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Coexistence results for catalysts

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Summary. In this paper we consider a modification of Ziff, Gulari and Barshad's (1986) model of oxidation of carbon monoxide on a catalyst surface in which the reactants are mobile on the catalyst surface. We find regions in the parameter space in which poisoning occurs (the catalyst surface becomes completely occupied by one type of atom) and another in which there is a translation invariant stationary distribution in which the two atoms have positive density. The last result is proved by exploiting a connection between the particle system with fast stirring and a limiting system of reaction diffusion equations.

Mathematics Subject Classification: 60K35

1 Introduction

We will consider a model of a catalyst in which the state at time t, $\xi_t : \mathbf{Z}^d \to \{0,1,2\}$ where $0 =$ a vacant site, $1 =$ a carbon monoxide (CO) molecule attached to the surface, and $2 =$ a single oxygen atom (O) attached to the surface. Letting $\|x\|_1 = |x_1| + |x_2| + \ldots + |x_d|$ and declaring that x and y are neighbors if $\|x-y\|_1 = 1$ we can formulate the model as follows:

(i) Carbon monoxide molecules land at vacant sites at rate p.
(ii) An adjacent pair of vacant sites becomes occupied by two oxygen atoms at rate $q/2d$. This reflects the fact that oxygen molecules O_2 need two adjacent vacant sites to land, and when they land separate into two oxygen atoms attached to the surface.
(iii) Adjacent carbon monoxide molecules and oxygen atoms react at rate $r/2d$ producing a carbon dioxide molecule and resulting in two vacant sites.

The rate r is much larger than the rates p and q so Ziff et al. (1986) found it convenient to take $r = \infty$. In this case (think about the limit $r \to \infty$) when a reactant lands it instantaneously checks its neighbors to see if one is occupied by the opposite type and if so, reacts with one of them chosen at random. When $r = \infty$, one can by changing the time scale assume that $q/2 = 1 - p$. Ziff et al. (1986) investigated this system starting from all sites vacant in the physically relevant case $d = 2$ and found the following

$$\begin{array}{rl} p \leqq p_1 \approx 0.389 & \text{poisoning to all O} \\ p_1 < p \leqq p_2 & \text{coexistence of CO and O} \\ p > p_2 \approx 0.525 & \text{poisoning to all CO} \end{array}$$

Here poisoning to all O means that $P(\xi_t(x) = 2) \to 1$, and coexistence of CO and O means that there is an equilibrium state in which 1's and 2's are present at a positive density. A remarkable result of their simulations is that at p_2 the system has a *first order* phase transition. The density of CO in equilibrium jumps from less than 0.25 to 1. See Fig. 1.

Proving that poisoning occurs for small p or for p close to 1 is not too hard. However, it seems very difficult to attack the more interesting problem of proving that the coexistence region exists since there is no parameter that is small (or large) in this regime. Two things that make this problem especially hard are:

(a) computer simulations indicate that there is no coexistence in $d = 1$;
(b) if one simplifies the model by supposing that 2's land one at a time then there never is coexistence (see Grannan and Swindle (1991); Mountford and Sudbury (1992)).

While waiting for an insight that will tell us why the combination of a diatomic molecule and $d > 1$ allows for coexistence to occur, it is natural to modify the model to introduce a large parameter. Bramson and Neuhauser (1992) have done this by replacing the oxygen atom by an $N \times N$ polymer that consists of N^2 identical atoms arranged in a square of side N. They prove the possibility of coexistence by showing that when N is large the first phase disappears (i.e., $p_1 = 0$) and $p_2 > 0$.

In this paper we will pursue a different modification. We use the observation (see Engel and Ertl (1979)) that carbon monoxide molecules are highly mobile on the catalyst surface and that oxygen atoms have some mobility as an excuse to introduce *stirring*. That is, for each pair of neighbors x and y we exchange the values at x and y at rate v, i.e., we change the configuration from ξ to $\xi^{x,y}$ where

$$\xi^{x,y}(z) = \begin{cases} \xi(y) & \text{if } z = x \\ \xi(x) & \text{if } z = y \\ \xi(z) & \text{otherwise} \end{cases} .$$

Our main motivation for introducing stirring is that if we consider the system on $\varepsilon \mathbf{Z}^d$, set $v = \varepsilon^{-2}$, and let $\varepsilon \to 0$ then, using results of Durrett and Neuhauser (1993), the particle system will converge to the solution of a reaction diffusion equation and we can use results about the limiting p.d.e. to prove that coexistence occurs for small ε.

Before stating our coexistence result, we will give two results about poisoning. We do this mainly to show that when r is fixed and ε is small we have, in contrast to Bramson and Neuhauser, a situation in which all three phases (poisoning to O, poisoning to CO, and coexistence) are present in the (p,q) plane.

Theorem 1 *Suppose $v \geqq 0$, $p > q$ and ξ_0 is translation invariant. Then for all x and neighbors y of x we have $P(\xi_t(x) = 0) \to 0$ and $P(\xi_t(x) = 1, \xi_t(y) = 2) \to 0$.*

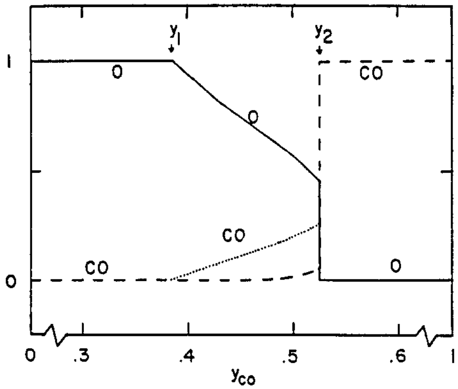

Fig. 1.

So for any K the probability that ξ_t is either $\equiv 1$ or $\equiv 2$ on $[-K,K]^d$ approaches 1 as $t \to \infty$.

To explain the condition $p > q$ note that (ii) can be formulated as:

(ii′) At rate $q/2$ an oxygen molecule attempts a landing at x. It picks a neighbor y at random. If both x and y are vacant, the two sites become occupied by oxygen atoms; otherwise no change occurs.

