

## Asymptotic behavior of Brownian polymers

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**Summary.** We consider a system that models the shape of a growing polymer. Our basic problem concerns the asymptotic behavior of  $X_t$ , the location of the end of the polymer at time  $t$ . We obtain bounds on  $X_t$  in the (physically uninteresting) case that  $d = 1$  and the interaction function  $f(x) \geq 0$ . If, in addition,  $f(x)$  behaves for large  $x$  like  $Cx^{-\beta}$  with  $\beta < 1$  we obtain a strong law that gives the exact growth rate.

### Introduction

Let  $B_t$  be a Brownian motion and  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be Lipschitz continuous. In this paper we will be concerned with processes of the form

$$(1.1) \quad X_t = B_t + \int_0^t ds \int_0^s du f(X_s - X_u).$$

If  $f(x) = \psi(x)x/\|x\|$  and  $\psi(x) \geq 0$ ,  $X_t$  is a continuous analogue of a process invented by Diaconis and studied by Pemantle (1988a,b) so stealing a metaphor from that paper (and changing the sign) we can think of  $X_t$  as the trajectory of a tourist who wants to stay away from places she has visited before. For a more serious physical motivation one can think of  $X_t$  as describing a growing polymer in which newly added units are repelled by existing ones. As a polymer model, (1.1) has two serious weaknesses: (i) the repulsion does not prevent self-intersections, and (ii) while real polymers can rearrange themselves to minimize energy, ours cannot. However, in contrast to Edward's model (see Westwater (1980a)) the existence of the process presents no problem. Theorem 11.2 of Rogers and Williams implies that (1.1) has a pathwise unique strong solution. Furthermore, one can hope that

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results about the end to end displacement of our polymer will give some insight into the behavior of more realistic models.

In this paper we will be concerned with the asymptotic behavior of  $X_t$  as  $t \rightarrow \infty$ . The reader will see this problem presents some interesting mathematical challenges. Most of our results are confined to one dimension but the first one is general.

**Theorem 1.** *Suppose  $\|f(x)\| \leq M$  and  $f$  has compact support. There is a constant  $\Gamma < \infty$  so that*

$$\limsup_{t \rightarrow \infty} \|X_t\|/t \leq \Gamma \quad \text{a.s.}$$

To see that there is something to prove, notice that the cumulative drift,  $X_t - B_t$ , at time  $t$  might be as large as  $Mt^2/2$ . Indeed in Theorem 4 we will see that for any  $\alpha < 2$ , there are examples with bounded  $f$  in which  $X_t$  is of order  $t^\alpha$ . Although Theorem 1 is not obvious, it is not difficult to prove. The key observation is that if  $f(x) = 0$  for  $\|x\| \geq K$  and  $\|X_t\|$  grows too quickly then many annuli  $\{x: (n - 1)K < \|x\| < nK\}$  must be crossed quickly. However after a fast crossing the drift is small and the chances of another fast crossing are not very good.

To get lower bounds on  $X_t/t$  we have to impose some strong assumptions.

**Theorem 2.** *Suppose  $d = 1, f \geq 0$  and  $f(0) > 0$ . Then there is constant  $\gamma > 0$  so that*

$$\liminf_{t \rightarrow \infty} X_t/t \geq \gamma .$$

The condition  $f(0) > 0$  is too strong;  $f \neq 0$  should be sufficient. A more interesting problem is to strengthen the conclusion.

**Conjecture 1.** *Suppose  $f$  has compact support,  $f \geq 0$ , and  $f \neq 0$ . Then there is a constant  $\mu > 0$  so that*

$$X_t/t \rightarrow \mu \quad \text{a.s.}$$

To see why this should be true consider

$$Y_t = B_t + \int_0^t ds \int_{(s-T)^+}^s du f(Y_s - Y_u) ,$$

i.e., a modification in which we look back only  $T$  units in time. It is easy to check that  $Z_t = \{Y_t - Y_{t-s}: s \in [0, T]\}$  (defined for  $t \geq T$ ) is an exponentially ergodic Harris chain with state space  $C([0, T], \mathbb{R})$  and hence  $Y_t/t \rightarrow v$  almost surely where  $v$  is the average drift in equilibrium. By analogy, it seems reasonable to guess that  $\{X_t - X_{t-s}: s \in [0, \infty)\}$  converges rapidly to equilibrium and Conjecture 1 holds. Carrying out this approach seems difficult. Perhaps the result can be proved using ideas from Kusuoka (1985).

The assumption  $f \geq 0$  is undesirable since it says that our tourist avoids familiar territory by moving North. The situation becomes very complicated when  $f$  has values of both signs.

