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ASYMPTOTIC CRITICAL VALUE FOR A COMPETITION MODEL[1]

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In this paper we study a model of the competition of annual and perennial plants proposed by Crawley and May. Specifically, we calculate the asymptotic behavior of the critical value for coexistence in the biologically reasonable limit in which annual seeds are dispersed over a large number of sites. Our results are related to earlier work of Durrett and Swindle and prove a conjecture made in that paper.

1. Introduction. In this paper we will discuss a slight modification of a model of competition between annuals and perennials, which was introduced by Crawley and May (1987). In their own words, the model may be described as follows:

1. There are two plant species: (a) an annual invading only by seed; and (b) a perennial, invading only by lateral spread (through the production of "ramets").
2. The plants exist in a spatially uniform environment in which habitable sites (cells) are distributed in a hexagonal pattern. This is the simplest tesselation of the plane, and is selected for convenience rather than as a quantitatively accurate description of the spatial spread of real plants.
3. The size of a cell is such that it can accommodate a single individual of the annual species or a single ramet of the perennial species.
4. The time unit of the model is taken to represent one generation of the annual plant.
5. In any one generation, the perennial is capable of occupying only those cells that are immediately adjacent to it; it may, however, occupy any or all of its 6 first order neighboring cells in one generation.
6. In competition, perennial ramets always exclude the annual.
7. The annual has no effect on the demography of the perennial.
8. In any generation, the order of events is as follows: (a) death of the perennial ramets; (b) birth of the perennial ramets (occupation of empty cells); and (c) recruitment of annuals from seed.
9. Recruitment of annuals by seed can only occur in empty cells (i.e., into cells not containing a surviving or newly born perennial ramet).

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10. The probability of recruitment by annuals in any given empty cell is a function of the number of seeds produced in the previous generation. Specifically, we assume for each cell that recruitment occurs with probability 1 − exp(−mean number of seeds per cell), and that the entire crop of annual seeds is mixed and distributed at random over all cells whether empty or not.
11. Death of perennial ramets occurs in each generation with probability a, independent of the age of the ramet.
12. For each empty cell, the probability of being invaded by a perennial ramet from a given neighboring cell containing a surviving ramet is b, and if k out of the 6 first order neighbors contain surviving ramets, the probability that a cell is invaded is given by $1 - (1 - b)^k$.
13. To minimize edge effects, the universe has wrap-around margins, so that the upper neighboring row of the top row is the bottom row (and vice versa), and the left hand neighboring column of the leftmost column is the rightmost column (and vice versa).

The hexagonal lattice is nice because it is the geometry that allows us to pack in the largest number of circles per unit area. However, there is very little difference between the qualitative behavior of interacting particle systems on the hexagonal and on the square lattice, so for simplicity we will formulate the model on the square lattice, and since there is nothing special about two dimensions we will formulate the process in d dimensions. A second, more substantial, change that we will make is to formulate the model on all of Z^d, and replace item 10 by

10′. An annual plant at x will for each y with $\|y - x\|_\infty \le M$ send a seed to y with probability $c/(2M)^d$. The birth events from each x and to each y are independent.

With a little more work we could prove our results with 10′ replaced by the more realistic

10″. An annual plant at x will send a seed to y with probability $c\psi((y - x)/M)/M^d$, where ψ is a nonnegative continuous function with $\int\psi(z)\,dz = 1$. The birth events from each x and to each y are independent.

However, we will not give the details of the proof at this level of generality.

If we replace 10 by 10′ and use the labels, 0, 1 and 2 to indicate a site that is vacant, occupied by an annual or occupied by a perennial, then the rules above define a discrete time Markov process $\xi_t\colon \mathbf{Z}^d \to \{0, 1, 2\}$, which may be described algorithmically as follows:

(i) Each 2 independently survives with probability $1 - a$ and if it does, then independently and with probability b gives birth onto each of its $2d$ nearest neighbors. The sites occupied by a surviving 2 or at least one 2 offspring will be occupied by a 2 at the next time.

(ii) Independently for each x with $\xi_n(x) = 1$ and y with $\|y - x\|_\infty \le M$, there is a 1 offspring sent to y with probability $c/(2M)^d$. Any site not in state 2 at time $n + 1$ and receiving at least one 1 offspring will be in state 1.

(iii) Sites that are not the location of a surviving 2 or an offspring 1 or 2 will be in state 0.

Let $\eta_t = \{x: \xi_t(x) = 2\}$. Since 2's do not see any difference between 1's and 0's, η_t is a Markov process and, in fact, η_t is a discrete time version of the contact process; see Liggett (1985) or Durrett (1988). Well known results imply that if we let $\bar{\eta}_t$ be the process of 2's starting with all sites occupied by 2's at time 0, then for all A, $P(\bar{\eta}_t \cap A \neq \emptyset)$ decreases to a limit and hence $\bar{\eta}_t$ converges in law to an equilibrium distribution that is called the *upper invariant measure*. Let $\rho = \lim_{t\to\infty} P(x \in \bar{\eta}_t)$. ρ will be zero if a is too large or b is too small and in this case we say that the 2's *die out*. In this paper we are interested in the question of whether 1's and 2's can coexist in equilibrium, so we will suppose throughout the paper that the 2's *survive*, that is, $\rho > 0$.

To investigate the question of whether or not coexistence will occur, we will look at what happens when the range M is large. In this context the assumption that M is large is quite reasonable because the distance annual seeds are dispersed is large in comparison to the distance between our sites. To derive a condition for coexistence that will become exact in the limit as $M \to \infty$, we follow Crawley and May (1987) and observe that if we assume that the 2's are in equilibrium, then in the limit as $M \to \infty$ the fraction of sites not occupied by 2's that are occupied by 1's will satisfy

$$v_{t+1} = 1 - \exp(-c(1-\rho)v_t). \tag{1.1}$$

To explain this, we note that the number of sites occupied by 1's in the neighborhood of a fixed site x is about $v_t(1-\rho)(2M+1)^d$ and each such site has a probability of $c/(2M)^d$ of sending a seed to x. Thus, if M is large, the number of seeds that land at x has approximately a Poisson distribution with mean $cv_t(1-\rho)$ and the probability of at least one seed landing at x is $1 - \exp(-c(1-\rho)v_t)$.

Now the function $f(p) = 1 - \exp(-\alpha p)$ is increasing and concave with $f'(0) = \alpha$, and hence has a fixed point $p_\alpha \in (0,1)$ if and only if $\alpha > 1$. This leads us to our first result:

THEOREM 1. *Suppose* $c(1-\rho) > 1$. *If M is large, then there is a translation invariant stationary distribution in which the density of* 2*'s is ρ and the density of* 1*'s is close to* $(1-\rho)p_{c(1-\rho)}$, *that is, within a given* $\varepsilon > 0$ *of this value if* $M \geq M_\varepsilon$.

To construct the stationary distribution we start from an initial state ξ_0^{12} in which the 2's are in their equilibrium distribution and we have 1's at all the sites not occupied by 2's. Repeating the proof of (2.3) from Durrett and Moller (1991), one can see that as $t \to \infty$, $\xi_t^{12} \Rightarrow \xi_\infty^{12}$, a translation invariant stationary distribution. (Here $\Rightarrow$ denotes weak convergence, which in this setting is just convergence of finite dimensional distributions.) To show that when M is large, the density of 1's in equilibrium is close to its "mean field" value, we use the following observation. Scale space by $1/M$ and write our process as ξ_t: $Z^d/M \to \{0,1,2\}$.