Thus we have two oxygen atoms attempting to land at rate $q/2$ versus one carbon monoxide molecule attempting landings at rate p, and when $p > q$ the landing rate for carbon monoxide exceeds that of oxygen. We conjecture that if $p > q$ and 1's have a positive density in the translation invariant initial state ξ_0 then the system converges to all 1's but our argument for Theorem 1, which is simple enough to give in detail in the introduction, does not allow us to prove this.

Proof. It suffices to prove that $P(\xi_t(x) = 0) \to 0$ to get the other two conclusions since (a) there is a $\delta > 0$ so that if y is a neighbor of x

$$P(\xi_{t+1}(x) = 0) \geqq \delta P(\xi_t(x) = 1, \xi_t(y) = 2),$$

and (b) if there is no 0 in $[-K,K]^d$ and no two adjacent sites with a 1 and a 2 then the whole cube must be $\equiv 1$ or $\equiv 2$.

To prove that $P(\xi_t(x) = 0) \to 0$ we begin by observing that if we write $y \sim x$ to denote y is a neighbor of x then

$$\frac{d}{dt}P(\xi_t(x) = 1) = pP(\xi_t(x) = 0) - \frac{r}{2d}\sum_{y\sim x} P(\xi_t(x) = 1, \xi_t(y) = 2)$$

$$\frac{d}{dt}P(\xi_t(x) = 2) = \frac{q}{2d}\sum_{y\sim x} P(\xi_t(x) = 0, \xi_t(y) = 0) - \frac{r}{2d}\sum_{y\sim x} P(\xi_t(x) = 2, \xi_t(y) = 1).$$

There are no terms involving the stirring since it does not change the density of 1's or 2's. If ξ_0 is translation invariant then so is ξ_t and hence the last term in each of the two equations are the same. Subtracting and using the fact that $P(\xi_t(x) = 0, \xi_t(y) = 0) \leqq P(\xi_t(x) = 0)$ we have

$$\frac{d}{dt}\{P(\xi_t(x)=1)-P(\xi_t(x)=2)\}\geqq(p-q)P(\xi_t(x)=0)\,.$$

Integrating from 0 to T it follows that

$$(p-q)\int_0^T P(\xi_t(x)=0)dt\leqq\{P(\xi_t(x)=1)-P(\xi_t(x)=2)\}|_0^T\leqq 2\,.$$

The 2 on the right hand side of the last inequality does not depend on T, so if $p > q$ we have

$$\int_0^\infty P(\xi_t(x)=0)dt\leqq 2/(p-q)\,.$$

From the last conclusion it follows easily that $P(\xi_t(x)=0)\to 0$. Just observe that since 0's disappear at a rate $\leqq(p+q)$, we have

$$P(\xi_{t+s}(x)=0)\geqq e^{-(p+q)s}P(\xi_t(x)=0)$$

so if $P(\xi_t(x)=0)$ does not go to 0, it would not be integrable.

Proving convergence to all 2's is made difficult by the fact that if $p=0$ then the oxygen atoms cannot completely fill space since they will leave isolated vacant sites. Nonetheless we can prove

Theorem 2 *Suppose $v>0$, $p>0$, and $0<r\leqq\infty$ are fixed. If q is large enough and ξ_0 contains infinitely many 2's then $P(\xi_t(x)=2)\to 1$.*

Sketch of Proof. We begin by considering the system with $q=\infty$, i.e., an adjacent pair of vacant sites is immediately filled with a pair of 2's (and if several pairs become vacant simultaneously we fill randomly chosen pairs of sites until no vacant pairs remain). We will show that for any $\delta>0$ we can pick a large L so that with probability at least $1-\delta/2$ an event we call *tripling* occurs: if $[-L,\ L]^d$ is completely occupied by 2's at time 0 then $[-3L,\ 3L]^d$ is completely occupied by 2's at all times $2L^2\leqq t\leqq 4L^2$ and the event that guarantees tripling is measurable with respect to the flips that occur in $\mathcal{A}=[-6L,\ 6L]^d\times[0,\ 4L^2]$. (See Fig. 3 for a picture.) Since this is a statement about a finite space time block, it follows by "continuity" that if q is large then tripling occurs with probability at least $1-\delta$. Once this is established one can compare with a mildly dependent oriented percolation process exactly as in Sects. 3 and 4 of Durrett (1992) to conclude that if $[-L,\ L]^d$ is completely occupied by 2's at time 0 then there is an $a>0$ so that with positive probability we will have $[-at,\ at]^d$ completely occupied for all large t. To get Theorem 2 from this, we note that if there are infinitely many 2's in the initial configuration we will eventually find a translate of $[-L,\ L]^d$ filled with 2's that gives rise to a linearly growing set of 2's.

The system with $q=\infty$ is simple because vacant sites cannot be created. To see this note that 0's are only produced by a reaction of a 1 and a 2 but this leaves an adjacent pair of vacant sites that is immediately filled by a pair of 2's. 0's cannot be created but they are easily destroyed - any 0 is subject to being landed on by a 1 at rate p. From the last observation it follows easily that if L is large then with probability at least $1-\delta/4$ there are no 0's in $\mathcal{B}=[-4L,4L]^d\times[L^2,4L^2]$. To prove that with probability at least $1-\delta/4$ there are no 1's in $\mathcal{C}=[-3L,3L]^d\times[2L^2,4L^2]$ we note that when there are no 0's in $\mathcal{B}$ any 1 in $\mathcal{C}$ must have started outside of $\mathcal{B}$ and moved into $\mathcal{C}$ by stirring, however this is unlikely. $[-L,L]^d$ is filled with 2's at time 0 and these 2's cannot be destroyed. So after stirring they lead to a positive density of 2's in $\mathcal{B}$ that the 1 must swim through for about L^2 units of time without undergoing a reaction. □

Remark. By interchanging the roles of 1's and 2's in the last argument one can show that for fixed values of v, q, and r, poisoning to all 1's occurs for large p. We will not give the details since Theorem 1 suggests strongly that this result should be true if one assumes only that $p > q$.

