsequence. The result, call it π, is a stationary distribution. (See Liggett,[15] Proposition 1.8, on page 10). To see that π is nontrivial, we observe that if L is large the law of large numbers implies $P(0 \in V_0)$ is close to 1 and if ε is small a well-known percolation result (see Durrett,[3] Section 10) implies $\liminf P(0 \in W_{2n})$ is close to 1. From the last result it follows easily that π is not concentrated on the absorbing states and the proof of Theorem 1 is complete.

3. PERCOLATION LEMMAS

In this section we will prove some results about oriented percolation that will be the key to the proofs. To explain our motivation let $S_2(x, r)$ be the set of sites in the houses in $B_2(x, r)$ and note: (i) Until the time τ defined in the proof of Lemma 2 all the sites in $S_2(0, (r+\Delta) M_1)$ flip independently; (ii) if we have two realizations of the single-site Markov chain that are driven by the same Poisson process, as in Step 3 above, then the probability they will agree by time L is at least $1 - C\exp(-\delta L)$ for some $C, \delta \in (0, \infty)$. From the last two observations it follows that if we have two initial configurations ξ_0 and ζ_0 in which city 0 is occupied and L is large then ξ_L and ζ_L will with high probability agree on $S_2(0, (R_0+\Delta) M_1)$ at time L. Iterating the last argument and arguing as in Step 3, it is easy to see (details will be given in Section 4), that if L is large then the coupled region $\{x: \xi_t(x) = \zeta_t(x)\}$ dominates a 1-dependent oriented percolation process with density close to 1. The last comparison was good enough to construct a stationary distribution but is not enough to prove uniqueness. We want the coupled region to expand and cover the whole space, but the percolation process has a positive density of closed sites.

Having explained the problem, we turn our attention now to the solution. We begin by defining oriented percolation on $\mathbb{Z}^d$. Each site $z \in \mathbb{Z}^d$ is independently designated as open or closed with probabilities p and $1-p$. In the usual oriented percolation model one is allowed to move up from x to $x + e_i$ where e_i is the ith unit vector but here we are interested in "dual clusters" so we say there is an open dual path from x to y if there is a sequence of open points $z_0 = x, ..., z_n = y$ so that $z_m - z_{m-1} \in \{-e_1, ..., -e_d\}$ for $1 \leqslant m \leqslant n$. The cluster containing x, C_x, is the set of points that can be reached from x by an open dual path. We say that x is *wet* if $|C_x| = \infty$, and *dry* otherwise.

The aim of this section is to show that if p is close enough to 1 then the dry sites do not percolate. To define percolation for dry sites, we say there is a dry path from x to y if there is a sequence of dry sites $z_0 = x, ...,$ $z_n = y$ so that $\|z_m - z_{m-1}\|_\infty \leqslant 1$ for $1 \leqslant m \leqslant n$. This definition is motivated by the proofs of Theorems 2A and 3. It is a little more lenient than necessary but easier to state than the exact condition.

To rule out percolation of dry sites we will use a "contour argument." To build up the contour we will start with the unit cube $Q = [-1/2, 1/2]^d$ and then orient the faces assigning $+1$ to the top faces (i.e., ones pierced by segments from 0 to e_i) and -1 to the other "bottom" faces. If the top face is pierced by the segment from 0 to e_i we call e_i the site above the face and 0 the site below. These definitions are extended to other cubes $x + Q$ by translation.

Let D_0 be the cluster of dry sites containing the origin, i.e., all the y that can be reached from 0 by a dry path. We define the dry region $\mathscr{D}_0$ to be the union of the $y + Q$ for all the $y \in D_0$ and define the contour associated with D_0 to be the algebraic sum of all the faces of the cubes that make up $\mathscr{D}_0$. That is, plus and minus faces cancel and disappear from the sum. The first ingredient in the contour argument is the following:

Lemma 4. If the contour has n faces then there are at least $n/2d$ closed sites.