PROPOSITION 1. *Suppose we consider a sequence of initial conditions ξ_0^M in which the 2's are in equilibrium and the density of 1's converges weakly to $(1-\rho)u(x)\,dx$, where $0 \le u(x) \le 1$ is continuous. That is, for any C^∞ function ϕ with compact support,*

$$\frac{1}{M^d} \sum_{\substack{x \in Z^d \\ \xi_0^M(x/M)=1}} \phi\left(\frac{x}{M}\right) \to \int \phi(y)(1-\rho)u(y)\,dy$$

in probability. Then as $M \to \infty$,

$$P(\xi_1^M(x) = 1 | \xi_1^M(x) \neq 2) \to 1 - \exp\left(-\frac{c}{2^d}\int_{\|y-x\|\le 1}(1-\rho)u(y)\,dy\right)$$

and asymptotically the states of sites not occupied by 2's are independent.

Here we have used a superscript M to indicate that we are considering a sequence of initial conditions. In most cases below this will be true, but we will suppress the dependence on M.

It is easy to see that the state at time 1 satisfies the assumptions on the state at time 0, so the iteration

$$Qu(x) = 1 - \exp\left(-\frac{c}{2^d}\int_{\|y-x\|\le 1}(1-\rho)u(y)\,dy\right) \tag{1.2}$$

describes the evolution of the densities of 1's up to some fixed time. Weinberger (1982) has analyzed a class of discrete iterations that include Qu and his results allow us to conclude that:

PROPOSITION 2. *Suppose $c(1-\rho) > 1$. There is a convex set D with $0 \in D^o$, the interior of D, so that if we start from an initial function $u(x) \in [0,1]$ that is positive on a set of positive measure, then for any compact $E \subset D^o$,*

$$\liminf_{n\to\infty} \inf_{x \in nE} Q^n u(x) \ge p_{c(1-\rho)}.$$

If we assume that $u(x)$ has compact support, then we can prove a convergence theorem that says $Q^n u(x) \approx p_{c(1-\rho)} 1_{nD}$.

Combining Propositions 1 and 2 with a "block argument," which we will now sketch, gives Theorem 1. The actual details will be somewhat different and fully explained later, so if you find this sketch confusing you can skip to the next paragraph. Let $\gamma > 0$, $\pi = p_{c(1-\rho)}$ and suppose that $u_0(x) = \pi - 2\gamma$ for $x \in [-1,1]^d$. Proposition 2 implies that if N is large, then $Q^N u(x) \ge \pi - \gamma$ for $x \in [-3,3]^d$. Break space into blocks $J(k) = \prod_{j=1}^d (k_j\beta, (k_j+1)\beta)$, where $k \in \mathbf{Z}^d$ and β is small, pick $\delta > 0$ so that $c(1-\rho-\delta) > 1$ and say that $[-1,1]^d$ is rich in 1's if in the particle system on $\mathbf{Z}^d/M$ there are at least $\sigma(1-\rho-\delta)(\beta M)^d$ 1's in each $J(k)$ contained in $[-1,1]^d$. If β is small and M is large, an initial state rich in 1's is like taking $u_0(x) = \pi - 2\gamma$ in the integral equation. Using the connection between the particle system and the integral equation suggested in Proposition 1, it is not hard to show that if M is large, then with high probability $-2e_1 + [-1,1]^d$ and $2e_1 + [-1,1]^d$ will be rich in 1's at time N. The last conclusion shows that if we start with one "pile of particles," then we will have with high probability two piles of particles at later times. A comparison with mildly dependent oriented percolation, invented by Bramson and Durrett (1988) and surveyed in Durrett (1991b) and Durrett (1993), now shows that the density of 1's does not go to 0. Once this is established, a standard trick (take the Cesaro average of the distribution at times 0 to T and extract a convergent subsequence) produces a stationary distribution.

Readers who are familiar with the first author's previous work may have already noted that: (1) if we wanted to simply conclude that when $c(1-\rho) > 1$ and M is large, $P(\xi_\infty^{12}(x) = 1) > 0$, then the methods of Durrett and Swindle (1991) would have sufficed; (2) the result as given in Theorem 1 could be proved using the methods of Durrett and Moller (1991). One reason for presenting a new proof here is that, while the argument of Durrett and Moller is based on a duality that holds only for the contact process, the proof given in Section 2 does not rely on the structure of the process of 2's. The same conclusion holds if we assume that the 2 process is translation invariant, finite range and attractive, if we start the 2's in the upper invariant measure and let ρ be its density. The second and main reason for the existence of this paper is that we can prove a converse to Theorem 1.

THEOREM 2. *Suppose $c(1-\rho) < 1$. If M is large, then the* 1*'s die out. That is, if ξ_0 contains infinitely many* 1*'s and* 2*'s, then for any x we have $\xi_t(x) \neq 1$ for all t sufficiently large.*

A corollary of this result is that there are no stationary distributions in which 1's and 2's are both present.

REMARK. The proof of Theorem 2, after some minor modifications, proves the converse of the result in Durrett and Swindle (1991); if $\beta_1 < \beta_2^2$ and M is large, then the 1's die out. We get explicit information about the limiting critical region in this case since both contact processes have long range interactions.

The first ingredient in the proof of Theorem 2 is the following observation, which we formulate for the scaled process ξ_t: $Z^d/M \to \{0,1,2\}$. Pick $\delta > 0$ so that $\mu = c(1-\rho+\delta) < 1$. We say that the space time region $\mathscr{B} = [-2M, 2M]^d \times [\kappa M^2/2, 2\kappa M^2]$ is *good for 2's* (here κ is a small positive number to be chosen later) if in each cube $[x_1 - 1, x_1 + 1] \times \cdots \times [x_d - 1, x_d + 1] \times \{t\}$ contained in $\mathscr{B}$ there are at least $(\rho - \delta)(2M+1)^d$ 2's. The equilibrium distribution for the 2's has exponentially decaying correlations. So by computing a high enough moment and using Chebyshev's inequality, it is easy to see that when the 2's are in equilibrium, then with high probability the region $\mathscr{B}$ will be good for 2's. Now when $\mathscr{B}$ is good for 2's, the 1's are dominated by a subcritical branching process and an easy estimate shows that, even if we suppose that there are 1's at all the sites outside $\mathscr{B}$ at all times, then when M is large, with high probability there will be no 1's in $[-M, M]^d \times [\kappa M^2, 2\kappa M^2]$.

The last paragraph gives one key ingredient in our block construction. The second is a consequence of work of Bezuidenhout and Grimmett (1990) and Durrett and Schonmann (1987): if J is a large integer and if somewhere in $[-1, 1]^d$ there is a translate of $[-J/M, J/M]^d$ that is completely occupied by 2's at time 0, then with high probability (a) $\mathscr{B}$ is good for 2's and (b) for each vector v with components $\in \{-1, 1\}$ there is a translate of $[-J/M, J/M]^d$ in $vM + [-1, 1]^d$ at time κM^2 that is completely occupied by 2's. The last result and a block construction give us a linearly growing region with no 1's and prove Theorem 2.

