Stochastic Models of Growth and Competition

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In this paper we will describe recent results on four interacting particle systems that model the growth and competition of plant species or the spread of an epidemic or forest fire. In each system there is a collection of sites, the d-dimensional integer lattice, that at each time $t \in [0, \infty)$ can be in one of a finite number of states, so the state of the process at time t is a function $\xi_t : \mathbb{Z}^d \to \{0, 1, \ldots, k\}$. The time evolution is described by declaring that each site changes its state at a rate that depends upon the states of a finite number of neighboring sites. Here, we say that something happens at rate r if the probability of an occurrence between times t and $t + h$ is $rh + o(h)$.

1. The Basic Contact Process

In this model $\xi_t : \mathbb{Z}^d \to \{0, 1\}$, we think of 0 as vacant and 1 as occupied by a "particle," and the system evolves as follows:

(i) Particles die at rate one, give birth at rate β.
(ii) A particle born at x is sent to a y chosen at random from the $2d$ nearest neighbors $\{y : \|x - y\|_1 = 1\}$.
(iii) If y is occupied then the birth is suppressed.

Rule (iii) says that there can be at most one particle per site. This is a reasonable constraint if you are thinking of the spread of a plant species but this realism makes the model very difficult to analyze. Let ξ_t^A be the state at time t when initially $\xi_0^A(x) = 1$ if and only if $x \in A$, and let $\tau^A = \inf\{t : \xi_t^A \equiv 0\}$. If there are no particles then none can be born, so $\xi_t^A \equiv 0$ for all $t \geq \tau^A$. In words, the "all 0" state is an *absorbing state* and we say the system *dies out* at time τ^A.

The first question to be addressed is "When does the system have positive probability of not dying out starting from a single occupied site?" or "When is $P(\tau^{\{0\}} = \infty) > 0$?" It suffices to use a single occupied site as an initial configuration since $P(\tau^{\{0\}} = \infty) = 0$ implies $P(\tau^A = \infty) = 0$ for all finite A. Now, increasing β improves the chances for survival, so it should be clear that there is a critical value

$$\beta_c = \inf\{\beta : P(\xi_t^0 \not\equiv 0 \text{ for all } t) > 0\}.$$

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If we delete rule (iii) from the definition, the resulting system is called a branching random walk and has $\beta_c = 1$. That is, in order for a branching random walk to survive it is sufficient to have a birth rate larger than the death rate. Since in the contact process some of the birth rate will be wasted on occupied sites, this proves the easy half of the following result.

Theorem 1A. $1 < \beta_c(\mathbb{Z}^d) \leq 4$.

The lower bound is due to Harris (1974), the upper bound to Holley and Liggett (1978). Both bounds are reasonably accurate. Numerical results (see Brower, Furman, and Moshe (1978)) suggest that $\beta_c(\mathbb{Z}) \approx 3.299$ and $\beta_c(\mathbb{Z}^2) \approx 1.645$, and it has been shown (see Holley and Liggett (1981) or Griffeath (1983)) that $\beta_c(\mathbb{Z}^d) \to 1$ as $d \to \infty$.

Once it was established that $\beta_c \in (0, \infty)$, attention turned to "What does the process look like when it does not die out?" To answer this question we begin by introducing a special property of the contact process called *duality*

$$P(\xi_t^A(x) = 0 \text{ for } x \in B) = P(\xi_t^B(x) = 0 \text{ for } x \in A).$$

An immediate consequence of duality is that if we start from $\xi_0^1(x) \equiv 1$ then $\xi_t^1 \Rightarrow \xi_\infty^1$. Here, $\Rightarrow$ is short for *converges weakly* and means that

$$P(\xi_t^1(x) = 0 \text{ for } x \in B) \to P(\xi_\infty^1(x) = 0 \text{ for } x \in B)$$

for all finite sets B. To prove the weak convergence we set $A = \mathbb{Z}^d$ in the duality equation to get

$$P(\xi_t^1(x) = 0 \text{ for } x \in B) = P(\xi_t^B(x) \equiv 0)$$

which increases to a limit as $t \to \infty$, since "all 0" is an absorbing state. It follows from standard results (see Chapter 1 of Liggett (1985)) that ξ_∞^1 is a stationary distribution for the contact process, i.e., if we start the process with this distribution it has this distribution for all time.