**Conjecture 2.** *Suppose  $f$  has compact support,  $xf(x) \geq 0$ , and  $f(-x) = -f(x)$  then  $X_t/t \rightarrow 0$  a.s.*

Before the reader declares that this is an obvious consequence of symmetry we would like to observe that there is no zero-one law, so one might have  $X_t/t \rightarrow c > 0$

on a set of probability 1/2. Indeed the last behavior occurs in Westwater's polymer model (see Kusuoka (1985)), but computer simulations of related discrete systems suggest that for our process the following scenario is more likely.  $X_t$  grows (or decreases) linearly for a while until a fluctuation overcomes the drift, which is  $O(1)$ , and brings the process well below its maximum. At this point the push from above is larger than that from below and the process tends to decrease for a while. Once the process gets well inside the initial increasing segment things get complicated but can be visualized if one thinks of the graph of the local time at time  $t$  as a mountain range and  $X_t$  as a Brownian ant that drifts downhill and drops sand at rate 1. We have no idea whether  $X_t$  satisfies the central limit theorem or displays more interesting behavior but suspect that this will be very difficult to resolve.

The problems we encountered in the compactly supported case become somewhat simpler when  $f$  is not integrable, for then the drift grows with time. Suppose

- (A1)  $|f(x)| \leq M$
- (A2)  $f(x)$  is decreasing for  $x \in [q, \infty)$
- (A3)  $x^\beta f(x) \rightarrow l > 0$  as  $x \rightarrow \infty$  with  $0 < \beta < 1$

Letting  $x_t = T^{-\alpha} X(tT)$  and  $W_t = T^{-1/2} B(tT)$  we can rewrite (1.1) as

$$(1.2) \quad x_t = T^{1/2-\alpha} W_t + T^{2-\alpha} \int_0^t ds \int_0^s du f(T^\alpha(x_s - x_u)) .$$

If we take  $\alpha = 2/(1 + \beta)$  so that  $2 - \alpha = \alpha\beta$  and let  $T \rightarrow \infty$  we see that the limit, if all goes well, should satisfy

$$(1.3) \quad x_t = \int_0^t ds \int_0^s du \frac{l}{(x_s - x_u)^\beta} .$$

One solution is  $x_t = c_0 t^\alpha$  where  $c_0$  satisfies

$$(1.4) \quad \alpha c_0^{\beta+1} = \int_0^1 \frac{l du}{(1 - u^\alpha)^\beta} .$$

Our first result says that this argument provides an upper bound

**Theorem 3.** *Suppose (A1)–(A3) hold and  $\alpha$  and  $c_0$  are as above. Then*

$$\limsup_{t \rightarrow \infty} X_t/t^\alpha \leq c_0 .$$

Theorem 3, when suitably reformulated, holds in  $\mathbb{R}^d$ . A more interesting extension would be to prove

**Conjecture 3.** *Suppose  $d = 1$  and  $f(x) = x/(1 + |x|^{\beta+1})$  with  $0 < \beta < 1$ . Then with probability  $\frac{1}{2}$*

$$X_t/t^\alpha \rightarrow c_0 .$$

To see the difficulties involved the reader should try the following much simpler open question.

**Problem.** Under the hypotheses of Conjecture 3,

$$\sup |X_t| = \infty \text{ a.s.}$$

Our final result gives some support for Conjecture 3.

**Theorem 4.** *Suppose (A1)–(A3) hold,  $f \geq 0$ , and  $f(0) > 0$ . Then*

$$X_t/t^\alpha \rightarrow c_0 \quad \text{a.s.}$$

The rest of the paper is devoted to proofs. Theorem  $k$  is proved in section  $(k + 1)$ . Sections 2–4 are independent of each other and can be read in any order but the proof in Sect. 5 depends on results in Sects. 3 and 4. In what follows  $c_0$  is the constant in (1.4) and  $c$  is used for constants either slightly larger or smaller than  $c_0$ , so we use  $D$  to denote dumb constants whose values are unimportant.

### 2 Upper bound for compactly supported $f$

In this section we assume that

- (i)  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is Lipschitz continuous,
- (ii)  $f(x) = 0$  for  $\|x\| \geq 1$ ,
- (iii)  $\|f(x)\| \leq K$  for all  $x$ .

We will prove the following result that after rescaling space and then changing time to make the Brownian motion have variance  $t$  gives the version stated in the introduction.

**Theorem 1.** *There is a constant  $\gamma > 0$  so that*

$$\limsup_{t \rightarrow \infty} \|X_t\|/t \leq 2/\gamma \quad \text{a.s.}$$

*Proof.* Consider the one dimensional s.d.e.

$$(2.1) \quad dY_t = dB_t + \left\{ \frac{d-1}{2(Y_t+2)} + 7\gamma K \right\} dt \quad Y_0 = 0$$

and impose reflecting boundary conditions at 0. We can choose the parameter  $\gamma > 0$  so small that if  $\tilde{H} = \inf\{t: Y_t = 2\}$  then

$$(2.2) \quad P(\tilde{H} \geq 5\gamma) \geq 1/2.$$

Now let  $H_n = \inf\{t: \|X_t\| = 2n\}$ ,  $\tau_n = H_n - H_{n-1}$ ,  $\mathcal{G}_n = \mathcal{F}_{H_n}$  where  $\mathcal{F}_t$  is the filtration generated by the Brownian motion and define events  $G_n = \{\tau_n \geq 5\gamma\}$ ,  $F_n = \{\tau_n \leq 2\gamma\}$ . When  $F_n$  happens we speak of a *fast* crossing from  $2n - 2$  to  $2n$ . When  $G_n$  happens we speak of a *slow* crossing from  $2n - 2$  to  $2n$ .