At this point the reader may be wondering: How do Theorems 1 and 2 relate to the work of Crawley and May? The main difference is that they considered a system that satisfied 10 rather than 10′ and hence were not concerned with the existence of stationary distributions. By considering the limit of large systems and reasoning with the Poisson distribution, Crawley and May derived equation (1.1) and concluded that $c(1 - \rho) > 1$ was needed for the annuals to survive. Unfortunately, they also applied the mean field reasoning that led to (1.1) to the contact process of 2's and computed the critical value and equilibrium density of the discrete time contact process, not realizing that their computations were incorrect.

The rest of the paper is devoted to the proof of Theorem 1 in Section 2 and of Theorem 2 in Section 3. These sections are independent of each other and can be read in any order.

2. Proof of Theorem 1. Let $Y = Z^d/M = \{x/M\colon x \in Z^d\}$ and consider $\xi_t\colon Y \to \{0, 1, 2\}$ so that we can more easily let $M \to \infty$. We begin by constructing the process from a collection of 0, 1 random variables, that we will refer to as a *graphical representation*. For x in Y let $\{U_n^x\colon n \in Z\}$ be a collection of i.i.d. 0, 1 random variables with $P(U_n^x = 1) = 1 - a$. If $U_n^x = 1$, then we draw a 2-arrow from $(x, n - 1)$ to (x, n); this means that if there is a 2 at x at time $n - 1$ it survives to time n. For $x, y \in Y$, with $\|x - y\| = 1/M$, let $\{T_n^{2,x,y}\colon n \in Z\}$ be a collection of i.i.d. 0, 1 random variables with $P(T_n^{2,x,y} = 1) = b$. If $T_n^{2,x,y} = 1$ we draw a 2-arrow from $(x, n - 1/2)$ to (y, n); this means that if x is occupied by a 2 at time $n - 1$ that survives up to time n, then there is a birth from x to y at time n. We put the tail of the arrow at height $n - 1/2$ rather than at height n to prevent particles just born at time n from giving birth at time n.

For $x, y \in Y$ with $\|y - x\|_\infty \le 1$, let $\{T_n^{1,x,y}\colon n \in Z\}$ be a collection of i.i.d. 0, 1 random variables with $P(T_n^{1,x,y} = 1) = c/(2M)^d$. If $T_n^{1,x,y} = 1$ we draw a 1-arrow from $(x, n - 1)$ to (y, n); this means that if x is occupied by a 1 at time $n - 1$, then there is a birth attempted to y at time n, which will succeed if y is not occupied by a 2 at time n. We assume that the $\{U_n^x\}$ and $\{T_n^{i,x,y}\}$ $(i = 1, 2)$ are mutually independent. Note that this construction allows us to start the process at any integer (possibly negative) time.

We will prove Theorem 1 by using a block construction. To do so, we need the following definitions. Let $\mathscr{L} = \{(m, n) \in \mathbf{Z}^2: m + n \text{ is even}\}$ be the *renormalized lattice*. Let $e_1 = (1, 0, \dots, 0)$ be the first unit vector. Let L, N and T be positive integers to be chosen later, and define

$$\phi(m, n) = (2mLe_1, nN) \quad \text{for } (m, n) \in \mathscr{L},$$
$$I = [-L, L]^d, \qquad I_m = 2mLe_1 + I,$$
$$\mathscr{B} = [-3L - N, 3L + N]^d \times [0, N], \qquad \mathscr{B}_{m,n} = \phi(m, n) + \mathscr{B},$$
$$\mathscr{A} = [-3L - N - T/M, 3L + N + T/M]^d \times [-T, N],$$
$$\mathscr{A}_{m,n} = \phi(m, n) + \mathscr{A}.$$

We will show that the parameters of our construction can be chosen so that: (i) if I_m is "rich in 1's" (a phrase we will define later) at time nN, then with probability $\geq 1 - \theta$ the cubes I_{m-1} and I_{m+1} are "rich in 1's" at time $(n + 1)N$, and (ii) the events we use to guarantee this are measurable with respect to the graphical representation $\mathscr{A}_{m,n}$. The box $\mathscr{A}_{m,n}$ intersects only finitely many other boxes, so if θ is small enough, then results in Durrett (1984) imply that the sites rich in 1's will dominate the wet sites in a supercritical oriented percolation on $\mathscr{L}$, and as we will explain at the end of this section, well-known percolation results will then allow us to construct the desired stationary distribution.

Let $0 < \beta, \delta < 1$ and for any k in $\mathbf{Z}^d$ define $J(k) = \prod_{j=1}^d (k_j \beta, (k_j + 1)\beta)$. Our first step is to show that with high probability there are not too many 2's in $\mathscr{B}_{m,n}$. Let G^2 ("good for 2's") be the event: on each space–time region $J(k) \times \{t\}$ included in $\mathscr{B}_{m,n}$ we have at most $(\rho + \delta)(\beta M)^d$ 2's, even if at time $nN - T$ we had 2's at all the sites of Z^d.

LEMMA 2.1. *Let $\varepsilon > 0$, N, β and δ be fixed. We can pick T so that if M is large, then*

$$P(G^2) > 1 - \varepsilon.$$

PROOF. Let $\eta_t^{-T,Y}$ be the set of 2's at time t when the configuration at time $-T$ has a 2 at each site of $Z^d/M = Y$. We can pick T so that at time 0, $P(x \in \eta_0^{-T,Y}) \leq \rho + \delta/2$. It is important to note that T depends on δ but does not depend on M. For a fixed k in Z^d, let $|\eta_t^{-T,Y} \cap J(k)|$ be the number of 2's in $J(k)$ at time t. We have that

$$(2.1) \qquad E\Big(\big|\eta_t^{-T,Y} \cap J(k)\big|^2\Big) = \sum_{x \in J(k)} P\big(x \in \eta_t^{-T,Y}\big) + \sum_{\substack{x, y \in J(k) \\ x \neq y}} P\big(x \in \eta_t^{-T,Y}, y \in \eta_t^{-T,Y}\big).$$

Since this is a nearest neighbor discrete time model on a lattice with spacing $1/M$, if $|x_i - y_i| > 2(T + t)/M$ for some i, then the events $\{\eta_t^{-T,Y}(x) = 2\}$

and $\{\eta_t^{-T,Y}(y) = 2\}$ depend on two space-time regions that do not intersect and are therefore independent. So the right-hand side of (2.1) is less than

$$(\rho + \delta/2)(\beta/M)^d + (4t + 4T + 1)^d(\beta M)^d + \sum_{x, y \in J(k)} P(x \in \eta_t^{-T,Y}) P(y \in \eta_t^{-T,Y}).$$

Since the last sum is the square of $E|\eta_t^{-T,Y} \cap J(k)|$, it follows that

$$\mathrm{Var}(|\eta_t^{-T,Y} \cap J(k)|) \le (\rho + \delta/2)(\beta M)^d + (4T + 4t + 1)^d(\beta M)^d.$$

Now $E|\eta_t^{-T,Y} \cap J(k)| \le (\rho + \delta/2)(\beta M)^d$ so it follows from Chebyshev's inequality that

$$P(|\eta_t^{-T,M} \cap J(k)| > (\rho + \delta)(\beta M)^d)$$

$$\le \frac{(\rho + \delta/2)(\beta M)^d + (4T + 4t + 1)^d(\beta M)^d}{((\delta/2)(\beta M)^d)^2}. \tag{2.2}$$