At the other extreme, the point mass on the "all 0" state, δ_0, is a trivial stationary distribution. Letting $B = \{y\}$ and $t \to \infty$ in the duality relation gives

$$P(\xi_\infty^1(y) = 0) = P(\tau_t^{\{y\}} < \infty),$$

so $\xi_\infty^1 = \delta_0$ if the contact process dies out, but is a nontrivial stationary distribution if the contact process survives. The next result, called the *complete convergence theorem* implies that ξ_∞^1 is the only nontrivial stationary distribution.

Theorem 1B. $\xi_t^A \Rightarrow P(\tau^A < \infty)\, \delta_0 + P(\tau^A = \infty)\, \xi_\infty^1$.

In words, when the process dies out it looks dead, but when it survives and t is large it looks like the system starting from all sites occupied.

The last result took fifteen years to evolve to its current form. Harris (1974), Griffeath (1978), Durrett (1980), Durrett and Griffeath (1982), and Durrett and

Schonmann (1987) proved increasingly more general results before Bezuidenhout and Grimmett (1990) finished the problem and in addition proved

Theorem 1C. *When* $\beta = \beta_c$, $P(\tau^{\{0\}} = \infty) = 0$.

In words, the contact process dies out at the critical value. For applications (including some we will make in this paper) it is worthwhile to note that all the results in this section hold if (ii) is replaced by

(ii) A particle born at x is sent to a y chosen at random from $x + \mathcal{N}$.

if we assume $\mathcal{N}$ is (a) *symmetric* with respect to reflection in any coordinate plane, and (b) *irreducible*, i.e., the group generated by $\mathcal{N}$ is $\mathbb{Z}^d$.

2. Multitype Contact Processes

It is well known, even to mathematicians, that there is more than one type of plant, so it is natural to generalize the contact process to have two (or more) types of particles. In this model, the state at time t $\xi_t : \mathbb{Z}^d \to \{0, 1, 2\}$ and we think of 0 as vacant and 1 and 2 as occupied by pine and maple trees respectively. With this in mind we formulate the evolution as follows:

(i) Particles of type i die at rate one, give birth at rate β_i.
(ii) A particle born at x is sent to a y chosen at random from $x + \mathcal{N}$ where $\mathcal{N}$ is symmetric and irreducible.
(iii) If y is occupied then the birth is suppressed.

When only one type of particle is present the system reduces to the basic contact process so if $\beta_1, \beta_2 > \beta_c(\mathbb{Z}^d)$ then there are three trivial equilibria: δ_0, μ_1 and μ_2, where μ_i is the limit starting from $\xi_t(x) \equiv i$. The main question to be answered about the new system is: "Is there a nontrivial stationary distribution?", i.e., one that concentrates on configurations that contain both 1's and 2's. The first result is a negative one.

Theorem 2A. *If* $\beta_1 > \beta_2$ *then there are no nontrivial translation invariant stationary distributions.*

Here translation invariant means that that the distribution is invariant under spatial shifts. This result and the others in this section are from Claudia Neuhauser's (1990) thesis. We conjecture that Theorem 2A holds without the assumption of translation invariance but that assumption is often difficult to remove. Note that Harris proved Theorem 1B for translation invariant initial distributions in 1974 but the general case was settled 15 years later.