To prove the result it suffices to show that  $\liminf_{n \rightarrow \infty} H_n/n \geq \gamma$ . The first step in doing this is to observe that if  $H_N \leq \gamma N$ , then at least half of the  $\tau_1, \dots, \tau_N$  must be smaller than  $2\gamma$ , so there are at least  $N/2$  fast crossings from  $2n - 2$  to  $2n$ . The second step will be to show that after a fast crossing from  $2n - 2$  to  $2n$  there is probability at least  $1/2$  that the crossing from  $2n$  to  $2n + 2$  will be slow, so that the total time to get to  $2N$  will be larger than  $\gamma N$  with high probability.

To carry out the second step note that the process  $R_t = \|X_t\| - 2n$  satisfies

$$(2.3) \quad dR_t = dB_t + \left\{ \frac{d-1}{2(R_t+2n)} + \int_0^t \frac{X_u}{\|X_u\|} \cdot f(X_t - X_u) du \right\} dt.$$

Now suppose that  $F_n$  has happened and  $n \geq 1$ . Then for any  $t \in [H_n, H_n + 5\gamma]$ , the drift in (2.3) is bounded above by  $(d - 1)(2(R_t + 2))^{-1} + 7\gamma K$  while the process is in  $[2n, 2n + 2]$ . A standard comparison theorem for stochastic differential equations (see e.g. Rogers and Williams V.43) implies that we can build a process  $Y'$  identical in law to  $Y$  so that

$$Y'_t \geq \|X(H_n + t)\| - 2n \quad \text{for } t \in [0, 5\gamma]$$

so  $P(G_{n+1} | \mathcal{G}_n) \geq 1/2$  on  $F_n$ . The desired conclusion now follows from a result of Dubins and Freedman (1965) (see e.g. Durrett (1990) p. 220)

Suppose  $G_n$  is adapted to  $\mathcal{G}_n$  and let  $p_n = P(G_n | \mathcal{G}_{n-1})$ . Then

$$(2.4) \quad \sum_{m=1}^n 1_{G_m} \Big/ \sum_{m=1}^n p_m \rightarrow 1 \quad \text{a.s. on } \left\{ \sum_{m=1}^{\infty} p_m = \infty \right\}.$$

To derive the desired conclusion now we let  $N_n$  be the number of fast crossings in the first  $n$  trials and  $N_\infty = \lim N_n$ . On  $\{N_\infty < \infty\}$  it is clear that  $\liminf_{n \rightarrow \infty} H_n/n \geq 2\gamma$ . On  $\{N_n = \infty\}$  we can apply (2.4) to conclude that the number of slow crossings  $M_n$  satisfies

$$\liminf_{n \rightarrow \infty} M_n/(N_n/2) \geq 1 \quad \text{a.s.}$$

The last result implies that for  $n \geq n_0$ ,

$$M_n \geq \frac{4}{5} \frac{N_n}{2}.$$

For  $n \geq n_0$  either  $N_n \geq n/2$  (in which case  $M_n \geq n/5$  and  $H_n \geq \gamma n$ ) or  $N_n < n/2$  (in which case there are more than  $n/2$  crossings that take more than  $2\gamma$  units of time and  $H_n \geq \gamma n$ ). In all cases we have  $\liminf H_n/n \geq \gamma$  and the proof is complete.

### 3 Lower bound when $f$ is positive at 0

In this section we consider the one-dimensional situation with  $f \geq 0$ . We do not require that  $f$  have compact support but we do need  $f$  to be positive near 0. Again, scaling space and time gives the result in the introduction.

**Theorem 2.** Let  $A = \inf\{f(x) : |x| \leq \frac{1}{2}\}$ . Then

$$\liminf_{t \rightarrow \infty} X_t/t \geq A^{1/2}/2.$$

*Proof.* We will prove this result by getting a lower bound on the total drift up to time  $t$  and then observing that the contribution of the Brownian motion can be ignored. Indeed in this argument Brownian motion could be replaced by any process with  $B_t/t \rightarrow 0$  as  $t \rightarrow \infty$ . Let  $g(x) = A$  when  $|x| \leq 1/2$  and  $g(x) = 0$  otherwise.

$$\begin{aligned} \int_s^t ds \int_0^s du f(X_s - X_u) &\geq \int_0^t ds \int_0^s du g(X_s - X_u) \\ &= \frac{1}{2} \int_0^t ds \int_0^t du g(X_s - X_u) = \frac{1}{2} J(\mu) \end{aligned}$$

where  $\mu$  is the occupation measure  $\mu(C) = \int_0^t 1_C(X_s) ds$  and

$$J(v) \equiv \int v(dx) \int v(dy) g(x - y).$$

If  $v$  is a probability measure supported in  $[0, n]$  then

$$J(v) \geq A \left( v([0, 1/2])^2 + \sum_{k=1}^{2n-1} v((k/2, (k + 1)/2])^2 \right) \geq A/2n$$

by the Cauchy-Schwarz inequality. Let  $b_t = \sup\{X_s : s \leq t\}$  and  $a_t = \inf\{X_s : s \leq t\}$ . Then