Let K_β be the number of regions of the type $J(k) \times \{t\}$ in $\mathscr{B}$. K_β does not depend on M, so if we pick M large enough, the right-hand side of (2.2) is less than ε/K_β. Applying the inequality $P(\bigcap_i A_i) \ge 1 - \sum_i P(A_i^c)$ with $A_i =$ "the ith region does not have too many 2's" finishes the proof of Lemma 2.1. □

The next step is to show that if G^2 occurs in a box, then the 1's are supercritical there. Inspired by Proposition 1, but not relying on the truth of that result, we define for a measurable function u taking values in $[0,1]$,

$$Qu(x) = 1 - \exp\left(-\frac{c}{2^d}\int_{\|y-x\|\le 1}(1 - \rho - \delta)u(y)\,dy\right).$$

Intuitively, $Qu(x)$ gives a lower bound on $P(\xi_1(x) = 1 | \xi_1(x) \ne 2)$ when the configuration at time 0 is good for 2's and a site x not occupied by 2's at time 0 is set equal to 1 with probability $u(x)$. To get a lower bound on the evolution of the 1's we will use a closely related operator R defined for functions u that are constant on each $J(k)$, for $k \in Z^d$. For k in Z^d, let $K(k) = \prod_{i=1}^d ((k_i + 1)\beta - 1, k_i\beta + 1)$ and note that each site in $J(k)$ can receive births from any site in $K(k)$. If $x \in J(k)$ we set

$$Su(x) = 1 - \exp\left(-\frac{c}{2^d}\int_{K(k)}(1 - \rho - \delta)u(y)\,dy\right) - \alpha,$$

$$Ru(x) = (Su(x))^+ = \max\{Su(x), 0\},$$

where $\alpha > 0$. The next lemma will explain these definitions.

Suppose that $u(x)$ is supported in $[-L, L]^d$. Let G_m^1, $0 \le m \le N$, be the event that at time m there are at least $(\beta M)^d(1 - \rho - \delta)R^m u(k)$ 1's in each $J(k) \subset [-3L - N, 3L + N]^d$, where $R^m u(k)$ is short for the value of $R^m u$ on $J(k)$ and we set $R^0 u = u$.

LEMMA 2.2. *Let $\varepsilon > 0$, N, δ, β and α be fixed. If M is large, then*

$$P(G_N^1 | G_0^1, G^2) \geq 1 - \varepsilon.$$

PROOF. It suffices to show that for $0 \leq m \leq N - 1$,

$$P(G_{m+1}^1 | G^2, G_m^1) \geq 1 - \varepsilon/N, \tag{2.3}$$

for then it follows that

$$P(G_N^1 | G^2, G_0^1) = \frac{P(G_N^1 \cap G^2 \cap G_0^1)}{P(G^2 \cap G_0^1)} \geq \prod_{m=0}^{N-1} \frac{P(G_{m+1}^1 \cap G^2 \cap G_m^1)}{P(G^2 \cap G_m^1)}$$

$$\geq \left(1 - \frac{\varepsilon}{N}\right)^N \geq 1 - \varepsilon,$$

the last inequality following from the inequality $P(\cap A_i) \geq 1 - \sum_i P(A_i^c)$ applied to independent events with probability $(1 - \varepsilon/N)$.

Given G_m^1, the number of 1's at time m in each $J(k)$ is at least

$$(\beta M)^d (1 - \rho - \delta) R^m u(k) = M^d \int_{J(k)} (1 - \rho - \delta) R^m u(y)\, dy, \tag{2.4}$$

since $R^m u(y)$ is constant on each $J(k)$. For x in $J(k)$, the set of sites that may attempt to give birth onto x includes $K(k)$. So given G_m^1, the total number of 1's at time m that may send a 1 to x is larger than the right-hand side of (2.4) with $J(k)$ replaced by $K(k)$, and hence

$$P(\xi_{m+1}(x) = 1 | \xi_{m+1}(x) \neq 2, G_m^1)$$

$$\geq 1 - \left(1 - \frac{c}{(2M)^d}\right)^{M^d \int_{K(k)} (1-\rho-\delta) R^m u(y)\, dy}$$

Since $(1 - x) \leq e^{-x}$ it follows that

$$\begin{aligned} &P(\xi_{m+1}(x) = 1 | \xi_{m+1}(x) \neq 2, G_m^1) \\ &\quad \geq 1 - \exp\left(-\frac{c}{2^d} \int_{K(k)} (1 - \rho - \delta) R^m u(y)\, dy\right). \end{aligned} \tag{2.5}$$

Now let $F_k = \{x \in J(k): \xi_{m+1}(x) \neq 2\}$. If we condition on the number of 1's in $K(k)$ at time m and ignore births from outside that set, then the events $\{\xi_{m+1}(x) = 1\}$ with $x \in F_k$ are independent and have a probability that is estimated in (2.5). Given G^2, there are at least $(1 - \rho - \delta)(\beta M)^d$ sites not occupied by a 2 in each $J(k) \times \{m + 1\}$ included in $\mathscr{B}$. Let $N_t(k)$ be the number of 1's in $J(k)$ at time t. (2.5) implies that

$$E(N_{m+1}(k) | G_m^1, G^2)$$

$$\geq (1 - \rho - \delta)(\beta M)^d \left(1 - \exp\left(-\frac{c}{2^d} \int_{K(k)} (1 - \rho - \delta) R^m u(y)\, dy\right)\right)$$

and since a 0, 1 valued random variable has variance $\leq 1/4$, we have

$$\mathrm{Var}(N_{m+1}(k) | G_m^1, G^2) \leq (\beta M)^d / 4.$$

Using Chebyshev's inequality it follows that

$$(2.6)\qquad \begin{aligned} P\big(N_{m+1}(k) &\le (1-\rho-\delta)(\beta M)^d SR^m u(k)|G^2, G^1_m\big) \\ &\le \frac{(\beta M)^d/4}{\big((1-\rho-\delta)(\beta M)^d \alpha\big)^2} \end{aligned}$$

and since $N_{m+1}(k) \ge 0$, the last inequality holds with SR^m replaced by R^{m+1}. If M is large enough, then the upper bound in (2.6) is smaller than ε/N divided by the number of $J(k)$ in $[-3L-N, 3L+N]$, proving (2.3) and completing the proof of Lemma 2.2. □

REMARK. The proof of Lemma 2.2 generalizes easily to prove Proposition 1.

The next step is to analyse the behavior of the iterates Q^m, using results of Weinberger (1982). His basic assumptions are given on pages 361–362. Most of these are trivial to verify for Q. The only one that requires any thought at all is his (3.1): there are constants $\pi_0 < \pi_1$ such that $Q\pi_0 = \pi_0$, $Q\pi_1 = \pi_1$, $Q\gamma > \gamma$ for γ in (π_0, π_1). For our operator Q, π_0 is always 0 and π_1 exists if and only if $c(1-\rho-\delta) > 1$. Taking δ sufficiently small so that the preceding inequality holds [we are assuming that $c(1-\rho) > 1$], Theorems 6.2 and 6.4 in Weinberger (1982) imply that:

PROPOSITION 3. *Suppose $c(1-\rho) > 1$. There is a convex set D with $0 \in D^o$, the interior of D, so that for any $\sigma > 0$ there is an L such that if $\inf_{x\in[-L,L]^d} u(x) \ge \sigma$, then for any compact $E \subset D^o$,*

$$(2.7)\qquad \liminf_{n\to\infty} \inf_{x\in nE} Q^n u(x) \ge \pi_1.$$

PROOF. Once it is established that D^o is not empty, convexity and symmetry imply $0 \in D^o$, and the convergence result follows from Theorem 6.2 in Weinberger (1982). To show that $D^o \ne \varnothing$, we use Theorem 6.4, which states that $D^o \ne \varnothing$ if there is a positive γ such that for all continuous u with $0 \le u \le \gamma$, we have that

$$(2.8)\qquad Qu(x) \ge \int u(x-y)\, l(dy),$$

where l is a nonnegative bounded measure on R^d which does not concentrate on any hyperplane of R^d and such that $\int l(dy) > 1$.