As mentioned earlier, we will prove coexistence by considering the system ξ_t^ε on $\varepsilon\mathbf{Z}^d$, setting $v = \varepsilon^{-2}$ and letting $\varepsilon \to 0$ (fast stirring). The motivation for doing this comes from a result of De Masi et al. (1986). (For a proof that encompasses our application see Durrett and Neuhauser (1993)).

Proposition 1 *Suppose that $\xi_0^\varepsilon(x)$ are independent and let $u_i^\varepsilon(x,t) = P(\xi_t^\varepsilon(x) = i)$. If $u_i^\varepsilon(x,0) = g_i(x)$ is continuous then as $\varepsilon \to 0$, $u_i^\varepsilon(x,t) \to u_i(x,t)$ the bounded solution of*

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + p(1 - u_1 - u_2) - ru_1u_2 \tag{1.1}$$

$$\frac{\partial u_2}{\partial t} = \Delta u_2 + q(1 - u_1 - u_2)^2 - ru_1u_2$$

with $u_i(x,0) = g_i(x)$.

Proposition 1 is easy to understand. Pure stirring has product measures as its stationary distributions. When ε is small, stirring operates at a fast rate and keeps the system close to a product measure. The rate of change of the densities can then be computed by assuming that adjacent sites are independent, or, in the language of physics, the densities evolve according to the equations of "mean field theory" (see Dickman (1986)).

Remark. If you go back and look at our justification, it is clear that a better description would be to have the two species moving according to the simple exclusion process at different rates. However, the existence of the hydrodynamic limit becomes a very complicated problem and the limiting equation contains density dependent diffusion constants that cannot be computed explicitly. (Jeremy Quastel, private communication. See Quastel (1992) for some related results.)

The first step in using Proposition 1 to understand the behavior of the particle system with small ε is to let $u_i(x,0) = c_i$ in (1.1) to get a dynamical system $v_i(t) = u_i(x,t)$

$$\frac{dv_1}{dt} = p(1 - v_1 - v_2) - rv_1v_2 \tag{1.2}$$

$$\frac{dv_2}{dt} = q(1 - v_1 - v_2)^2 - rv_1v_2 \, .$$

The first step in analyzing (1.2) is to look for equilibria $(\bar{v}_1, \bar{v}_2)$, which in this case satisfy

$$0 = p(1 - \bar{v}_1 - \bar{v}_2) - r\bar{v}_1\bar{v}_2 \tag{1.3}$$

$$0 = q(1 - \bar{v}_1 - \bar{v}_2)^2 - r\bar{v}_1\bar{v}_2 \, .$$

This system has two trivial solutions $\bar{v}_1 = 1,\ \bar{v}_2 = 0$ and $\bar{v}_1 = 0,\ \bar{v}_2 = 1$. Subtracting the two equations and then substituting the result into the first equation we have that if $\bar{v}_1 + \bar{v}_2 < 1$ then

$$p/q = 1 - \bar{v}_1 - \bar{v}_2 \quad p^2/qr = \bar{v}_1\bar{v}_2 \,. \tag{1.4}$$

From the first equation it is clear that $p < q$ is necessary for the existence of an equilibrium with $\bar{v}_1 + \bar{v}_2 < 1$. Combining this with the observation that the largest product xy among nonnegative numbers with $x + y = a$ occurs when $x = y = a/2$ we see that necessary and sufficient conditions for the existence of a nontrivial equilibrium are

$$p < q \quad p^2/qr \leqq (1 - p/q)^2/4 \quad (\text{or } 4qp^2/r \leqq (q-p)^2)\,. \tag{1.5}$$

Substituting the first equation in (1.4) into the second to get a single quadratic equation, and solving it follows that when the conditions in (1.5) hold there are two (possibly equal) equilibria (α, β) and (β, α) with

$$\alpha = \frac{(q-p) - \sqrt{(q-p)^2 - 4qp^2/r}}{2q} \tag{1.6}$$

$$\beta = \frac{(q-p) + \sqrt{(q-p)^2 - 4qp^2/r}}{2q}\,.$$

To see which of the four equilibria $(1, 0)$, $(0, 1)$, (α, β) and (β, α) will correspond to equilibrium states of the particle system we begin by looking at their stability. Figure 2 shows a picture of trajectories of the dynamical system (1.2) when $p = 1/4$, $q = 3/4$, and $r = 1$, in which case $\alpha = 1/6$ and $\beta = 1/2$. As that picture shows (1,0) and (α, β) are attracting fixed points and (β, α) and $(0, 1)$ are saddle points. The last result holds in general and can be easily checked by linearizing about the fixed points.

The equilibria (β, α) and $(0, 1)$, being saddle points, are not serious candidates for equilibria of the particle system with fast stirring. To decide whether the equilibrium densities will be near (α, β) or $(1, 0)$ we take a look at the speed of the "travelling wave" connecting these points. To prove the existence of the travelling wave we need the following result of Volpert and Volpert (1988). For a treatment of closely related problems using the Conley index, see Feinberg and Terman (1991).

Proposition 2 *There is a constant σ and monotone functions U_i with*

$$U_1(x) \to \alpha \text{ as } x \to -\infty, \quad U_1(x) \to 1 \text{ as } x \to \infty$$
$$U_2(x) \to \beta \text{ as } x \to -\infty, \quad U_2(x) \to 0 \text{ as } x \to \infty$$

so that $u_i(t, x) = U_i(x - \sigma t)$ is a solution of (1.1) *in $d = 1$.*

Such a solution is called a travelling wave because the functions $u_i(t, x)$ keep their shape and move with speed σ which may be positive, negative or 0.

Our intuition, based on results for scalar equations in Durrett and Neuhauser (1993), is that

(a) when $\sigma > 0$ and the stirring is fast there will be an equilibrium for the particle system in which the densities of particles of types 1 and 2 are close to α and β;
(b) when $\sigma < 0$ there will be no nontrivial equilibrium when the stirring is fast.