$$\begin{aligned} X_t - B_t &= \int_0^t ds \int_0^s du f(X_s - X_u) \\ &\geq \frac{At^2/2}{2(1 + b_t - a_t)} \geq \frac{t^2 A}{4(1 + b_t - \inf_{s \leq t} B_s)}, \end{aligned}$$

since  $X_t \geq B_t$ . On the other hand

$$X_t - B_t \leq 1 + b_t - \inf_{s \leq t} B_s,$$

so it follows that

$$(3.1) \quad 1 + b_t - \inf_{s \leq t} B_s \geq tA^{1/2}/2.$$

The last inequality implies that

$$(3.2) \quad \liminf_{t \rightarrow \infty} b_t/t \geq A^{1/2}/2.$$

To strengthen this to the conclusion of Theorem 2 observe that if  $s < t$

$$X_t - X_s = B_t - B_s + \int_s^t du \int_0^u dv f(X_u - X_v),$$

so taking  $s$  to be the first time  $\max_{r \leq t} X_r$  is attained it follows that

$$(3.3) \quad X_t - b_t \geq \inf_{s \leq t} (B_t - B_s),$$

and

$$(3.4) \quad \liminf_{t \rightarrow \infty} X_t/t \geq \liminf_{t \rightarrow \infty} b_t/t.$$

Combining (3.4) with (3.2) completes the proof of Theorem 2. For results in Sect. 5 we will need a simple extension of the results above.

$$X_t - X_s \geq B_t - B_s + \int_s^t du \int_s^u dv f(X_u - X_v).$$

If we let  $b_t^s = \max_{s \leq r \leq t} X_s$  and repeat the proofs of (3.1) and (3.3) it follows that

$$(3.5) \quad 1 + b_t^s - X_s - \inf_{s \leq r \leq t} (B_r - B_s) \geq (t - s)A^{1/2}/2$$

$$(3.6) \quad X_t - b_t^s \geq \inf_{s \leq r \leq t} (B_t - B_r).$$

Adding (3.5) and (3.6) gives

$$(3.7) \quad X_t - X_s \geq (t - s)A^{1/2}/2 + \inf_{s \leq r \leq t} (B_t - B_r) + \inf_{s \leq r \leq t} (B_r - B_s) - 1 .$$

### 4 Upper bounds for fat tailed f

We begin by proving a fact about Brownian motion that will be very important in this section and the next.

**Lemma 4.1** *Let  $b < 1/2$ . Then for any  $k < \infty$*

$$P \left( \sup_{0 \leq s < t \leq 1} \frac{|B_t - B_s|}{(t - s)^b} > x \right) \leq Cx^{-k} .$$

*Proof.* Lemma 1.1 in Garsia Rodemich Rumsey (1970) implies that if (i)  $\psi, \phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are continuous and increasing, (ii)  $\phi(0) = 0$ , and (iii)  $f: [0, 1] \rightarrow \mathbb{R}$  is continuous and satisfies

$$\int_0^1 ds \int_0^1 dt \phi \left( \frac{|f(s) - f(t)|}{\phi(|s - t|)} \right) \equiv A < \infty$$

then for  $s, t \in [0, 1]$

$$|f(s) - f(t)| \leq \rho(|s - t|) \equiv 8 \int_0^{|s-t|} \psi^{-1}(A/x^2) d\phi(x)$$

If we take  $f(s) = B_s, \psi(x) = |x|^k, \phi(x) = |x|^{1/2}$  and  $k > 2$  then the modulus of continuity

$$\rho(r) = 8 \int_0^r A^{1/k} x^{-2/k} \frac{1}{2} x^{-1/2} dx = C_k A^{1/k} r^{1/2 - 2/k} ,$$

where  $C_k$  is a constant that depends only on the value of  $k$ . If we pick  $k$  large enough so that  $1/2 - 2/k > b$  it follows that

$$\sup_{0 \leq s < t \leq 1} |B_s - B_t| / (t - s)^b < C_k A^{1/k} .$$

$EA = E|B_1 - B_0|^k < \infty$  so Chebyshev's inequality implies  $P(A > x^k) < x^{-k}EA$ , and the desired result follows.

Turning now to the proof of Theorem 3, we begin by recalling that the assumptions on  $f$  throughout this section are

- (i)  $|f(x)| \leq M$
- (ii)  $f(x)$  is decreasing for  $x \in [q, \infty)$
- (iii)  $x^\beta f(x) \rightarrow l > 0$  as  $x \rightarrow \infty$  where  $0 < \beta < 1$ .