Proof. It is immediate from the definition that (i) there are an equal number of plus and minus faces and (ii) the site above a plus site must be wet and hence the site below the face must be closed. Obviously the same closed site can sit below d plus faces but this is the worst that can happen. □

The next thing we have to do is count the number of contours. Define $\partial\mathscr{D}_0$ to be the union of all the faces that appear in the contour with a nonzero weight.

Lemma 5. If $\mathscr{D}_0$ is finite then $\partial\mathscr{D}_0$ is connected (as a closed subset of $\mathbb{R}^d$).

Proof. We begin by observing that since wet sites are necessarily part of an infinite cluster of wet sites, there cannot be any finite collections of wet sites and $\partial\mathscr{D}$ is what is usually referred to as the external boundary. All the ideas needed to prove this result can be found in Kesten,[14] p. 143–151. The somewhat gruesome details are left to the reader. In defense of our sin of omission we would like to observe that the only purpose for Lemma 5 is to prove the estimate in Lemma 6 and for this a much weaker conclusion would suffice.

From Lemma 5 it follows immediately that we have the following:

Lemma 6. There are a constants C, μ that only depend on the dimension so that if A_n is the number of contours with n boundary faces that contain a given face then $A_n \leqslant C\mu^n$.

Proof. Embed the collection of boundary faces into $\mathbb{R}^d$ by identifying each boundary face with its midpoint, and make this set of points into a graph Γ by connecting any two points for which the corresponding faces intersect. Now each point in Γ has the same number of neighbors, say v. Lemma 5 shows that after the orientations are discarded the contour is a connected subset of the graph Γ. The desired conclusion is now a "simple exercise" in combinatorics, but following Kesten we will use percolation to get the desired estimate. Consider site percolation on Γ in which sites are open with probability a and closed with probability $1-a$. An induction argument shows that any cluster of n open sites has at most $n(v-2)+2$ closed sites on the boundary. [Adding a new open site removes 1 boundary site and adds at most $(\nu-1)$ others.] Using C_o to denote the cluster containing a fixed point o and B_n the number of clusters of size n containing o, it is clear that

$$1 \geqslant P(|C_o| = n) \geqslant B_n a^n (1-a)^{n(\nu-2)+2}$$

so $B_n \leqslant (1-a)^{-2}(a(1-a)^{\nu-2})^{-n}$. The last inequality is valid for all a. Taking $a = 1/(\nu-1)$ to minimize the right-hand side gives a bound on the number of contours when orientation is ignored. Since $A_n \leqslant 2^n B_n$ the desired result follows.

Combining Lemmas 4 and 6 now gives the following:

Theorem 5. If p is close enough to 1 then the dry sites do not percolate.

Proof. Consider first a modified system in which all sites outside $[-K, K]^d$ are wet. We will get an upper bound (independent of K) on the probability that the contour associated with the (necessarily finite) cluster of dry sites containing the origin contains n faces. The first step is to observe that if the cluster reaches $\{x: x_1 = n+1\}$ then by looking at the highest points (lexicographically ordered) of the intersections with $\{x: x_1 = m\}$ for $1 \leqslant m \leqslant n+1$ we conclude that there are $n+1$ faces, a contradiction, so the cluster is contained in $\{x: \|x\|_\infty \leqslant n\}$. Now, there are at most $d(2n+2)^d$ plus faces to pick to start our contour, and once the starting face is picked, Lemma 6 bounds the number of contours. Combining the last observation with Lemma 4 shows that the probability of having a contour with n faces in $\{x: \|x\|_\infty \leqslant n\}$ is at most $d(2n+2)^d C\mu^n(1-p)^{n/2d}$. If $(1-p)^{1/2d} < 1/\mu$ then

$$\pi_k = \sum_{n=k}^{\infty} d(2n+2)^d C\mu^n (1-p)^{n/2d}$$

approaches 0 as $k \to \infty$. Since π_k is a bound on the probability the cluster of dry sites containing the origin escapes from $\{x: \|x\|_\infty \leqslant k\}$, the result follows. □

The proof of Theorem 5 generalizes immediately to mildly dependent percolation processes.