To check that (2.8) holds, let $c_0 = c(1-\rho-\delta)$ and $b \in (1, c_0)$. It is easy to see that there is a $\gamma > 0$ such that if $0 \le x \le \gamma$, then

$$(2.9)\qquad 1 - \exp(-c_0 x) = c_0 \int_0^x e^{-c_0 x}\, dx \ge bx.$$

If u takes values in $[0, \gamma]$, then $0 \le (1/2^d)\int_{\|y-x\|\le 1} u(y)\, dy \le \gamma$ and it follows from (2.9) that

$$Qu(x) \ge \frac{b}{2^d} \int_{\|y-x\|\le 1} u(y)\, dy,$$

which is an inequality of the type (2.8) with a measure $l(dy)$ having total mass equal to $b > 1$. This checks (2.8), which proves that $D^o \neq \emptyset$ and completes the proof of Proposition 3. □

REMARK. By working harder, one can improve Proposition 3 to the convergence result given in Proposition 2 in the Introduction. We content ourselves to sketch the proof since we will not use the stronger result in what follows.

PROOF OF PROPOSITION 2. By looking at Qu we can suppose that the initial function is continuous and $u(x_0) > 0$ at some point x_0 which without loss of generality we can suppose is 0. The first step in proving the result in this case is to construct a continuous function v supported in $(-\gamma, \gamma)$ with $0 \le v \le \gamma$ so that $v < Qv$. Iterating, we get that $Q^n v$ increases to a limit w with $\inf w(x) > 0$. Arguments via contradiction allow us easily to conclude that $\inf w(x) = \pi_1$ and that the convergence occurs uniformly on compact sets. Now if u is continuous and $u(0) > 0$, then for small γ, u is larger than v. Monotonicity implies that $Q^k u \ge Q^k v$ and hence Proposition 3 can be applied to $Q^k u$ for some k. □

Proposition 3 gives us valuable information about $Q^m u$. To get from this to information about $R^m u$ we will use the next result:

$$\sup_{x \in R^d} |Q^m u(x) - R^m u(x)| \le (cd\beta + \alpha)(1 + c + \cdots + c^{m-1}). \tag{2.10}$$

PROOF OF (2.10). First note that since

$$|e^{-x} - e^{-y}| = \left| \int_x^y e^{-z}\, dz \right| \le |x - y| \quad \text{for } x, y \in [0, \infty), \tag{2.11}$$

we have

$$|Qu(x) - Ru(x)| \le \frac{c(1-\rho-\delta)}{2^d} \left| \int_{\|x-y\| \le 1} u(y)\, dy - \int_{K(k)} u(y)\, dy \right| + \alpha.$$

For $x \in J(k)$ we have $K(k) \subset \{y\colon \|y - x\| \le 1\}$ and the difference between the volumes of these two subsets is less than $(2d)2^{d-1}\beta$. Therefore,

$$\sup_{x \in R^d} |Qu(x) - Ru(x)| \le cd\beta + \alpha,$$

proving the result when $m = 1$.

To do the inductive step we note that a second application of (2.11) implies that if $|u(x) - v(x)| \le \gamma$ for all x, then for $y \in J(k)$,

$$\begin{aligned} |Ru(y) + Rv(y)| &\le |Su(y) - Sv(y)| \\ &\le \frac{c(1-\rho-\delta)}{2^d} \int_{K(k)} |u(x) - v(x)|\, dx \le c\gamma. \end{aligned}$$

So using the triangle inequality and the result for 1 and for $m-1$,

$$|Q^m u - R^m u| \le |QQ^{m-1}u - RQ^{m-1}u| + |RQ^{m-1}u - RR^{m-1}u|$$
$$\le (cd\beta + \alpha) + c\{(cd\beta + \alpha)(1 + c + \cdots + c^{m-2})\},$$

completing the proof of (2.10). □

Let $\gamma > 0$, $\sigma = \pi_1 - 2\gamma$, pick L as dictated by Proposition 3 and let

$$u_0(x) = \begin{cases} \sigma, & \text{when } x \in [-L, L]^d, \\ 0, & \text{otherwise.} \end{cases}$$

From Proposition 3, it follows that there exists N so that

$$\min_{x \in [-3L, 3L]^d} Q^N u_0(x) \ge \pi_1 - \gamma. \tag{2.12}$$

Picking α and β small enough we get from (2.12) and (2.10) that

$$\min_{x \in [-3L, 3L]^d} R^N u_0(x) \ge \pi_1 - 2\gamma. \tag{2.13}$$

We are now ready to compare with oriented percolation on $\mathscr{L} = \{(m,n) \in \mathbf{Z}^2\colon m + n \text{ is even}\}$. Recall the cubes $I_m = 2mLe_1 + [-L, L]^d$ and $J(k) = \prod_{j=1}^d (k_j\beta, (k_j+1)\beta)$ defined at the beginning of the section. We say that I_m is *rich in 1's* at time t if there are at least $\sigma(1-\rho-\delta)(\beta M)^d$ 1's in each $J(k)$ contained in I_m. Combining (2.13) with Lemmas 2.1 and 2.2, we see that if M is large and I_m is rich in 1's at time nN, then with probability at least $1-\theta$, I_{m+1} and I_{m-1} are rich in 1's at time $(n+1)N$, and furthermore that our good events are determined by the coin flips in the graphical representation in $\mathscr{A}_{m,n}$. A standard argument [see the end of Section 3 in Durrett and Neuhauser (1994) or part C of Section 4 in Durrett (1991b) or Section 4 of Durrett (1993)] now shows that the (m,n) for which I_m is rich in 1's at time nN dominates the set of wet sites in a supercritical oriented percolation.

To construct our stationary distribution, we let ξ_t^{12} denote our process starting with 2's in equilibrium, and 1's filled in on the sites not occupied by 2's. Now a result of Durrett and Moller (1991) mentioned in the Introduction implies that as $n \to \infty$, $\xi_n^{12} \Rightarrow \xi_\infty^{12}$. Since the density of 2's in equilibrium is ρ and $\sigma = \pi_1 - 2\gamma < 1 - \rho - \delta$, the set $W_0 = \{m \in 2\mathbf{Z}\colon$ the interval I_m is rich in 1's in $\xi_0^{12}\}$ contains a positive fraction of the even integers. Let W_k be the set of sites at time k in the comparison percolation process. It follows from results in the last paragraph and in Durrett (1984) that if M is large, then

$$\liminf_{n\to\infty} P(0 \in W_{2n}) \ge 1 - \gamma. \tag{2.14}$$

Now ξ_{2nN}^{12} is translation invariant, so it follows from (2.14) and what it means to be rich in 1's that

$$P(\xi_{2nN}^{12}(x) = 1) \ge (1-\rho-\delta)(\pi_1 - 2\gamma) P(0 \in W_{2n}). \tag{2.15}$$

Combining (2.14) and (2.15) we have

$$P(\xi_\infty(x) = 1) \geq (1 - \rho - \delta)(\pi_1 - 2\gamma)(1 - \gamma). \tag{2.16}$$

Now π_1 is what we called $p_{c(1-\rho-\delta)}$ in the Introduction and δ and γ are arbitrary, so we have shown that the equilibrium density is not much less than $(1 - \rho)p_{c(1-\rho)}$ when M is large.