Restricting our attention now to the special case $\beta_1 = \beta_2 > \beta_c(\mathbb{Z}^d)$, we have

Theorem 2B. *In dimensions* $d \le 2$, *for any initial configuration, we have* $P(\xi_t(x) = 1, \xi_t(y) = 2) \to 0$ *for all* $x, y \in \mathbb{Z}^d$, *so all stationary distributions are trivial.*

Theorem 2C. *In dimensions $d \geq 3$, there is a one parameter family of stationary distributions ν_θ, $\theta \in [0,1]$, and all translation invariant stationary distributions are convex combinations of the ν_θ.*

As in the voter model, (see Liggett (1985) Chapter V or Durrett (1988) Chapter 2), the dichotomy between the behavior in $d \leq 2$ and $d \geq 3$ comes from the fact that random walks are recurrent in the first case and transient in the second. The stationary distributions are constructed by starting the system from an initial product measure in which 1's have density θ and 2's have density $1-\theta$, i.e., $\xi_0(x)$ are independent and take values 1 and 2 with probabilities θ and $1-\theta$. The reader should note that while the basic contact process has a single nontrivial stationary distribution, the two color version has a one parameter family in $d \geq 3$.

3. Successional Dynamics

In this model we again have $\xi_t : \mathbb{Z}^d \to \{0,1,2\}$ but this time we think of 0 as vacant and 1 and 2 as occupied by a bush or tree respectively. With this interpretation in mind the dynamics are formulated as follows:

(i) Particles of type i die at rate one, give birth at rate β_i.
(ii) A particle born at x is sent to a y chosen at random from $\{y : \|x-y\|_1 \leq M\}$, where M is an integer.
(iii) If $\xi_t(y) \geq \xi_t(x)$ then the birth is suppressed.

In words, trees can give birth onto sites occupied by bushes but not conversely. In biological terms the two species are part of a successional sequence. When only one type of particle is present, the system reduces, as in the last example, to a contact process so if $\beta_1, \beta_2 > \beta_c$ then there are three trivial equilibria: δ_0, μ_1, and μ_2, where μ_i is the limit starting from $\xi_t(x) \equiv i$.

Again, the main question to be answered is: "Are there nontrivial stationary distributions?" or more briefly "Is coexistence possible?" Our first answer is

Theorem 3A. *If $d = 1$ and $M = 1$ then for any initial configuration we have $P(\xi_t(x) = 1, \xi_t(y) = 2) \to 0$ as $t \to \infty$ for all $x, y \in \mathbb{Z}$ so there is no coexistence.*

This result can be proved by drawing a picture of a "typical" realization of the process starting with a single 2

$$0\,0\,1\,0\,1\,0\,2\,0\,2\,2\,0\,2\,0\,0\,0\,2\,1\,0\,1\,0\,0\,1$$

and checking that since $M = 1$ there can never be a 1 between the leftmost and rightmost 2's. If the 2's do not die out, then the ends of the interval of 2's go to $-\infty$ and ∞ respectively (see Durrett (1980)) and the 1's get crowded out. In general either (a) all the 2's die out, or (b) some 2 starts an interval that grows forever. In either case $P(\xi_t(x) = 1, \xi_t(y) = 2) \to 0$ as $t \to \infty$.

We believe that coexistence is possible in all other cases

Conjecture 3A. *If $d > 1$ or $M > 1$ then coexistence is possible when $\beta_2 = \beta_c + \varepsilon$ and β_1 is large.*

The main trouble with proving this conjecture is that coexistence can only occur near the critical value. It is not hard to show that if $\beta_2 > \beta(d, M)$ then there is no coexistence for any $\beta_1 \leq \infty$. Somewhat surprisingly, this problem which is difficult to solve when $d = 1$ and $M = 2$, or $d = 2$ and $M = 1$ turns out to be more tractable when M is large. In addition to proving Theorem 3A, Durrett and Swindle (1990) have shown

Theorem 3B. *If $\beta_1 > \beta_2^2 > 1$ then coexistence occurs for large M.*

To explain the last conclusion we need to introduce the long range contact process, a modification of the basic contact process in which (ii) is changed to:

(ii) A particle born at x is sent to a y chosen at random from $\{y : \|x-y\|_1 \leq M\}$.