Let  $\alpha = 2/(1 + \beta) \in (1, 2)$ , observe  $2 - \alpha = \alpha\beta$ , and define  $c_0$  by

$$(4.1) \quad \alpha c_0^{1+\beta} = \int_0^1 \frac{l du}{(1 - u^\alpha)^\beta} .$$

Let  $\varepsilon > 0$  and  $c > c_0$ . Our aim is to show that if  $T$  is large  $P(X_T > ct^\alpha$  for some  $t \geq T) < \varepsilon$ . The first step in achieving this aim is to make some choices that for the

moment will seem rather mysterious. Their purposes will be revealed in the proof below. For the moment the reader should be content to check that such choices are possible. Once we have introduced the necessary definitions we will explain the idea behind the proof. Pick  $\theta > 1$  so that

$$(4.2) \quad \gamma \equiv c - \theta \frac{c_0^{1+\beta}}{c^\beta} > 0 .$$

Pick  $N \geq q$  large enough so that  $f(x) \leq \theta l x^{-\beta}$  for  $x \geq N$ . Choose  $\rho, g > 0$  and  $b \in (0, 1/2)$ , so that

$$(4.3) \quad v \equiv 2 - \alpha + 2\rho < \alpha ,$$

$$(4.4) \quad v < \alpha(\alpha - g - 1/2)/(\alpha - b) .$$

This is possible since  $2 - \alpha < \alpha$  and the right-hand side of (4.4) is  $\alpha$  when  $b = 1/2$  and  $g = 0$ . Note that  $\alpha < 2$  so  $v > 0$ . Let  $\varepsilon_T = T^{-\rho}$ , define  $\eta_T$  by the requirement

$$(4.5) \quad T^{2-\alpha} M(\eta_T + T^{-\alpha} N) = (c\gamma/2)\varepsilon_T^2 .$$

Note that the definition of  $\eta_T$  and (4.3) imply  $\eta_T \sim (c\gamma/2M)T^{-v}$  and let

$$(4.6) \quad \phi_0(t) = c \int_0^t (\alpha s^{\alpha-1}) \vee \varepsilon_T ds \quad \phi_1(t) = \phi_0(t) + \eta_T .$$

Let  $\tau = \inf\{t: x_t = \phi_1(t)\}$ , and let  $\sigma = \sup\{u < \tau: x_u = \phi_0(u)\}$ . We will show that  $P(\tau \leq 1)$  is small by getting an upper bound on the drift at times  $s \in [\sigma, \tau]$  which shows that the crossing from  $\phi_0$  at time  $\sigma$  to  $\phi_1$  at time  $\tau$  must be due to an abnormally large fluctuation in the Brownian motion. To bound the drift we let

$$v = \sup\{u: \phi_1(u) + T^{-\alpha} N < \phi_0(s)\}$$

with  $\sup \emptyset = 0$ . (Note that  $v$  depends on  $s$ .) Since  $x_t < \phi_1(t)$  for  $t < \tau$  and  $x_s > \phi_0(s)$ , using (i) and (ii) (and recalling  $N \geq q$ ) gives

$$(4.7) \quad \int_0^s f(T^\alpha(x_s - x_u)) du \leq M(s - v) + \int_0^v f(T^\alpha(\phi_0(s) - \phi_1(u))) du .$$

To estimate the integral on the right we observe that the choice of  $N$  and the definition of  $v$  imply

$$f(T^\alpha(\phi_0(s) - \phi_1(u))) \leq \theta l \{T^\alpha(\phi_0(s) - \phi_1(u))\}^{-\beta} \leq \theta l \{T^\alpha(\phi_1(v) - \phi_1(u))\}^{-\beta} .$$

Now

$$\phi_1(v) - \phi_1(u) = \phi_0(v) - \phi_0(u) \geq c(v^\alpha - u^\alpha) ,$$

so

$$(4.8) \quad \int_0^v f(T^\alpha(\phi_0(s) - \phi_1(u))) du \leq \frac{\theta}{c^\beta T^{\alpha\beta}} \int_0^v \frac{l du}{(v^\alpha - u^\alpha)^\beta} = \frac{\theta}{c^\beta T^{\alpha\beta}} v^{\alpha-1} \alpha c_0^{\beta+1} ,$$

by the definition of  $c_0$  given in (4.1). Now the convexity of  $\phi_0$  implies

$$(s - v)\phi'_0(v) \leq \phi_0(s) - \phi_0(v) = \eta_T + T^{-\alpha} N ,$$



and recalling the definition of  $\phi_0$  gives

$$(s - v) \leq (\eta_T + T^{-\alpha}N)/(c\varepsilon_T) .$$

Using the last inequality and  $v \leq s$  with (4.7) and (4.8) gives

$$\int_0^s f(T^\alpha(x_s - x_u)) du \leq \left( M(\eta_T + T^{-\alpha}N) \left/ \left( c\varepsilon_T + \frac{\theta}{c^\beta T^{\alpha\beta}} c_0^{\beta+1} \alpha s^{\alpha-1} \right) \right. \right) .$$

Recalling  $2 - \alpha = \alpha\beta$  and using the definition of  $\eta_T$  in (4.5), and  $\gamma$  in (4.2), we have

$$(4.9) \quad T^{2-\alpha} \int_0^s f(T^\alpha(x_s - x_u)) du \leq (\gamma/2)\varepsilon_T + (c - \gamma)\alpha s^{\alpha-1} .$$