The asymptotic upper bound on the equilibrium density is easy to prove. Define a sequence q_n by $q_0 = 1$ and

$$q_n = 1 - \exp(-c(1 - \rho)q_{n-1}) \quad \text{for } n \geq 1.$$

From Proposition 1, it follows easily that

$$\lim_{M\to\infty} P(\xi_n^{12}(x) = 1) = (1 - \rho)q_n.$$

Since $P(\xi_n^{12}(x) = 1)$ is a decreasing function of n and $q_n \downarrow p_{c(1-\rho)}$, it follows that

$$\limsup_{M\to\infty} P(\xi_\infty^{12}(x) = 1) \leq (1 - \rho)p_{c(1-\rho)},$$

completing the proof of Theorem 1. □

3. Proof of Theorem 2. As in Section 2, we will let $Y = \mathbf{Z}^d/M = \{x/M: x \in \mathbf{Z}^d\}$ and consider $\xi_t: Y \to \{0, 1, 2\}$, so that we can more easily let $M \to \infty$. Before we can explain the idea behind the proof we need to introduce some notation. We will describe the intuition and give the proof first in the case $d = 1$ and then at the end of the section indicate the modifications needed to treat the general case. Let $\kappa > 0$ be a constant to be chosen in the proof of Lemma 3.1, let $T = \kappa M^2$ and define

$$\mathscr{B} = [-2M, 2M] \times [T/2, 2T], \qquad \mathscr{D} = [-M, M] \times [T, 2T].$$

We will show that we can pick J so that if M is large and some translate of $I = [-J/M, J/M]$ that lies in $[-1, 1]$ is fully occupied by 2's at time 0, then with high probability (i) there are no 1's in $\mathscr{D}$ and (ii) there are translates of I in $[-M - 1, -M + 1]$ and in $[M - 1, M + 1]$ that are fully occupied by 2's at time T. (See Figure 1.) Let $\mathscr{L} = \{(m, n): m + n \text{ is even}\}$ and say that (m, n) is wet if there is a translate of I in $[mM - 1, mM + 1]$ that is fully occupied by 2's at time nT, and the events in (i) and (ii) above occur translated by mM in space and by nT in time. The events we use to guarantee that (i) and (ii) occur will be chosen to have a finite range of dependence, so results from Durrett (1984) imply that if M is large, then the points on the renormalized lattice $\mathscr{L}$ at which these good events occur are sufficiently dense so that oriented percolation occurs. Our events have been chosen so that when percolation occurs, the dead regions $(mM, nT) + \mathscr{D}$ overlap and it follows easily that the 1's die out. In $d = 1$ this can be done by noting that on the renormalized lattice the leftmost and rightmost sites of the wet region at time n go to infinity linearly and there can be no 1's inside the vee-shaped region made by the dead strips corresponding to the leftmost and rightmost paths. In $d > 1$ this simple argument breaks down but can be replaced by a lemma

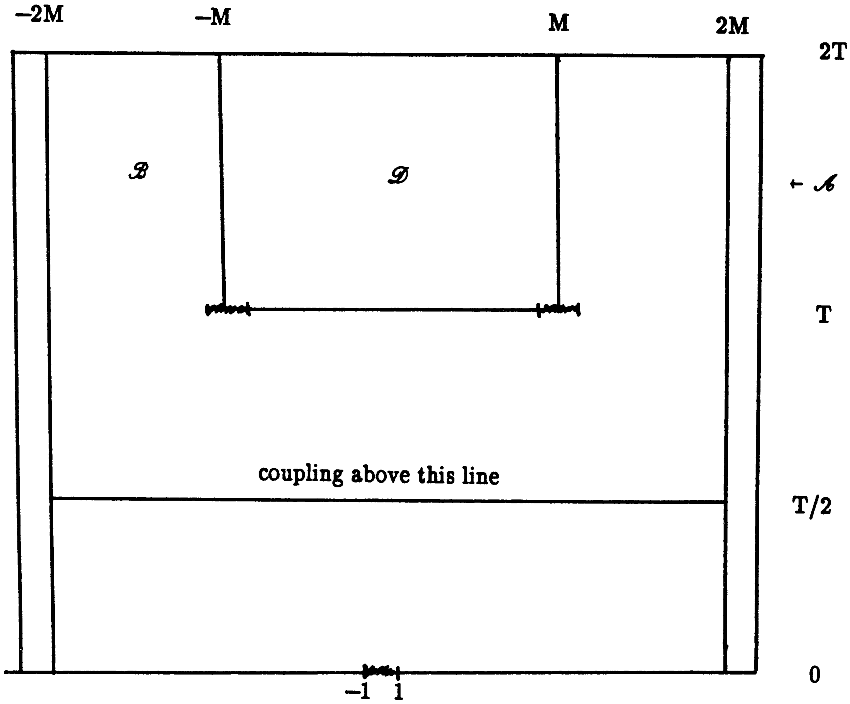

FIG. 1.

from Section 3 of Durrett (1992). We will give the details of these arguments at the end of the section.

To turn the last paragraph into a proof, we will first estimate the probability of our good event and then give the details of the comparison with oriented percolation on $\mathscr{L}$. Let $\eta_t^{t_0, A}$ be the set of sites occupied by 2's at time t when we start with A occupied at time t_0. When $t_0 = 0$ we omit it from our notation. The key to our construction is to prove the following.

($\star$) *Let $\varepsilon > 0$. We can find a constant κ and large integers J and M_0 so that for all $M \geq M_0$ and all translates I of $[-J/M, J/M]$ that lie in $[-1, 1]$, the following events happen with probability at least $1 - 10\varepsilon$. If we start with 2's on I at time 0, then*:

(i) $\eta_t^I(x) = \eta_t^Y(x)$ *for all* (x, t) *in* $[-2M, 2M] \times [T/2, 2T]$;

(ii) *there are translates of* $[-J/M, J/M]$ *in* $[-M-1, -M+1]$ *and in* $[M-1, M+1]$, *that are contained in* η_T^I;

(iii) *at all times between* $T/2$ *and* $2T$ *there are at least* $(\rho - \delta)(2M+1)$ *2's in each interval* $[x-1, x+1]$ *included in* $[-2M, 2M]$;

(iv) *even if there is a 1 at each site of* $[-2M, 2M] \cap I^c$ *at time 0 and 1's at all the sites outside* $\mathscr{B}$ *at all times* $\leq 2T$, *then there are no 1's in* $\mathscr{D}$.

PROOF. Let G_1, G_2, G_3 and G_4 be the good events described in (i), (ii), (ii) and (iv). We will estimate the probabilities of these events in Lemmas 3.1–3.4. To prepare for our remarks at the end of the section, we would like to observe that all these proofs easily generalize to $d > 1$ if we replace the intervals by suitable cubes.

LEMMA 3.1. *For M large enough*, $P(G_1) \geq 1 - \varepsilon$.