If we write $\beta_c(M)$ to indicate the dependence of the critical value on M and use ξ^1_∞ to denote the limit starting from all 1's then we have

Theorem 3C. *As $M \to \infty$, $\beta_c(M) \to 1$. Furthermore, if $\beta > 1$ then ξ^1_∞ converges weakly to a product measure with density $(\beta - 1)/\beta$.*

This result (for the neighborhood $\{y : \|x - y\|_\infty \leq M\}$) was proved by Bramson, Durrett, and Swindle (1989) who identified the rate at which $\beta_c(M)$ approached 1. A simpler and more general proof, which does not give the right rate, can be found in Durrett (1989).

To explain the condition in Theorem 3B, observe that $\eta_t = \{x : \xi_t(x) = 2\}$ is a long range contact process, so if M is large and we are in equilibrium, η_t is approximately a product measure with density $(\beta_2 - 1)/\beta_2$. If the 2's were exactly that product measure, a 1 would die at rate $1 + \frac{\beta_2-1}{\beta_2}\beta_2$, (the second term representing births onto the site by 2's) and give birth at rate β_1/β_2 (the site must not be occupied by a 2 for a successful birth to occur). So for coexistence to occur we need $1 + \frac{\beta_2-1}{\beta_2}\beta_2 < \beta_1/\beta_2$ or $\beta_1 > \beta_2^2$. The careful reader will have noted that we have just argued the condition is necessary while Theorem 3B proves it is sufficient. Having faith in the heurisitc argument we make

Conjecture 3B. *If $\beta_1 < \beta_2^2$ then there is no coexistence for large M.*

Remark. The heuristic argument generalizes easily to show that if the two particles die at different rates then we need

$$\delta_1 + \frac{\beta_2 - \delta_2}{\beta_2}\beta_2 > \beta_1 \frac{\delta_2}{\beta_2}$$

and the proof of Theorem 3B generalizes to show that this condition is sufficient. It is natural to generalize the multitype contact process in this way but we do not know how to prove any results in that generality. The naive guess is that

$\beta_1/\delta_1 > \beta_2/\delta_2$ is right hypothesis for Theorem 2A. We believe this is correct but have no idea how to prove it.

Having discussed the existence of nontrivial stationary distributions, we turn to the question of uniqueness. Durrett and Møller (1991) have proved a "complete convergence theorem." To state their result let δ_0, μ_1, and μ_2 be the trivial stationary distributions mentioned at the beginning of this section. Let μ_{12} be the nontrivial stationary distribution constructed in Theorem 3B. Let $\eta_t = \{x : \xi_t(x) = 1\}$, $\zeta_t = \{x : \xi_t(x) = 2\}$, $\tau_1 = \inf\{t : \eta_t = \emptyset\}$, and $\tau_2 = \inf\{t : \zeta_t = \emptyset\}$.

Theorem 3C. *If $\beta_1 > \beta_2^2 > 1$ and M is large then*

$$\begin{aligned}\xi_t \Rightarrow\, & P(\tau_1 < \infty, \tau_2 < \infty)\, \delta_0 + P(\tau_1 = \infty, \tau_2 < \infty)\, \mu_1 \\ & + P(\tau_1 < \infty, \tau_2 = \infty)\, \mu_2 + P(\tau_1 = \infty, \tau_2 = \infty)\, \mu_{12}\,.\end{aligned}$$

In words if the 1's and/or 2's die out we end up with a trivial stationary distribution in which one or zero types of particles are present. If both the 1's and 2's survive and t is large, the system looks like μ_{12} so that is the only nontrivial stationary distribution. The value of M required for Theorem 3C is larger than that for Theorem 3B which is enormous. With more work this difference might be eliminated but the interesting problem is to show

Conjecture 3C. *The complete convergence theorem holds whenever coexistence occurs.*