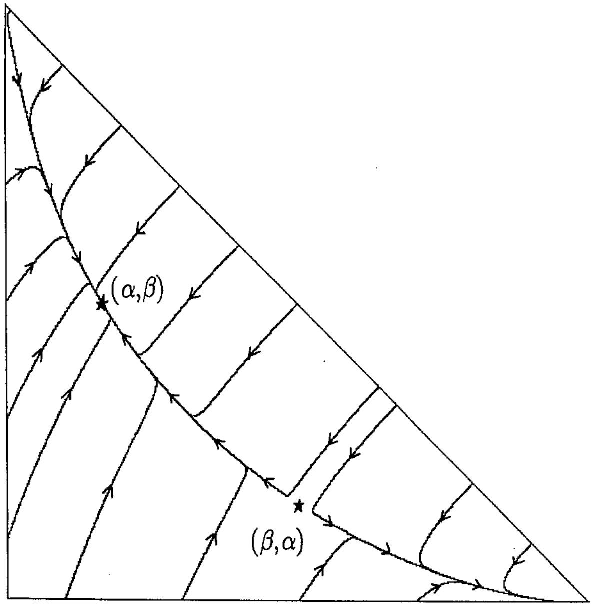

Fig. 2.

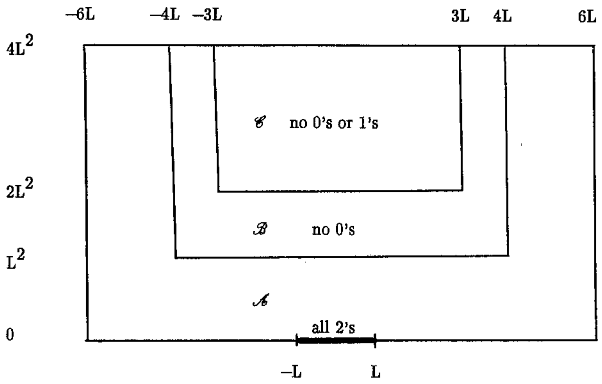

Fig. 3.

The main work in turning our intuition into a proof of (a) is to prove a (rather weak) convergence result. Here $\|x\|_\infty = \sup_i |x_i|$ and $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$

Proposition 3 *Suppose* $\sigma > 0$. *There are constants* $\eta > 0$, $c > 0$ *and* $L < \infty$ *so that if* $\|u(x,0) - (\alpha,\beta)\|_\infty \leqq \eta$ *when* $x \in [-L,L]^d$ *then*

$$\lim_{t\to\infty} \sup_{|x|\leqq ct} \|u(x,t) - (\alpha,\beta)\|_\infty = 0 .$$

In words, $u(x,t)$ is close to (α,β) on a ball that grows at a linear rate. Once Proposition 3 is established, we can let $c_1 = \alpha$, $c_2 = \beta$ and pick

$$c_i - \eta < A_i < a_i < c_i < b_i < B_i < c_i + \eta ,$$

to get

(*) There are constants $A_i < a_i < b_i < B_i$, L, and T so that if $u_i(x,0) \in (A_i, B_i)$ when $x \in [-L,L]^d$ then $u_i(x,T) \in (a_i, b_i)$ when $x \in [-3L, 3L]^d$.

This is the PDE input one needs for the argument in Sect. 3 of Durrett and Neuhauser (1993), (compare (*) with their Lemma 3.2), and it follows from the argument given there that

Theorem 3 *Suppose that $\sigma > 0$. If ε is small then our system has a translation invariant stationary distribution in which the densities of 1's and 2's are close to α and β respectively.*

In the case of scalar equations $\partial u/\partial t = \Delta u + f(u)$ one can use an integration by parts trick to relate the sign of σ to the sign of an integral of f (see the derivation of (1.7) in Durrett and Neuhauser (1993)). That trick fails for systems of equations. However, we can use a variant of this trick to prove that Theorem 3 is not vacuous.

Proposition 4 *If $p \leqq r/20$ and $q \geqq 20r$ then $\sigma > 0$.*

The rest of the paper is devoted to proofs. Theorem 2 is proved in Sect. 2, Proposition 3 in Sect. 3, and Proposition 4 in Sect. 4. This paper had its roots in conversations the authors had with Eric Grannan at the AMS regional meeting in Irvine, California in November 1990. We are grateful to Phil Holmes at Cornell and to Robert Gardner at U of Massachusetts at Amherst for their help in coping with the reaction-diffusion equation, and to Claudia Neuhauser for her extensive comments on the first draft of this paper.

2 Convergence to all 2's

In view of the sketch of the proof given in the introduction, it suffices to consider the system with $q = \infty$ and show that

(*) for any $\delta > 0$, if L is large then with the probability at least $1 - \delta/2$ *tripling* occurs: if $[-L,L]^d$ is completely occupied by 2's at time 0 then $[-3L,3L]^d$ is completely occupied by 2's at all times $2L^2 \leqq t \leqq 4L^2$ and the event that guarantees tripling is measurable with respect to the flips that occur in $\mathcal{A} = [-6L, 6L]^d \times [0, 4L^2]$.

To prove (*) it is convenient to construct the process from a graphical representation, i.e., a collection of independent Poisson processes. For each $x \in \mathbf{Z}^d$, let U_n^x be a Poisson process with rate p, and for each $1 \leqq i \leqq d$, let $S_n^{x,i}$ and $T_n^{x,i}$, be Poisson processes with rates v and r/d respectively. At time $S_n^{x,i}$ we exchange the values at x and $x + e_i$ where e_i is the ith unit vector, i.e., $(e_i)_j = 1$ if $i = j, 0$ otherwise. At time $T_n^{x,i}$ a reaction between atoms at x and $x + e_i$ occurs producing two vacant sites if these sites are occupied by two different atoms. At time U_n^x a 1 lands at x if it is vacant. If $q = \infty$ we add the rule that an adjacent pair of vacant sites is immediately filled with a pair of 2's (and if several pairs become vacant simultaneously we fill randomly chosen pairs of sites until no vacant pair remain). If $q < \infty$ we introduce Poisson processes $V_n^{x,i}$ with rate $q/2d$ at which times two 2's land if x and $x + e_i$ are vacant.