Lemma 7. Consider a dependent percolation process on $\mathbb{Z}^d$ in which

$$P(z_i \text{ is closed for } 1 \leqslant i \leqslant m) \leqslant \varepsilon^m$$

whenever $\|z_i - z_j\|_2 > V$ for all $i \neq j$. Let $\alpha = |\{z \in \mathbb{Z}^d: \|z\|_2 \leqslant V\}|$. If $\varepsilon^{1/2d\alpha} < 1/2\mu$ then the probability the cluster of dry sites containing the origin escapes from $\{x: \|x\|_\infty \leqslant n\}$ is at most $Cn^d 2^{-n}$, where C depends only on the dimension.

Proof. When the contour has size m, Lemma 4 guarantees the existence of at least $m/2d$ closed sites. There is a subset of the closed sites of size at least $m/2d\alpha$ that are all separated by distance $> V$. Since the cluster cannot escape from $\{x: \|x\|_\infty \leqslant n\}$ without having at least n faces, it follows that the probability of interest is at most

$$\sum_{m=n}^{\infty} d(2m+2)^d \, C\mu^m \varepsilon^{m/2d\alpha} \leqslant C' n^d 2^{-n}$$

The constant C in Lemma 6 only depends on d, so C' does as well. □

To use Lemma 7 to study the multicolor processes we need to rotate the integer lattice into a more convenient position. Let $D = d+1$ and let A be a $D \times D$ matrix so that (i) if x has $x_1 + \cdots + x_D = 1$ then $(Ax)_D = 1$ and (ii) if x and y are orthogonal then so are Ax and Ay. Let $e_1, \ldots, e_D$ be the D unit vectors and let $v_i = Ae_i$. Let $\mathscr{L}_D = \{Az: z \in \mathbb{Z}^D\}$ and make $\mathscr{L}_D$ into a graph by drawing an oriented arc from x to $x + v_i$ for $1 \leqslant i \leqslant D$. In words, we have rotated and scaled the D-dimensional lattice so that the steps x to $x + e_i$ of the usual oriented percolation model on $\mathbb{Z}^D$ now increase the last coordinate by 1. When $d = 1$ and $D = 2$, we must have $v_1 = (1, 1)$ and $v_2 = (-1, 1)$ so $\mathscr{L}_2 = \mathscr{L}$. To help visualize the lattice for $d \geqslant 2$ and explain why we have not given a formula for A, we note that when $d = 2$ and $D = 3$ one possibility is

$$v_1 = (1/\sqrt{2}, \sqrt{3/2}, 1)$$

$$v_2 = (1/\sqrt{2}, -\sqrt{3/2}, 1)$$

$$v_3 = (-\sqrt{2}, 0, 1)$$

Let u_i, $1 \leqslant i \leqslant D$ be the vectors in $\mathbb{R}^d$ that consist of the first d components of the v_i. It is easy to see that the vectors u_i are the vertices of a simplex and are all at the same distance U from the origin.

We come now to the final ingredient for the proofs of Theorems 2A and 3. For $(z, n) \in \mathscr{L}_D$, let $C_{z,n}$ be the collection of (y, m) that can be reached by a dual path of open sites. Let G_n be the set of z so that $0 \in C_{z,n}$ and let H_n be the set of z so that $|C_{z,n}| = \infty$.

Lemma 8. Consider a dependent percolation process on $\mathscr{L}_D$ in which

$$P(z_i \text{ is closed for } 1 \leqslant i \leqslant m) \leqslant \varepsilon^m$$

whenever $\|z_i - z_j\|_2 > V$ for all $i \neq j$. Let $\alpha = |\{z \in \mathscr{L}_D: \|z\|_2 \leqslant V\}|$. If $\varepsilon^{1/2d\alpha} < 1/2\mu$ then there is a $\beta > 0$ so that on $\{G_n \neq \emptyset$ for all $n\}$ we have $G_n \supset H_n \cap B_2(0, n\beta)$.