The last inequality gives an upper bound on the drift that is smaller than  $\phi'_0(s)$ . To complete the proof we will now bound the contribution of the Brownian motion. Using Lemma 4.1 with  $k > 1/g$ , the Borel Cantelli lemma, and Brownian scaling it follows that (almost surely) for large  $T$

$$T^{-1/2} |B_{sT} - B_{tT}| \leq T^g(t - s)^b \quad \text{for all } 0 \leq s < t \leq 1$$

Hence for large  $T$ ,

$$(4.10) \quad \begin{aligned} T^g(\tau - \sigma)^b T^{1/2-\alpha} &\geq T^{-\alpha}(B_{T\tau} - B_{T\sigma}) \\ &\geq x_\tau - x_\sigma - (\gamma/2)\varepsilon_T(\tau - \sigma) - (c - \gamma)(\tau^\alpha - \sigma^\alpha) , \end{aligned}$$

by (1.2) and (4.9). Now

$$\phi_1(\tau) - \phi_0(\sigma) = \eta_T + c \int_\sigma^\tau (\alpha s^{\alpha-1}) \vee \varepsilon_T ds ,$$

so the right-hand side of (4.10) is at least

$$(4.11) \quad \begin{aligned} \eta_T + c \{ (\tau^\alpha - \sigma^\alpha) \vee \varepsilon_T(\tau - \sigma) \} - (\gamma/2)\varepsilon_T(\tau - \sigma) - (c - \gamma)(\tau^\alpha - \sigma^\alpha) \\ \geq \eta_T + \frac{\gamma}{2} \{ (\tau^\alpha - \sigma^\alpha) \vee (\varepsilon_T(\tau - \sigma)) \} \end{aligned}$$

since for  $a, b, c, x, y > 0$ ,  $c(x \vee y) - ax - by \geq (c - (a + b)) \cdot (x \vee y)$  (consider two cases:  $x \geq y, x < y$ .) Combining (4.10) and (4.11), recalling the definition of  $\eta_T$ , and using

$$\tau^\alpha - \sigma^\alpha = \int_\sigma^\tau \alpha s^{\alpha-1} ds \geq \int_0^{\tau-\sigma} \alpha s^{\alpha-1} ds = (\tau - \sigma)^\alpha$$

gives

$$(4.12) \quad T^g(\tau - \sigma)^b T^{1/2-\alpha} \geq DT^{-\nu} + (\gamma/2)(\tau - \sigma)^\alpha .$$

We will now show that our choice of  $\nu$  makes this inequality impossible for large  $T$ . To do this we observe that using  $D$  to denote dumb constants whose values are unimportant

$$\begin{aligned} T^g h^b T^{1/2-\alpha} &\leq DT^{-\nu} \quad \text{when } h \leq D' T^{-(\nu+g-\alpha+1/2)/b} \\ T^g h^b T^{1/2-\alpha} &\leq (\gamma/2) h^\alpha \quad \text{when } h \geq D T^{-(\alpha-g-1/2)/(\alpha-b)} \end{aligned}$$

Our choice of  $v$  and  $b$  in (4.4) implies that

$$(4.13) \quad \frac{v}{b} < \frac{\alpha(\alpha - g - 1/2)}{b(\alpha - b)} = \frac{\alpha - g - 1/2}{\alpha - b} + \frac{\alpha - g - 1/2}{b}$$

so

$$\frac{v + g - \alpha + 1/2}{b} < \frac{\alpha - g - 1/2}{\alpha - b}$$

and the inequality in (4.12) is impossible for large  $T$ . This shows that when  $|B_t - B_s| \leq T^g |t - s|^b$  for all  $0 \leq s \leq t \leq 1$  it is impossible to have  $x_t = \phi_1(t)$  for  $t \leq 1$  and the proof is complete.

### 5 Lower bound for fat tailed f

Throughout this section we will suppose

- (i)  $|f(x)| \leq M$
- (ii)  $f(x)$  is decreasing for  $x \in [q, \infty)$
- (iii)  $x^\beta f(x) \rightarrow l > 0$  as  $x \rightarrow \infty$  where  $0 < \beta < 1$
- (iv)  $f(x) \geq 0$  and  $f(0) > 0$ .

The proof of Theorem 4 is similar to that of Theorem 3 but requires more computation. As in the last section we begin by making a number of choices whose purposes will become clear later. After we have enough definitions we will explain the idea behind the proof. Let  $\varepsilon > 0$  and  $c \in (0, c_0)$ . We choose  $\delta > 0$  to satisfy

$$(5.1) \quad \gamma \equiv (1 - \delta)^2 c_0^{1+\beta} c^{-\beta} - c > 0,$$

and then pick  $N$  large enough so that

$$(5.2) \quad f(x) \geq l(1 - \delta)x^{-\beta} \quad \text{for } x \geq N.$$

Next choose  $b \in (0, 1/2)$ ,  $g > 0$ ,  $0 < \lambda < v < \alpha$  to satisfy

$$(5.3) \quad -\frac{g\alpha}{1 - b} + \frac{\alpha}{2(1 - b)} > \lambda$$

$$(5.4) \quad -\frac{g}{1 - b} + \alpha - \frac{1 - 2b}{2(1 - b)} > \lambda$$

$$(5.5) \quad \frac{1/2 - g}{(1 - b)} - v\beta > \lambda(\alpha - 1)/\alpha$$

$$(5.6) \quad v < \{\alpha - g - 1/2 - \lambda b(\alpha - 1)/\alpha\}/(1 - b).$$

To see that such choices can be made, note that when  $\lambda = 0$ ,  $g = 0$ , and  $b = 1/2$  we can pick  $v$  small enough so that inequalities (5.3)–(5.6) hold strictly.