Even when $q = \infty$ it is not hard to use the argument in Harris (1972) to show that our recipe gives rise to a well defined process. Our main reason for introducing this construction is that it allows us to work backwards in time. To facilitate this it is useful to decorate our construction in the style of Griffeath (1979). At stirring times $s = S_n^{x,i}$ we draw an undirected line segment from (x, s) to $(x + e_i, s)$ and we write a δ at each end point. We say that (X_s, s), $a \leqq s \leqq b$ is a *stirring path* if it is (a) piecewise constant, (b) at each discontinuity there is a δ at (X_s, s) and at (X_{s-}, s), (c) there are no δ's at any (X_s, s) where the path is continuous. In words, the path jumps only at stirrings and does not ignore any stirrings that touch the path. We say that $(X_s, t - s)$ with $0 \leqq s \leqq t_0$ is a *dual stirring path* if (X_{t-r}, r), $t - t_0 \leqq r \leqq t$ is a stirring path.

When $q = \infty$, 0's cannot be created, only destroyed, so if there is a 0 at x at time t then there is a dual stirring path $(X_s, t - s)$, $0 \leqq s \leqq t$ with $X_0 = x$ so that $(X_s, t - s)$ is always occupied by a 0. Since 1's land at rate p this is very unlikely if t is large. This observation leads easily to

Lemma 2.1 *If L is large then with probability at least* $1 - \delta/4$ *there are no 0's in* $\mathcal{B} = (-4L, 4L)^d \times [L^2, 4L^2]$ *and the event that guarantees this is measurable with respect to the Poisson arrivals in* $\mathcal{A} = [-6L, 6L]^d \times [0, 4L^2]$.

The first step in proving this result is to define a *fast path* as a stirring path that moves a distance more than L in time $\leqq L$ and ends in $\mathcal{B}$ and to prove

Lemma 2.2 *If L is large then with probability at least* $1 - \delta/8$ *there is no fast path.*

Proof. If there is a fast path then working backwards in time from the end of the path, there will be a dual stirring path $\pi_s = (X_s, t - s)$ that starts at time $t = L^2$ or at the location of some δ, (x_0, t), in the graphical representation in $\mathcal{B}$ and exists from $x_0 + [-L, L]^d$ by time L. The number of starting points at time $t = L^2$ is smaller than $(8L)^d$. The number of such stirring δ's is 2 times a Poisson with parameter $A_1 L^{d+2}$ and hence, with probability at least $1 - C\exp(-\gamma L^{d+2})$, there are at most $4A_1 L^{d+2}$ such starting points. A standard large deviations results implies that the probability a simple random walk starting from 0 and moving at a fixed rate (here $2dv$) will exit $[-L, L]^d$ by time L is smaller than $Ce^{-\gamma L}$. Combining this with the previous observation it follows that the probability of a fast path is at most

$$Ce^{-\gamma L^{d+2}} + (8^d + 4A_1)L^{d+2}Ce^{-\gamma L} ,$$

and this proves Lemma 2.2. □

Proof of Lemma 2.1 By considering the first time there is a 0 in $\mathcal{B}$ we can find a dual stirring path that was always occupied by 0's that either starts at time L^2 or at a stirring point in $\mathcal{B}$ and goes backwards to time 0 or to a point outside of $\mathcal{A}$. When there are no fast paths, any such path must have duration at least L and hence has probability at most e^{-pL} to avoid having a 1 land on it. As in the previous proof, the number of stirring starting points is 2 times a Poisson with parameter $A_1 L^{d+2}$ and hence with probability at least $1 - C\exp(-\gamma L^{d+2})$ there are at most $4A_1 L^{d+2}$ such starting points. Adding the fewer than $(8L)^d$ starting points at time L^2 and recalling our estimate on the probability of having a fast path, it follows that the probability of a 0 in $\mathcal{B}$ is at most

$$\frac{\delta}{8} + Ce^{-\gamma L^{d+2}} + (8^d + 4A_1)L^{d+2}e^{-pL}$$

and this proves Lemma 2.1. □

Lemma 2.3 *If L is large then with probability at least $1 - \delta/2$ there are no 0's or 1's in $\mathcal{C} = [-3L, 3L]^d \times [2L^2, 4L^2]$ and the event that guarantees this is measurable with respect to the Poisson arrivals in $\mathcal{A} = [-6L, 6L]^d \times [0, 4L^2]$.*

Proof. We can, in view of Lemma 2.1, suppose that there are no 0's in $\mathcal{B}$ and in view of Lemma 2.2 suppose that there are no fast paths. We would like to emphasize that we are not conditioning on the occurrence of these events which could change the personality of our process but instead are looking at the part of the probability space where these events occur.

Since there are no 0's in $\mathcal{B}$, any 1 in $\mathcal{C}$ must have started outside $\mathcal{B}$ and moved into $\mathcal{C}$ without reacting with a 2. To prove that this is unlikely we note that the first time such a 1 entered $\mathcal{C}$, must have been at time $2L^2$ or at a stirring time. Let (x, t) be one of these space time points and let $(X_s^0, t - s)$ be the dual stirring path starting from $X_0^0 = x$. Since there are no fast paths we know that $X_s^0 \in [-4L, 4L]^d$ for $0 \leqq s \leqq L$. Our goal is to prove that with probability at least $1 - C\exp(-\gamma L^\beta)$

$$|\{0 \leqq s \leqq L : \xi_{t-s}(X_s^0 + e_1) = 2\}| \geqq a_2 L^\beta$$

(here $e_1 = (1, 0, \ldots, 0)$ is the first unit vector) so the probability that a 1 can travel along this path without reacting with a neighboring 2 is $\leqq \exp(-ra_2 L^\beta / 2d)$.