Proof. The result is easy in $d = 1$. In that case results in Section 3 of Durrett[3] show that if $G_n \neq \emptyset$, $l_n = \inf G_n$, and $r_n = \sup G_n$ then $G_n = H_n \cap [l_n, r_n]$ and results in Section 11 of Durrett[3] show that

$$\liminf r_n/n, \ \liminf -l_n/n \geqslant \beta > 0$$

To get the result in $d > 1$ we have to work harder. A simple argument using the methods in Section 10 of Durrett[3] shows that we have percolation in each plane $\Pi = \{z + mv_i + nv_j: m, n \in \mathbb{Z}\}$. With this in hand, Lemma 8 can be proved using the methods in Durrett and Griffeath.[5] Referring to that paper we see that the conclusion of Lemma 5 is (6) on page 539, which as demonstrated on pages 541–542, is a simple consequence of the exponential estimates (4) and (5) on page 539. The proof of (4) given on page 546 generalizes easily to the current situation. The proof of (5) given on pages 547–549 must be adapted to the geometry of $\mathscr{L}_D$ but no new ideas are required so we will not repeat the proof here. □

4. UNIQUENESS RESULTS

In this section we will prove Theorems 2A–2C.

4.1. Proof of Theorem 2A

The first ingredient is some "general nonsense." The multicolor processes are Feller processes with a compact state space so well-known results of Rosenblatt[16] imply the following:

Lemma 9. The collection of stationary distributions is a nonempty simplex, i.e., a convex set in which each element can be written as a convex combination of its extreme points in a unique way.

Lemma 10. Any two extreme points are mutually singular.

We will now use the last two lemmas and the coupling in Step 4 in Section 2 to prove

Lemma 11. Let μ and ν be two extreme points and suppose that in each measure, city 0 is occupied with positive probability. Then $\mu = \nu$.

Here city 0 refers to the ball $B_2(0, 2K)$ of houses where $K = R_0 M_1$ was defined in Step 3 in Section 2. It is clear that if some city $B_2(x, 2K)$ is occupied with positive probability in a stationary distribution μ then by using the construction in Step 3 in Section 2 to expand the occupied region we can conclude that city 0 is occupied with positive probability so Lemma 11 implies Theorem 2A.

Proof. In view of Lemma 10 it suffices to show that μ and ν are not mutually singular. Our hypotheses imply that realizations ζ_0 and ξ_0 of the two stationary distributions can be constructed on the same space so that with positive probability city 0 is occupied in both processes. We will now use a modification of the construction in Step 3 in Section 2 to show that if the evolution of the two processes is determined by the same graphical representation then the "coupled region" $\{x: \xi_t(x) = \zeta_t(x)\}$ will expand to cover the whole space with positive probability. Since the distributions of ξ_t and ζ_t are independent of t, this implies that the two measures are not mutually singular and ves the desired result.

Let $K = R_0 M_1$ and $T = R_0/\Delta$, which we suppose is an integer. If $(x, n) \in \mathscr{L}_D$ we say that city x is occupied at time n if all the houses in $B_2(xK/U, K)$ are occupied at time nT in ξ_{nT} and in ζ_{nT}. The scaling is chosen so that the points $u_i K/U$ lie on the boundary of $B_2(0, K)$. (u_i and U were defined right before Lemma 8.) At the beginning of Section 3 we argued that if (x, n) is occupied then with high probability at time $(nT+1)L$ all the houses in $B_2(xK/U, K + \Delta M_1)$ will be occupied in ζ and ξ and $\zeta = \xi$ in $S_2(xK/U, K + \Delta M_1) =$ the set of sites in $B_2(xK/U, K + \Delta M_1)$. Iterating the last argument T times we conclude that if (x, n) is occupied then with high probability all the $(x + u_i, (n+1)\,TL)$ will be occupied and $\zeta = \xi$ on the corresponding cities. Here, by "high probability," we mean that if $\varepsilon > 0$ and L is large then the event has probability at least $1 - \varepsilon$ for all possible values of ξ_{nT} and ζ_{nT}. Now if we condition on ζ_{nT} and ξ_{nT}, the "very good" events that produce the coupling are measurable with respect to the Poisson points in $A_{x,n} = B_2(xK/U, 2K) \times [nTL, (n+1)\,TL]$. Many of these cylinders intersect but there is a V so that if $\|(x, n) - (y, m)\|_2 > V$ then $A_{x,n} \cap A_{y,m} = \varnothing$.