Let  $x_t = T^{-\alpha}X(tT)$ ,  $a_T = T^{-\lambda}$ ,  $\eta_T = T^{-\nu}$  and

$$\begin{aligned} \phi_0(t) &= ct^\alpha - a_T & \phi_1(t) &= \phi_0(t) - \eta_T \\ \tau &= \inf\{t: x_t = \phi_1(t)\} \\ \sigma &= \sup\{u < \tau: x_u = \phi_0(u)\} \\ \rho &= \inf\{t: x_t = \phi_0(t)\} \\ t_T &= \inf\{t: \phi_0(t) = 0\} = c^{-1/\alpha}T^{-\lambda/\alpha}. \end{aligned}$$

Using Lemma 4.1 with  $k > 1/g$ , the Borel Cantelli lemma, and Brownian scaling it follows that (almost surely) for large  $T$

$$(*) \quad T^{-1/2}|B_{sT} - B_{tT}| \leq T^g(t - s)^b \quad \text{for all } 0 \leq s < t \leq 1$$

Our aim will be to show that if  $(*)$  holds and  $T$  is large then  $\tau \leq 1$  is impossible. We will do this in two steps. First, we will show  $\rho \geq t_T$  and then we will show that  $t_T \leq \sigma < \tau \leq 1$  is impossible. In each part of the proof we will use the assumption  $f(0) > 0$  to give a lower bound on the rate of growth of  $X_t$ .

**Lemma 5.1.** *If  $(*)$  and  $T \geq T_0(g, b)$ , we must have  $\rho \geq t_T$ .*

*Proof.* When  $t < t_T$ ,  $\phi_0(t) \leq 0$ . To get a lower bound on  $x_t$  we observe that (3.7) with  $s = 0$  and  $t = u$  implies

$$X_u \geq \frac{1}{2}uA^{1/2} + \inf_{r \leq u} (B_u - B_r) + \inf_{r \leq u} B_r - 1.$$

Changing the time scale  $u = tT$ , using  $(*)$ , and dividing by  $T^\alpha$  gives

$$x_t = T^{-\alpha}X(tT) \geq \psi(t) \equiv T^{-\alpha}(tTA^{1/2}/2 - 2T^{1/2}T^gt^b - 1).$$

Let  $\kappa = (4/A^{1/2})^{1/(1-b)}$ . When  $t = \kappa T^{(g-1/2)/(1-b)}$ ,

$$tTA^{1/2}/2 = 2T^{1/2}T^gt^b.$$

So if  $u_T = 2\kappa T^{(g-1/2)/(1-b)}$ ,  $t \geq u_T$ , and  $T$  is large then  $\psi(t) > 0$ . To take care of  $[0, u_T]$  we notice that over this interval

$$\begin{aligned} \psi(t) &\geq -T^{-\alpha}(2T^{1/2}T^gu_T^b + 1) = -T^{-\alpha}(DT^gT^{(1-2b)/2(1-b)} + 1) \\ \phi_0(t) &\leq D'T^{-\alpha/2(1-b)} - T^{-\lambda}. \end{aligned}$$

Now, (5.3) guarantees that for large  $T$ ,  $\phi_0(T) \leq -T^{-\lambda}/2$ , so using (5.4) we see that for large  $T$ ,  $\psi(t) > \phi_0(t)$  for  $t \in [0, u_T]$  and the proof of Lemma 5.1 is complete.

To finish the proof of Theorem 4 now it suffices to show:

**Lemma 5.2** *If  $(*)$  and  $T \geq T_1(g, b)$  then it is impossible to have  $t_T \leq \sigma < \tau \leq 1$ .*

*Proof.* Suppose that  $t_T \leq \sigma < \tau < 1$  and let  $\sigma < t < \tau$ . We want to get a lower bound on the drift at time  $t$ . To do this using (ii) we have to know  $X_t - X_s \geq q$  so our first step is to observe that (3.7) says

$$X_t - X_s \geq (t - s)A^{1/2}/2 + \inf_{s \leq r \leq t} (B_t - B_r) + \inf_{s \leq r \leq t} (B_r - B_s) - 1,$$

so it follows from the proof of Lemma 5.1 that if  $t - s \geq u_T$  and  $W_t$  satisfies (\*) then

$$(5.7) \quad X_{t,T} - X_{s,T} \geq (DT^{((1-2b)/2(1-b))+(1+b)\theta} - 1) \geq q$$

for large  $T$ . Let  $v = t - u_T$ . To estimate the drift of  $x_t = T^{-\alpha}X(tT)$  we observe that  $x_t \leq \phi_0(t)$  and if  $s \leq v$ ,

$$q \leq T^\alpha(x_t - x_s) \leq T^\alpha(\phi_0(t) - \phi_1(s)),$$

so (ii) and (iv) imply

$$(5.8) \quad \int_0^t f(T^\alpha(x_t - x_s))ds \geq \int_0^v f(T^\alpha(\phi_0(t) - \phi_1(s)))ds.$$