To examine the states of the sites $(X_s^0 + e_1, t - s)$ we will follow their dual stirring paths. In order to have the dual stirring paths that we examine behave like independent random walks, we will only look at the sites $(X_s^0 + e_1, t - s)$ at times that are fairly well separated. This is the intuition behind the following construction. At times $t - kL^\alpha + 1$ for $1 \leqq k \leqq L^\beta$ we begin watching the site $x_k + e_1$ where $x_k = X_{kL^\alpha - 1}^0$. If this site is not occupied by one of our earlier chosen particles and if no stirring event affects x_k or $x_k + e_1$ between time $t - kL^\alpha$ and $t - kL^\alpha + 1$ we add a new particle to our collection and let X_s^k denote its position at time $t - s$ for $kL^\alpha \leqq s \leqq t$; otherwise we add no particle to our collection. For reasons that will emerge in the proof we pick $\alpha = 1/5$ and $\beta = 1/15$ to satisfy

$$\alpha + \beta \leqq 1 \quad \beta - (\alpha/2) < 0 \quad 0.36 + 0.3\beta < 0.4 .$$

We will show that there is a constant $a_1 > 0$ so that with probability $\geqq 1 - L^{-(d+3)}$ (a number chosen to reflect the fact that it is very likely that there are fewer than $A_2 L^{d+2}$ starting points to worry about):

(i) we start at least $a_1 L^\beta$ particles, and
(ii) the particles X^k we start can be coupled to independent particles Y^k so that the discrepancy is never more than $L^{0.8}$.

We will now show that (i) and (ii) easily imply the desired result before embarking on their somewhat lengthy proofs. Since X_k^0 is not a fast path and $\alpha + \beta \leqq 1$, all the Y^k start inside $[-4L, 4L]^d$ and at times $\geqq 2L^2 - L^{\alpha+\beta} \geqq L^2$. So they each have a probability $\geqq b_1$ of tracing back to $[-L/2, L/2]^d$ at time 0 and staying inside $[-5L, 5L]^d$. Since the Y^k are independent, a standard large deviations estimate implies that if $a_2 = a_1 b_1 / 2$ then with probability at least $1 - C\exp(-\gamma L^\beta)$ at least $a_2 L^\beta$ of the Y^k's are *good*, that is, they trace back to $[-L/2, L/2]^d$ at time 0 and stay inside $[-5L, 5L]^d$. The coupling result in (ii) then implies that the X^k associated with these good Y^k trace back to $[-L, L]^d$ at time 0 and stay inside $(-6L, 6L)^d$, so $x_k + e_1$ is occupied by a 2 from time $t - kL^\alpha$ to $t - kL^\alpha + 1$. We have arranged things so that no stirring affects $x_k + e_1$ or x_k between times $t - kL^\alpha$ and $t - kL^\alpha + 1$, so there is a probability $\geqq b_2 > 0$ that a reaction between these two sites will occur removing the 1. The possibilities of a reaction are independent for different values of k so the probability that no reaction occurs is smaller than $\exp(a_2 L^\beta \ln(1 - b_2))$. Our error probabilities add up to something less than $4L^{-(d+3)}$ when L is large, so taking into account the number of possible starting points it follows that with high probability there is no 1 in $\mathcal{C}$.

To complete the proof, it is enough to prove (i) and (ii). We begin with the

Proof of (i). Let E_k be the event that we add a new particle at the kth stage, and let $\mathcal{F}_j$ denote all the information in the graphical representation between times t and $t - jL^\alpha + 1$.

Lemma 2.4 *There is a constant $a_0 > 0$ so that if L is large then*

$$P(E_k | \mathcal{F}_{k-1}) \geqq a_0 \ .$$

Proof. It is clear that there is a positive probability, independent of $\mathcal{F}_k$, of no stirring affecting x_k or $x_k + e_1$ between times $t - kL^\alpha$ and $t - kL^\alpha + 1$. To prove the result then it suffices to establish a lower bound on the probability that none of our earlier chosen particles sits at $x_k + e_1$. To do this we begin by considering the behavior of the distance $D_s = \|Y_s - Z_s\|_1$ between two particles Y_s and Z_s moved by stirring. We claim that D_s is stochastically larger than $\|S_s\|_1$ where S_s is a d dimensional simple random walk that starts at 0 and jumps at rate $4dv$. To verify this claim note that

(a) if $D_s > 1$ or $D_s = 1$ and the stirring involves only one of X_s and Y_s then D_s behaves like $\|S_s\|_1$,
(b) if $D_s = 1$ stirrings of the pair (X_s, Y_s) (which leave D_s unchanged) happen at rate v while $\|S_s\|_1$ has jumps from 1 to 0 at rate $2v$.

With the last two observations in mind one can easily construct a coupling of the two processes that has $\|S_s\|_1 \leqq D_s$. Combining the last comparison with well known facts about S_s we have

$$P(D_s = 1) \leqq P(\|S_s\|_1 \leqq 1) \leqq C/(1+s)^{1/2} \ .$$

The last estimate applies to $\|X_s^0 - X_s^j\|_1$ for any value of j. So the probability that $x_k + e_1$ is occupied is at most $CL^{\beta - (\alpha/2)}$. The last quantity approaches 0 as $L \to \infty$ because $\beta - (\alpha/2) < 0$, so the proof is complete. □

Lemma 2.5 *Let $N_k = \sum_{j=1}^k 1_{E_j}$ be the number of particles we generate in the first k tries and let $a_1 = a_0/2$. Then*

$$P(N_k \leqq a_1 k) \leqq 2e^{-ka_1^2/2} \ .$$

Proof. Define M_k by $M_0 = 0$ and for $k \geqq 1$

$$M_k = M_{k-1} + 1_{E_k} - P(E_k | \mathcal{F}_{k-1}) \ .$$

M_k is a martingale with respect to $\mathcal{F}_k$ with $| M_k - M_{k-1} | \leqq 1$ so Azuma's inequality (see (4.1) on p. 159 in McDiarmid (1989)) implies that for any $t > 0$

$$P(|M_k| > t) \leqq 2e^{-t^2/2k} \ .$$

Lemma 2.4 implies that $N_k \geqq M_k + a_0 k$ so

$$P(N_k \leqq a_1 k) \leqq P(|M_k| \geqq a_1 k) \leqq 2e^{-ka_1^2/2} .$$

Having established that there are lots of paths, the next step is to show that they can be closely coupled to independent random walks Y^k.