To compare the coupled region with oriented percolation, we need to define the random variables $\omega(x, n)$ that indicate whether the sites are open (1) or closed (0). To do this we consider two cases. If (x, n) is occupied we let $\omega(x, n) = 1$ if the "very good" event in $A_{x,n}$ occurs, and $=0$ otherwise. If (x, n) is vacant we define $\omega(x, n)$ by flipping a coin with probability ρ of heads (1) and $1 - \rho$ of tails (0). For $(x, n) \in \mathscr{L}_D$, let $C_{x,n}$ be the collection of (y, m) that can be reached by a (dual) path of open sites, i.e., $m \leqslant n$ and there is a sequence $z_0 = x, \dots, z_{n-m} = y$ so that $\omega(z_k, k) = 1$ for $0 \leqslant k \leqslant n - m$ and $z_{k-1} - z_k \in \{u_1, \dots, u_D\}$ for $1 \leqslant k \leqslant n - m$. Let G_n be the set of z so that

$(0, 0) \in C_{z,n}$. It is easy to see that if $z \in G_n$ and $n \geq 1$ then ζ_{nTL} and ξ_{nTL} agree at all the sites in $S_2(zK/U, K)$. However, a positive fraction of the sites are not very good so we need to work harder.

Let $G = \bigcup_n (G_n \times \{n\})$ and observe that discrepancies cannot appear at a site when all of its neighbors within distance N (the range of the interaction) are the same. Let w_i, $1 \leq i \leq D(D-1)$ be the vectors that have one $+1$ component, one -1 component, and $D-2$ zeros. These are the vectors with integer coordinates in the hyperplane $x_1 + \cdots + x_D = 0$ that are closest to 0. Let $\mathscr{W} = \{Aw_i: 1 \leq i \leq D(D-1)\}$ and $\mathscr{V} = \{v_i: 1 \leq i \leq D\}$ where A and the v_i were defined before Lemma 8. A little thought reveals that if ζ_{nTL} and ξ_{nTL} disagree at some site in $S_2(zK/U, K)$ then z is *G-exposed* at time n, i.e., we can find a sequence (z_m, k_m), $0 \leq m \leq M$ of points not in G so that $z_0 = z$, $k_0 = n$, $k_M = 0$, k_m is nonincreasing, and $z_m - z_{m-1} \in \mathscr{V}$ when $k_m < k_{m-1}$ and $z_m - z_{m-1} \in \mathscr{W}$ when $k_m = k_{m-1}$. Here we use the fact that the cities were defined using $B_2(zK/U, K)$ and if $w \in \mathscr{W}$ then there are $v_i, v_j \in \mathscr{V}$ so that $w + v_i = v_j$, and hence $\|w/U\|_2 \leq 2$.

To make connection with results in Section 3, let H_n be the set of z for which $|C_{z,n}| = \infty$ and $H = \bigcup_n (H_n \times \{n\})$. Defining *H-exposed* in the obvious way and applying the transformation A^{-1} to return to $\mathbb{Z}^d$ we see that $A^{-1}H$ are the wet sites and the H-exposed sites are a subset of the dry sites defined there. To connect G and H we observe that Lemma 8 shows that there is an $\beta > 0$ so that on $\{|G| = \infty\}$, we have $G_n \supset H_n \cap B_2(0, n\beta)$ for large n. Lemma 7 gives an exponential upper bound on the radius of dry clusters so it follows from the Borel–Cantelli lemma that, on $\{|G| = \infty\}$, we have no G-exposed sites in $B_2(0, n\beta/2)$ for large n. The last conclusion implies that with positive probability the coupled region grows at a linear rate and the proof is complete. □

4.2. Proof of Theorem 2B

Here and for the rest of the paper we restrict our attention to the Greenberg–Hastings model. The next two lemmas prove conclusions (a) and (b).