Using the definition of the  $\phi_i$  and then (5.2) we see that for large  $T$  the right hand side of (5.8) is equal to

$$(5.9) \quad \int_0^v f(T^\alpha(ct^\alpha - cs^\alpha + \eta_T))ds \geq \frac{1 - \delta}{c^\beta T^{\alpha\beta}} \int_0^v \frac{l ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} \\ = \frac{1 - \delta}{c^\beta T^{\alpha\beta}} \left[ \int_0^t \frac{l ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} - \int_v^t \frac{l ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} \right].$$

Changing variables  $s = tu$  and using  $\alpha\beta = 2 - \alpha$

$$(5.10) \quad \int_0^t \frac{l ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} = t^{\alpha-1} \int_0^1 \frac{l du}{(1 - u^\alpha + \eta_T/ct^\alpha)^\beta} \\ \geq (1 - \delta)t^{\alpha-1}\alpha c_0^{1+\beta}$$

for large  $T$ , since  $t > t_T$  and  $v > \lambda$  imply  $\eta_T/ct^\alpha \leq \eta_T/ct_T^\alpha \rightarrow 0$ . To estimate the second integral on the right hand side of (5.9) we observe that  $t^\alpha - s^\alpha \geq 0$  implies

$$(5.11) \quad \int_v^t \frac{l ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} \leq c^\beta l(t - v)/\eta_T^\beta = lc^\beta u_T/\eta_T^\beta.$$

Combining (5.9)–(5.11) and using the definition of  $\gamma$  in (5.1) gives

$$(5.12) \quad \int_0^v f(T^\alpha(ct^\alpha - cs^\alpha + \eta_T))ds \geq \frac{(\gamma + c)t^{\alpha-1}\alpha}{T^{\alpha\beta}} - \frac{(1 - \delta)2lu_T}{T^{\alpha\beta}\eta_T^\beta}.$$

Now we use (1.2) and the relationship  $2 - \alpha = \alpha\beta$  to write

$$(5.13) \quad -T^{1/2-\alpha}(B_{T\tau} - B_{T\sigma}) = -x_\tau + x_\sigma + T^{2-\alpha} \int_\sigma^\tau \int_0^t f(T^\alpha(x_t - x_s))ds \\ \geq -c(\tau^\alpha - \sigma^\alpha) + \eta_T + (\gamma + c)(\tau^\alpha - \sigma^\alpha) \\ - (1 - \delta)lu_T\eta_T^{-\beta}(\tau - \sigma)$$

by (5.8) and (5.12). Letting  $\zeta_T = (1 - \delta)lu_T\eta_T^{-\beta}$ , then using the convexity of  $x^\alpha$  and the fact that  $\sigma \geq t_T$  we can write (5.13) as

$$(5.14) \quad -T^{1/2-\alpha}(B_{T\tau} - B_{T\sigma}) \geq \gamma(\tau^\alpha - \sigma^\alpha) + \eta_T - \zeta_T(\tau - \sigma) \\ \geq \eta_T + (\gamma\alpha\sigma^{\alpha-1} - \zeta_T)(\tau - \sigma) \\ \geq \eta_T + (\gamma\alpha t_T^{\alpha-1} - \zeta_T)(\tau - \sigma).$$

Now (ignoring constants)

$$t_T^{\alpha-1} = T^{-\lambda(\alpha-1)/\alpha} \quad \eta_T^{-\beta} = T^{\nu\beta} \quad u_T = T^{(g-1/2)/(1-b)}$$

and we have supposed (see (5.5))

$$\frac{g}{(1-b)} - \frac{1}{2(1-b)} + \beta\nu < -\lambda(\alpha-1)/\alpha$$

so  $\xi_T = o(t_T^{\alpha-1})$  and the right hand-side of (5.14) is positive for large  $T$ . Using (\*) now, it follows that for large  $T$

$$(5.15) \quad T^g T^{1/2-\alpha} |\tau - \sigma|^b \geq \eta_T + \frac{\gamma\alpha}{2} t_T^{\alpha-1} (\tau - \sigma).$$

To see that this is impossible for large  $T$  we notice that

$$T^g T^{1/2-\alpha} h^b \leq \eta_T \quad \text{when} \quad h \leq DT^{(-g-\nu-1/2+\alpha)/b}$$

$$T^g T^{1/2-\alpha} h^b \leq \frac{\gamma\alpha}{2} t_T^{\alpha-1} h \quad \text{when} \quad h \geq DT^{(g+1/2-\alpha+\lambda(\alpha-1)/\alpha)/(1-b)}$$

and (5.6) implies that

$$\frac{-g-\nu-1/2+\alpha}{b} > \frac{g+1/2-\alpha+\lambda(\alpha-1)/\alpha}{1-b}.$$

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**Note added in proof.** M. Cranston and T. Mountford have proved Conjecture 1.