Proof of (ii). Following the approach of part *b* of Sect. 2 of Durrett and Neuhauser (1993) we say that X^k is crowded at time s if $\|X_s^j - X_s^k\|_1 = 1$ for some $j \neq k$. When X^k is not crowded we define the increments of Y^k to be equal to those of X^k but when X^k is crowded we use an independent random walk to define the increments of Y^k. Since the movements of an X^k are independent of the other X^j when X^k is not crowded, the Y^k just defined are independent simple random walks.

To estimate the amount of time that X^k is crowded we will use the observation made in the proof of Lemma 2.4 : $\|X_s^j - X_s^k\|_1$ is stochastically larger than $\|S_s\|_1$, where S_s is a simple random walk that starts at 0 and takes steps at rate $4dv$. Let

$$H_t(x) = \int_0^t 1_{\{S_s = x\}} ds$$

be the occupation time of x up to time t. By considering the time of the first visit to x it is easy to see that $H_t(x)$ is stochastically smaller than $H_t(0)$. To estimate $H_t(0)$ with an error probability smaller than $L^{-(d+3)}$ we will compute moments $E\{H_t^m(0)\}$. The next estimate can be improved considerably in $d > 1$. We use this crude bound to avoid splitting the proof into cases.

Lemma 2.6 *There is a constant C so that* $E\{H_t^m(0)\} \leqq m! C^m (1+t)^{m/2}$ *and hence*

$$P(H_t(0) > (1+t)^{0.6}) \leqq m! C^m (1+t)^{-m/10} .$$

Proof. The second result is an immediate consequence of the first, which we will prove by induction on m. When $m = 1$ this estimate is well known. Writing $p_t(x,y)$ for the transition probability of simple random walk, letting $t_0 = 0$, and then using the result for $m-1$ and for 1 we have

$$\begin{aligned} E\{H_t^m(0)\} &= m! \int\limits_{0 \leqq t_1 < \ldots < t_m \leqq t} \prod_{i=1}^m p_{t_i - t_{i-1}}(0,0) dt_m \ldots dt_1 \\ &= m \int_0^t p_{t_1}(0,0) \{E H_{t-t_1}^{m-1}(0)\} dt_1 \\ &\leqq m(m-1)! C^{m-1} \int_0^t p_{t_1}(0,0)(1 + t - t_1)^{(m-1)/2} dt_1 \\ &\leqq m! C^{m-1} (1+t)^{(m-1)/2} \int_0^t p_{t_1}(0,0) dt_1 \leqq m! C^m (1+t)^{m/2} . \end{aligned}$$

The movements of Y^k when X^k is crowded are a simple random walk. Somewhat surprisingly.

Lemma 2.7 *The movements of* X^k *when it is crowded are a simple random walk.*

Proof. Intuitively this is true because we are using a predictable function to pick points out of a family of Poisson processes. We will now prove a result that makes the last sentence precise. For simplicity we move up (in time) rather than down. For $x \in \mathbf{Z}^d$ let $N_t^{x,v}$ be the stirrings of x and $x+v$, where $v = e_1, -e_1, \dots$ or $-e_d$. and let $\mathcal{F}_t$ be the σ-field generated by all stirrings up to time t. Let $h : [0,\infty) \to \mathbf{Z}^d \cup \{\Delta\}$ be a left continuous process adapted to $\mathcal{F}_t$. In our application $h_s = x$ if X_{s-}^k is crowded and at $x, h_s = \Delta$ if X_{s-}^k is not crowded. The left continuity of h_s is important here since the intervals on which $h_s = x$ will end with Poisson arrivals that move the path to another site and we don't want to miss these points! Let

$$\sigma_t = \int_0^t 1_{\{h_s \neq \Delta\}} ds$$

suppose $\sigma_\infty = \infty$ and let $\gamma_s = \inf\{t : \sigma_s \geqq t\}$ be the left continuous inverse of σ_t. Let $J_s^x = 1$ if $h_s = x$ and write

$$M_t^v = \sum_x \int_0^{\gamma_t} J_s^x dN_s^{x,v} .$$

We have written M_t^v as a sum of stochastic integrals since this makes it clear that its compensator $\langle M^v \rangle_t = vt$ and then it follows from Theorem 4.5 on p.103 of Jacod and Shiryaev (1987) that M_t^v is a Poisson process with rate v.

The last result implies that after the time change γ_t is applied, the times at which X_s^x is crowded and is stirred to $X_s^x + v$ are a Poisson process with rate v. To get from this to the full result we have to prove that the Poisson processes M_t^v are independent. To do this we turn the M_t^v into a random measure that puts a point at v when M_t^v jumps, note that the compensating measure is deterministic and then use Theorem 4.8 on p. 104 in Jacod and Shiryaev (1987). □

Combining Lemma 2.7 with Lemma 2.6 it follows that we can estimate the difference between X_s^k and Y_s^k by considering the behavior of a simple random walk S_s run for a random amount of time. To do this the next result is useful.

Lemma 2.8 *If $m > 1$ there is a constant $C_{m,d}$ so that*

$$P\left(\max_{0 \leqq s \leqq t} \|S_s\|_1 > (1+t)^{0.6}\right) \leqq C_{m,d}(1+t)^{-m/10} .$$

Proof. Using the triangle inequality, $P(\cup_i A_i) \leqq \sum_i P(A_i)$, and then Doob's inequality we have (here $| S_s^1 |$ is the absolute value of the first coordinate)

$$P\left(\max_{0 \leqq s \leqq t} \|S_s\|_1 > (1+t)^{0.6}\right) \leqq dP\left(\max_{0 \leqq s \leqq t} |S_s^1| > (1+t)^{0.6}/d\right)$$

$$\leqq d\left(\frac{m}{m-1}\right)^m \frac{d^m E|S_t^1|^m}{(1+t)^{0.6m}} \leqq C_{m,d}(1+t)^{-m/10} .$$

For the last inequality we use the well known fact that in the case of one dimensional simple random walk, the central limit theorem can be strengthened to conclude the convergence of moments $E(| S_t^1 |^m / t^{m/2})$ to those of the limiting normal distribution and this convergence implies $E(| S_t^1 |^m) \leqq C(1+t)^{m/2}$. □