Lemma 12. Let μ be a stationary distribution. If $h(\xi_0) = \mathbb{Z}^d$, μ almost surely, then all finite dimensional sets have positive probability, and hence occupied cities do.

Proof. The definition of h implies that for any $L < \infty$, $P(\xi_1(x) = 1$ for all $x \in [-L, L]^d) > 0$. From this it follows easily that $P(\xi(x) = \eta(x)$ for all $x \in [-L, L]^d) > 0$ for any η, and hence occupied cities have positive probability. □

Lemma 13. Consider $d=1$, let N be the range of the interaction, and suppose that $\theta \leqslant N$. Any stationary distribution has $h(\xi_0)=\mathbb{Z}$, μ almost surely.

Proof. If $h(\xi)$ contains θ consecutive sites then $h(\xi)=\mathbb{Z}$ so suppose it does not. Let $a \in h(\xi)$ and $b<a<c$ with $b, c \notin h(\xi)$ and $|b-c|>N$. Starting with initial distribution μ, there is positive probability that there are no 1's in (b, c) at time 1. However once this happens no 1's can be created in (b, c) because (i) the two endpoints see a number of 1's below threshold in $h(\xi_t) \supset \{x: \xi_t(x)=1\}$ for all t, and (ii) since $|b-c|>N$ the sites in (b, c) see fewer 1's in $h(\xi_t)$ than one of the points b or c does. □

4.3. Proof of Theorem 2C

Let $D=d+1$. We begin by observing that $\|(1/D,..., 1/D)\|_2=D^{-1/2}$ and $\|e_i-e_j\|_2=2^{-1/2}$ so the constant U defined in Step 3 in Section 2 is $(2D)^{1/2}$. For $(z, n) \in \mathscr{L}_D$ we say that z is occupied at time n if all sites in $B_2(z, 2U)$ are occupied at time $n\delta$. If we pick δ small and then λ large it follows that the occupied sites on $\mathscr{L}_D$ dominate the wet sites in a mildly dependent percolation process with p close to 1. Repeating the argument at the end of Step 3 in Section 2 shows that there is a nontrivial stationary distribution.

To prove the convergence theorem we begin by observing that if all sites in $B_2(z, 2U)$ are occupied then ξ_t must agree on $B_2(z, 2U)$ with the process starting from all 1's. It follows from the proof of Lemma 11 that if all sites in $B_2(0, 2U)$ are occupied at time 0 then with positive probability the coupled region grows linearly and covers the whole space. To get the coupling started we use the following:

Lemma 14. Suppose ξ is a configuration in which a birth is possible at a vacant site; then $h(\xi)=\mathbb{Z}^d$.

Proof. If a birth is possible at x then there must be 1's at $y, z \in B_p(x, N)$. Suppose without loss of generality that $x_1 \neq y_1$. Let u be the vector that has the first coordinate of y and the last $d-1$ from x. Since $\|x-u\|_p$ and $\|y-u\|_p$ are less than $\|x-y\|_p$ a birth is possible at u once x is occupied. Now once u is occupied births are possible at $x+e_1$ if $y_1>x_1$ and at $x-e_1$ if $y_1<x_1$. Once we have a pair of sites $v, v+e_1$ occupied the rest is easy. The conclusion is trivial if $d=1$, $N>1$. In $d>1$ when $N>2^{1/p}$ the birth rate is positive at $v+e_i$ and $v+e_1+e_i$, so $h(\xi)$ contains $\{v+me_i, v+e_1+me_i \text{ for } m \in \mathbb{Z}\}$. Repeating the last argument $(d-1)$ times gives the desired result. □

Lemma 14 implies that if births are possible at time t then there is probability at least $\delta > 0$ that at time $t+1$, all sites are occupied in some $B_2(z, 2U)$. From this it follows immediately that if births are possible at a sequence of times $t_n \to \infty$ then $\xi_t \Rightarrow \xi^1_\infty$. Conversely, if births are not possible for large t then the system dies out, i.e., $\xi_t \Rightarrow \delta_0$.