4. An Epidemic Model

Our fourth system is a process $\xi_t : \mathbb{Z}^2 \to \{0, 1, 2\}$ that has been used to model the spread of epidemics and forest fires. In the epidemic interpretation $0 =$ healthy, $1 =$ infected, $2 =$ removed $=$ immune or dead. In the forest fire interpretation, $0 =$ alive, $1 =$ on fire, and $2 =$ burnt. With these interpretations in mind, we formulate the dynamics as follows:

(i) A burning tree sends out sparks at rate β.
(ii) A spark emitted from x flies to one of the four nearest neighbors $\{y : \|y - x\|_1 = 1\}$ chosen at random. If the spark hits a live tree, the tree catches fire and begins immediately to emit sparks.
(iii) A tree remains on fire for an exponential amount of time with mean 1 then becomes burnt.
(iv) Burnt trees come back to life at rate α.

At first glance, the spontaneous re-appearance of trees may not seem reasonable. In the epidemic interpretation this is quite natural, however. Consider a disease like measles that upon recovery confers lifetime immunity. New susceptibles are born and immune individuals die. We combine the two transitions into the one in (iv) to keep a constant population size.

When $\alpha = \infty$, sites change instantaneously from 2 to 0 and the result is the contact process. At the other extreme, $\alpha = 0$, is the so-called "spatial epidemic with removal" in which regrowth is impossible. We begin by considering the behavior of our processes starting with a single burning tree at the origin in the midst of an otherwise virgin forest, i.e., $\xi_0^0(0) = 1$, $\xi_0^0(x) = 0$ for $x \neq 0$. Let $\eta_t^0 = \{x : \xi_t^0(x) = 1\}$, let $\zeta_t^0 = \{x : \xi_t^0(x) = 2\}$, and define a critical value by

$$\beta_c(\alpha) = \inf\{\beta : P(\zeta_t^0 \neq \emptyset \text{ for all } t) > 0\}.$$

Cox and Durrett (1988) considered the case $\alpha = 0$ and showed

Theorem 4A. *If $\beta > \beta_c(0)$ then there is a nonrandom convex set D so that on $\{\eta_t \neq \emptyset$ for all $t\}$ we have $\zeta_t^0 \approx \zeta_\infty^0 \cap tG$, and $\eta_t^0 \approx t\partial G$. To be precise, for any $\varepsilon > 0$ the following inequalities hold for large t*

$$\zeta_\infty^0 \cap (1-\varepsilon)tG \subset \zeta_t^0 \subset (1+\varepsilon)tG$$
$$\eta_t^0 \subset (1+\varepsilon)tG - (1-\varepsilon)tG\,.$$

In words, this result says that the fire expands linearly and has an asymptotic shape. The statement is made contorted by the fact that the set of trees that will ever burn, ζ_∞^0, is not all of $\mathbb{Z}^d$. Thus what we prove is that when t is large, ζ_t^0 is contained in $(1+\varepsilon)tG$ and (if nonempty) contains all the points of ζ_∞^0 in $(1-\varepsilon)tG$.

When $\alpha = 0$ the system cannot have a nontrivial stationary distribution but Durrett and Neuhauser (1991) have shown

Theorem 4B. *If $\beta > \beta_c(0)$ and $\alpha > 0$ then there is a nontrivial stationary distribution, i.e., one that assigns no mass to "all healthy" state.*

The last result illustrates some of the frustrations in "applied probability." The proof is intricate and required several months to put down on paper, but we have been repeatedly told by physicists and biologists that the conclusion is obvious. In view of our difficulties in proving existence the reader should not be surprised to learn that we have little to say about uniqueness.

Conjecture 4C. *If $\beta > \beta_c(\alpha)$ then there is a unique nontrivial stationary distribution.*

In the first three examples we have had varying degrees of success in identifying the set of stationary distributions. In each of those cases however there is a useful "duality equation" and we have not been able to find one here.

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