PROOF. This result is true because "in the supercritical contact process starting from a finite set, we get exponentially rapid convergence in the complete convergence theorem inside a linearly growing set." The remark in quotes is at the same time well known and also unknown, that is, it cannot be found in print. For completeness, we will indicate how it can be proved by combining, in a straightforward way, ideas from Bezuidenhout and Grimmett (1990) and Durrett and Schonmann (1987). It would take several pages to write out all the details, so we will content ourselves just to sketch the proof.

Since we are trying to prove a statement about the process of 2's, we can and will suppose that we are dealing with the ordinary discrete time contact process on $\mathbf{Z}$, which we denote by λ_t. Bezuidenhout and Grimmett (1990) proved that if the contact process is supercritical then we can pick a K, L and S so that if we start with a translate H of $[-K, K]$ occupied in $[-L, L]$, then with high probability at time S there will be translates of $[-K, K]$ occupied in $[-3L, -L]$ and in $[L, 3L]$ even if no births are allowed to occur on sites outside $[-4L, 4L]$.

Let $\hat{\lambda}_s^{t,B}$ be the usual dual process for the contact process starting from B occupied at time t and working backward s units of time, and recall that this has the same distribution as λ_s^B. Using the result quoted in the last paragraph and a "restart argument," as is done in Durrett and Schonmann (1987), one can show that there is an $a > 0$ so that for any finite set A and $|x| \leq at$,

$$(3.1) \qquad P\Big(\lambda_{t/2}^A \neq \varnothing,\ \hat{\lambda}_{t/2}^{t,\{x\}} \neq \varnothing,\ \lambda_{t/2}^A \cap \hat{\lambda}_{t/2}^{t,\{x\}} = \varnothing\Big) \leq Ce^{-\gamma t}.$$

Here and in what follows C and γ are positive finite constants (which in this case will depend on A) whose values are unimportant and will in general change from line to line. Let $\tau = \inf\{t\colon \lambda_t^A = \varnothing\}$ and $\lambda_t^A(x) = 1$ if $x \in \lambda_t^A$, 0 otherwise. Now $\lambda_t^A(x) \leq \lambda_t^Z(x)$, $\{\lambda_t^A(x) = 1\} = \{\lambda_{t/2}^A \cap \hat{\lambda}_{t/2}^{t,\{x\}} \neq \varnothing\}$, $\{\lambda_t^Z(x) = 1\} = \{\hat{\lambda}_t^{t,\{x\}} \neq \varnothing\}$ and $P(t/2 < \tau \leq t) \leq Ce^{-\gamma t}$. Combining the last four results with (3.1), it follows that

$$(3.2) \qquad P\big(\lambda_t^A \neq \varnothing,\ \lambda_t^A(x) \neq \lambda_t^Z(x)\big) \leq Ce^{-\gamma t}$$

when $|x| \leq at$. The last result says that, except for a set with exponentially small probability, either $\lambda_t^A = \varnothing$ or λ_t^A agrees with λ_t^Z on $[-at, at]$. This is a precise version of the statement in quotation marks at the beginning of the proof.

To complete the proof at this point, we have to improve (3.2) to a statement about the space–time box, but this is easy. Since there are only Ct^2 sites in $[t, 4t] \times [-at, at]$ and $P(t < \tau \le 4t) \le Ce^{-\gamma t}$, it follows that

$$P\big(\lambda_t^A \ne \varnothing, \lambda_s^A(x) \ne \lambda_s^Z(x) \text{ for some } t \le s \le 4t, |x| \le at\big) \le Ce^{-\gamma t}.$$

Let $A = [-J, J]$ and pick J large enough so that $P(\lambda_t^A \ne \varnothing \text{ for all } t) \ge 1 - \varepsilon/2$. Letting $\kappa = 4/a$ and setting $t = T/2 = \kappa M^2/2$, we have the desired result. (Recall that we have to scale space by M to go from λ_t to η_t.) □

LEMMA 3.2. *For M large enough*, $P(G_2) \ge 1 - 4\varepsilon$.

PROOF. Again, since we are trying to prove a statement about the process of 2's, we can and will suppose that we are dealing with the ordinary discrete time contact process on $\mathbf{Z}$, which we denote by λ_t. Let μ_2 be the upper invariant measure for λ_t. By the ergodic theorem, if λ_t^{eq} has distribution μ_2, then

$$\lim_{L\to\infty} \frac{1}{2L+1} \sum_{x=-L}^{L} 1_{\{x+[-J,J] \subset \lambda_t^{eq}\}} = P([-J, J] \subset \lambda_t^{eq}) > 0. \tag{3.3}$$

Returning to the process on the lattice $\mathbf{Z}/M$, we note that on G_1, at time T, η_t^I is coupled to η_t^Y whose law is "larger" than μ_2 in the sense that we can construct η_t^Y and a process η_t^{eq} with distribution μ_2 on the same probability space so that

$$\eta_t^Y \supset \eta_t^{eq}. \tag{3.4}$$

Combining (3.4) and (3.3) with Lemma 3.1, we see that with probability at least $1 - 2\varepsilon$ we can find the interval we seek in $[M - 1, M + 1]$. The same argument applies to $[-M - 1, -M + 1]$ and the proof is complete. □

LEMMA 3.3. *For M large enough*, $P(G_3) \ge 1 - 2\varepsilon$.

PROOF. In view of the result in Lemma 3.1, it suffices to show for the system started from all sites occupied that the event in (iii) happens with probability at least $1 - \varepsilon$.

As in the proof of Lemma 2.1, we will do a moment computation to estimate $|\eta_t^Y \cap [-1, 1]|$, but this time (a) there are CM^4 intervals to deal with and (b) the times involved are larger, so we have to (a) compute high moments to make the error probabilities small enough and (b) we have to use a trick to make the correlations small. The computation we are about to do is a straightforward generalization of one done in Section 13 of Durrett (1984). (There $l = 4$). We start with the trick referred to above: Let $s(M, t) = t - a \log M$ and let $\eta_t^{s(M,t),Y}$ be the process starting with all sites occupied at time $s(M, t)$. Monotonicity implies $\eta_t^{s(M,t),Y}(x) \ge \eta_t^Y(x)$ and it follows from duality and known results that

$$\begin{aligned} P\big(\eta_t^{s(M,t),Y}(x) = 1\big) - P\big(\eta_t^Y(x) = 1\big) &\le P(a \log M \le \tau^x < \infty) \\ &\le Ce^{-\gamma(a \log M)} \le CM^{-5} \end{aligned}$$

if a is large. The last bound implies that with high probability we have $\eta_t^{s(M,t),Y}(x) = \eta_t^Y(x)$ for all (x,t) in our space–time box, so using translation invariance in space and time and recalling that there are CM^4 intervals we are concerned with, the proof of the lemma will be complete when we show that

$$P\Big(\big|\eta^Y_{a\log M} \cap [-1,1]\big| \le (\rho - \delta)(2M+1)\Big) \le M^{-4.5}. \tag{3.5}$$