5. PROOF OF THEOREM 3

The proof follows the outline of Theorem 1 but this time we show that a large vacant ball grows.

Lemma 15. Consider the threshold contact automaton defined in Step 1 in Section 2 and suppose $\lambda > \frac{1}{2}$. There are constants R_0, Δ, and M_0, so that if $M \geq M_0$ and $\eta_0(x) = 0$ for $x \in B_2(0, rM)$ with $r \geq R_0$ then $\eta_1(x) = 0$ for $x \in B_2(0, (r+\Delta)M)$.

Proof. Use the constants of Lemma 1 for $1 - \lambda$.

Consider the two color Greenberg–Hastings model with threshold $\theta = bv_N$ where $b > \frac{1}{4}$ and pick $\lambda, \rho > \frac{1}{2}$ so that $\lambda\rho < b$. Our next step is the following:

Lemma 16. Let $\frac{1}{2} < \sigma < \rho$. There is a constant $\tau < \infty$ so that if we start with all sites in the contact process $= 1$ then at times $t \geq \tau$ the process is dominated by a product measure in which 1's have density σ.

Proof. The construction in Step 3 in Section 2 shows that the contact process is dominated by an independent flips process which always jumps from 0 to 1 at times $T^{0,x}_n$. In the independent flips process the density of 1's at time t is $(1 + e^{-2t})/2$.

Let $L = [N^{1/2}]$ and for $x \in \mathbb{Z}^d$ let $H_x = [x_1 L, (x_1+1)L) \times \cdots \times [x_d L, (x_d+1)L)$. As before we call H_x a *house* but now we say that it is *vacant* if it contains no 1's. Let $D = d+1$ and let $\mathscr{L}_D$ be the lattice introduced in Step 3 in Section 2. For $(z, n) \in \mathscr{L}_D$ we say that city z is vacant at time n if all the houses in $B_2(zL/U, 2L)$ are vacant at time nTL where $T = L/\Delta$. We call a house H_x *reasonable* at time n if there are at most σL^d 1's in H_x at all times $t \in [nTL, (n+1)\,TL]$. Recalling Lemma 16 and using Lemma 3 as in Step 2 in Section 2 it is easy to see that if L is large and $n \geq 1$ then with high probability all houses H_x, $x \in B_2(zL/U, 3L)$ are reasonable at time n. Now, if all the houses in H_x, $x \in B_2(zL/U, 3L)$ are reasonable at time n, if all the houses in $B_2(zL/U, 2L + k\Delta)$ are vacant at time $(nT+k)L$, and if L is large, then all the sites in $B_2(zL/U, 2L + (k+1)\Delta)$ see a number of 1's that is below threshold at all times in $[(nT+k)L, (nT+k+1)L]$ and no births occur. The probability a 1 will survive for L units of time is $\exp(-L)$ and there are only CL^d sites to worry about, so with high probability the vacant region will expand to $B_2(zL/U, 3L)$ by time $(n+1)\,TL$.

The definition in the last paragraph are those of Step 3 in Section 2 with $K=L$ and $T=L/\Delta$. Continuing to follow the developments there, let $A=B_2(0, 3KL)\times[-\tau, TL]$, and for $(z, n)\in\mathscr{L}_D$ let $\psi(zKL/U, nTL)$, and $A_{z,n}=\psi(z, n)+A$. If we condition on the state at time $nTL-\tau$ then the "good" event that makes a vacant ball centered at z grow and depopulate the neighboring cities is measurable with respect to the Poisson points in $A_{z,n}$. This time the space–time cylinders $A_{z,n}$ have the property that $A_{w,m}\cap A_{z,n}=\emptyset$ unless $|m-n|\leqslant 1$ and $\|w-z\|_2\leqslant 6U$. However, if L is large the vacant cities again dominate a mildly dependent percolation process in which open sites have probability close to 1.

To complete the proof of Theorem 3 now, we follow the proof of Lemma 11. For $(z, n)\in\mathscr{L}_D$ let $C_{z,n}$ be the collection of (y, m) that can be reached by a (dual) path of good sites, let G_n be the set of z so that $(0, d)\in C_{z,n}$, and let H_n be the set of z so that $|C_{z,n}|=\infty$. (We need a positive time so that Lemma 16 guarantees the density has dropped close to $\frac{1}{2}$. Time d is the first time $(0, n)\in\mathscr{L}_D$.) If we suppose that at time d city 0 is vacant then there can be a 1 in some house in $B_2(zK/U, 2K)$ at time n only if (z, n) is G-exposed. Using Lemmas 8 and 7 now as in the proof of Lemma 11 we conclude that with positive probability we have no G-exposed sites in $B_2(0, n\varepsilon/2)$ for large n. This shows that a large vacant ball will expand linearly with positive probability and Theorem 3 follows. For the proof of Theorem 4 in the next section we would like to observe that the "positive probability" in the last sentence is close to 1 if L is large. □

6. PROOF OF THEOREM 4

We begin with the lower bound on τ_N. This part of the result holds for a general multicolor system. Let σ_N be the first time there is an x and an i so that the number of neighbors of site x of color i is below threshold. Until time σ_N all sites flip independently. In a system of independent flips starting from a product measure in which all colors have density $1/\kappa$ it follows from Lemma 3 that the probability that at a fixed time t some site $x\in(\mathbb{Z} \bmod L)^d$ sees color i below threshold is at most $CL^d\exp(-\varepsilon N^d)$ for some C, $\varepsilon\in(0,\infty)$. Subdividing the time interval $[0, \exp(\varepsilon N^d/4)]$ into intervals $[k\alpha, (k+1)\alpha)$ where $\alpha=\exp(-\varepsilon N^d/2)$ we see that the probability two sites flip in the same interval is at most $\alpha^2 L^{2d}\exp(\varepsilon 3N^d/4)$ which converges to 0 as $N\to\infty$ since $(\log L)/N^d\to 0$. When we never have two flips in the same interval, the state at each time $t\in[k\alpha, (k+1)\alpha)$ agrees with the state at one endpoint. The probability some site will see some color below threshold at some time $k\alpha \in [0, \exp(\varepsilon N^d/4)]$ is at most

$$\exp(\varepsilon 3N^d/4)\, C\kappa L^d \exp(-\varepsilon N^d)$$

which also converges to 0 as $N \to \infty$. Combining the last two estimates proves $P(\sigma_N < \exp(\varepsilon N^d/4)) \to 0$ as $N \to \infty$.

To prove the upper bound on τ_N we observe that it follows from the proof of Theorem 3 that there is a constant R_0 so that if there are no occupied sites in $B_2(0, R_0 N)$ then with a probability that approaches 1 as $N \to \infty$ the vacant region will expand linearly and cover the whole space. Now if we let $\beta = e^{-2}$ be the probability that a site sees exactly one arrival in $T_n^{1,0,x}$ and no arrivals in $T_n^{0,1,x}$ in $[t, t+1)$ then the probability that all sites in $B_2(0, R_0 N)$ are vacant at time $t+1$ is at least $\beta^{|B_2(0, R_0 N)|}$ independent of the state at time t. Let T_N be the first time $B_2(0, R_0 N)$ is vacant. Standard arguments show that for any $\delta > 0$

$$P(T_N > \exp((2+\delta)\,|B_2(0, R_0 N)|)) \to 0 \qquad \text{as} \quad N \to \infty$$

Since $(\log L)/N^d \to 0$ and the hole grows linearly, it follows that

$$P(\tau_N > \exp((2+2\delta)\,|B_2(0, R_0 N)|)) \to 0 \qquad \text{as} \quad N \to \infty$$

completing the proof of Theorem 4. □

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