Let $\bar\eta(x) = \eta^Y_{a\log M}(x) - P(\eta^Y_{a\log M}(x) = 1)$. If there is an x_i so that $|x_i - x_j| > (2a\log M)/M$ for all $j \ne i$, then $\bar\eta(x_i)$ is independent of the $\bar\eta(x_j)$ and $E\prod_{k=1}^{2l}\bar\eta(x_k) = 0$. Let $S_M = \sum_{x\in[-1,1]}\bar\eta(x)$. As we will now explain, the last conclusion leads easily to

$$ES_M^{2t} \le C_l(2M+1)^l(4la\log M)^l. \tag{3.6}$$

To prove this, note that $|\prod_{k=1}^{2l}\bar\eta(x_k)| \le 1$ and in order for there to be a nonzero contribution, the points $x_1,\dots,x_{2l}$ must satisfy: (c) each x_i must have some other x_j within distance $2a\log M$ of it. To see that the right-hand side gives a bound on the number of sets of points that satisfy (c), note that if we draw an arc from i to j when $|x_i - x_j| \le 2a\log M$, then we get a graph with at most l connected components, and the triangle inequality implies that any two points in the same component are within a distance $(2l-1)2a\log M$. The number of sets that give rise to graphs with k components is at most $C_k(2M+1)^k(4la\log M)^{2l-k}$ and $k \le l$, so we have the desired bound. Taking $l = 5$ in (3.6) and using Chebyshev's inequality gives

$$P(S_M < -\delta(2M+1)) \le \big(CM^5(\log M)^5\big)/M^{10} \le M^{-4.5}$$

for large M. Since $P(\eta^Y_{a\log M}(x) = 1) \ge \rho$, we have proved (3.5) and the desired result follows. □

LEMMA 3.4. *For M large enough*, $P(G_4) \ge 1 - 3\varepsilon$.

PROOF. Pick $\delta > 0$ so that $\mu = c(1-\rho+\delta) < 1$ and fix a space–time configuration of 2's that satisfies (iii). In view of Lemma 3.3, it suffices to show that for each such configuration we have $P(G_4^c) \le \varepsilon$. To do this, we note that by virtue of (iii) it follows that for each site in the space–time box a 1 at that site will produce an average of $\mu < 1$ children that land in the box. The last observation suggests that we compare the set of 1's with a branching process. Let b_t be a process that evolves like ξ_t (with the same birth and death probabilities) but is more restrictive in that no births outside the space–time box $\mathscr{B}$ are allowed, and more liberal in that within the space–time box $\mathscr{B}$, a 1 for the process b_t can give birth to a 1 on a site already occupied by a 1. Let $b_t^{s,y}(x)$ be the number of 1's at x at time t for the process b_t beginning at time s with one 1 at y. We claim that for $(x,t) \in \mathscr{D}$,

$$P(\xi_t(x) = 1) \le \sum_{(y,s)\in\mathscr{C}} Eb_t^{s,y}(x), \tag{3.7}$$

where

$$\mathscr{C} = ([-2M-1, -2M] \times [T/2, 2T]) \cup ([2M, 2M+1] \times [T/2, 2T])$$
$$\cup([-2M, 2M] \times \{T/2\}).$$

To see this, note that a particle at x at time t must have a last ancestor in $\mathscr{C}$, and after that ancestor the line of descent stays in $\mathscr{B}$. To estimate the sum, note that $E(b_t^{s,y}(x)) = 0$ unless $s \le t - M$ (or $s = T/2$) since at least M births are needed to get a particle from outside $(-2M, 2M)$ to inside $[-M, M]$. Now $T/2 = kM^2/2 \ge M$ for large M, so if we let $N_t^{s,y} = \sum_x b_t^{s,y}(x)$, then for all the possible values of (s, y) we have

$$E(b_t^{s,y}(x)) \le EN_t^{s,y} \le \mu^{t-s} \le \mu^M.$$

The number of (s, y) pairs is smaller than CM^3 so

$$P(\xi_t(x) = 1) \le CM^3\mu^M$$

and the probability of a 1 somewhere in $\mathscr{D}$ is smaller than $CM^7\mu^M$, completing the proof of Lemma 3.4 and of (★). □

PROOF OF THEOREM 2. Let $\mathscr{L} = \{(m, n)\text{: } m + n \text{ is even}\}$ and say that (m, n) is wet if: (a) there is a translate of I in $[mM - 1, mM + 1]$ that is fully occupied by 2's at time nT; (b) there are no 1's in $(mM, nT) + \mathscr{D}$; and (c) there are translates of I in $[(m-1)M - 1, (m-1)M + 1]$ and in $[(m+1)M - 1, (m+1)M + 1]$ that are fully occupied by 2's at time $(n+1)T$. Lemmas 3.1–3.4 have shown that if (a) holds, then with high probability (b) and (c) also occur. To check that the events that we have used to guarantee that (b) and (c) occur have a finite range of dependence, suppose without loss of generality that $m = n = 0$. Now the evolution of η_t^I for $0 \le t \le T$ is determined by the coin flips in the space–time box $[-1 - (2T/M), 1 + (2T/M)] = [-1 - 2\kappa M, 1 + 2\kappa M]$, so all of our statements about 2's involve a finite range of dependence. As for the 1's, they are discussed only in Lemma 3.4, and that result is proved by considering a branching random walk that lives in $\mathscr{B} \cup \mathscr{C}$.

(★) implies that for any $\varepsilon > 0$, the events that guarantee (b) and (c) have probability at least $1 - 10\varepsilon$ when M is large. Let W_n be the set of m so that (m, n) can be reached from $(0, 0)$ by a path of wet sites that can only jump from (j, k) to $(j-1, k+1)$ or to $(j+1, k+1)$. Let $l_n = \inf W_n$ and let $r_n = \sup W_n$. If ε is small enough, then results in Section 10 and 11 of Durrett (1984) imply that with positive probability $W_n \ne \varnothing$ for all n (i.e., percolation occurs) and when this occurs we have

$$\limsup_{n\to\infty} l_n/n \le -a < 0, \qquad \liminf_{n\to\infty} r_n/n \ge a > 0.$$

Suppose that $W_n \ne \varnothing$, let $(i_0, 0), (i_1, 1), \dots, (i_n, n)$ be the leftmost path from $(0, 0)$ to (l_n, n) and let $(j_0, 0), (j_1, 1), \dots, (j_n, n)$ be the rightmost path from $(0, 0)$ to (r_n, n). The definition of a wet site implies that the regions $(i_k M, kT) + \mathscr{D}$ and $(j_k M, kT) + \mathscr{D}$ do not contain any 1's, and they overlap to form a connected vee-shaped region $\mathscr{V}$. Now $\mathscr{V}^c \cap \{\mathbf{R} \times (0,(n+2)T)\}$ consists of two unbounded components (the outside) and one or more bounded components (the inside). Since 1's can be dispersed at most a distance one and cannot appear spontaneously, there can be no 1's in the inside of $\mathscr{V}^c$.

The results in the last paragraph imply that if we can find a translate of the interval I in the initial configuration that is fully occupied by 2's, then with positive probability the 1's will die out. Now, if there are infinitely many 2's in the initial configuration, then by waiting J units of time we are sure to find an interval of 2's to begin our construction. If our initial try fails, then by waiting J units of time after it dies out we can find another interval to try again. Each trial has a positive probability of success independent of what happened earlier; so eventually we will succeed in creating a linearly growing region in which there are no 1's.

In $d > 1$ we cannot use our edge argument to show that the 1's die out, but must instead compare with a $(d+1)$ dimensional oriented percolation to show that if ε is small we get with positive probability a linearly growing cone that contains no 1's. The necessary percolation lemmas can be found in Section 3 of Durrett (1992). □

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