Lemmas 2.6 and 2.8 give us what we need to estimate the difference between the X^k and the Y^k. The t's we are interested in are related to stirring points in C and hence satisfy $4L^2 \geqq t \geqq 2L^2 - L^{\alpha+\beta} \geqq L^2$ since $\alpha + \beta \leqq 2$, so ignoring constants, we can think of $t = L^2$. Taking $m \geqq 5(d+5)$ in Lemma 2.6 it follows that the amount of time X^k is crowded by one X^j with $j \neq k$ is larger than $t^{0.6}$ with probability at most $Ct^{-(d+5)}/2$. Applying this result to each of the at most L^β other particles it follows that the total amount of time that any X^k is crowded is at most $t^{0.6+(\beta/2)}$ (recall $t = L^2$) with probability at least $1 - Ct^{-(d+5-\beta)/2}$. Now Lemma 2.8 implies that the maximum movement by a simple random walk in $t^{0.6+(\beta/2)}$ units of time is at most $t^{0.36+0.3\beta}$ with probability at least

$$1 - C_{m,d}t^{-(0.036+0.03\beta)m} \geqq 1 - Ct^{-(d+3)/2}$$

if we pick $m \geqq (d+3)/0.072$. Plugging $\beta = 1/15 < 1$ into the last two results, and recalling $0.36 + 0.3\beta < 0.4$, $t = L^2$ we have proved (ii). This completes the proof of Lemma 2.3 from which (*) and Theorem 2 follow.

3 Proof of Proposition 3

This result is proved by using a result of Fife and Tang (1981) and an argument of Gardner (1982). We begin with the result of Fife and Tang, which is a theorem for a general system $\partial_t v = \Delta v + f(v)$ with a *quasi-monotone* f, that is if $v_i = \tilde{v}_i$ and $v_j \geqq \tilde{v}_j$ when $j \neq i$ then $f_i(v) \geqq f_i(\tilde{v})$. To apply their result we change variables $v_1 = u_1$ and $v_2 = 1 - u_2$ in (1.1) to get

$$\frac{\partial v_1}{\partial t} = \Delta v_1 + p(v_2 - v_1) - rv_1(1 - v_2) \tag{3.1}$$

$$\frac{\partial v_2}{\partial t} = \Delta v_2 - q(v_2 - v_1)^2 + rv_1(1 - v_2)\,.$$

Letting f_1 and f_2 denote the reaction terms we have

$$\frac{\partial f_1}{\partial v_2} = p + rv_1 \qquad \frac{\partial f_2}{\partial v_1} = 2q(v_2 - v_1) + r(1 - v_2)\,.$$

The first expression $p + rv_1 \geqq 0$ when $v_1 \geqq -p/r$. The second is nonnegative when $-2qv_1 + (2q - r)v_2 \geqq -r$, i.e.,

$$v_2 \geqq \frac{2q}{2q - r}v_1 - \frac{r}{2q - r}\,.$$

(Eventually we will take $q \geqq 20r$ so we can suppose now that $2q > r$.) Now the set where $=$ holds is a line that contains $v_1 = v_2 = 1$ and has slope > 1. To see that this (just barely) contains the region of interest recall that we have changed variables $v_1 = u_1$, $v_2 = 1 - u_2$ so

$$\{(u_1, u_2) : u_1 \geqq 0, u_2 \geqq 0, u_1 + u_2 \leqq 1\} = \{(v_1, v_2) : v_1 \geqq 0, v_2 \leqq 1, v_2 - v_1 \geqq 0\}\,,$$

i.e., a triangle with vertices at $(0,0)$, $(0,1)$ and $(1,1)$.

Applying a slight generalization of Theorem 5 on p. 181 of Fife and Tang (1981) to (3.1) gives.

Lemma 3.1 *Let $v(x,t)$ be a solution of (3.1) with continuous initial data satisfying $0 \leqq v(x,0) \leqq 1$, and $v(x,t) \not\equiv 0$. There is a constant $c > 0$ so that*

$$\liminf_{t\to\infty} \inf_{\|x\|_\infty \leqq ct} v_1(x,t) \geqq \alpha$$

$$\liminf_{t\to\infty} \inf_{\|x\|_\infty \leqq ct} v_2(x,t) \geqq 1-\beta\,.$$

To explain the conclusion, note that our change of variables has moved the unstable critical point at (0,1) to (0,0), and the desired limit (α,β) to $(\alpha, 1-\beta)$. The result thus says that if we have an initial condition that lies strictly above the unstable critical point at (0,0) then the solution will grow up at least to the first critical point above the unstable one. This is a generalization to systems of the "hair trigger effect" of Sect. 3 of Aronson and Weinberger (1978).

Proof of (3.1). Fife and Tang assume (to use their formula number) that for each i there is a $\kappa_i > 0$ and a j (that may depend on i) so that for $v > 0$ in a neighborhood of 0

$$f_i(v) \geqq \kappa_i v_j\,. \tag{12}$$

However in (3.1), $f_1(v) = -(p+r)v_1 + (p+rv_1)v_2$ so this condition is not satisfied. Fortunately, for their proof it is enough that

(12′) there are $\theta_i > 0$ so that $f_i(w\theta) \geqq \kappa_i\theta_i w$ when $w > 0$ is small.

To see that (12′) suffices for their proof, observe that if we let

$$\underline{v}_i = \begin{cases} \varepsilon\theta_i(1-\rho^2)^\gamma & \text{when } 0 \leqq \rho \leqq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\rho = \|x\|_2^2$ and $\gamma > 2$ then (15) on page 181 in Fife and Tang (1981) holds with the right hand side multiplied by θ_i (and r replaced by ρ) and the rest of the argument for their Theorem 5 is exactly as before.
To check (12′) now we observe that if $v_1 = w$ and $v_2 = aw$ then

$$f_1(v) \geqq -(p+r)w + paw$$

$$f_2(v) = rw - raw^2 - qw^2(1-a)^2$$

so the desired conclusion holds if $a > (p+r)/p$. □

To prove the other half of the Proposition 3, we will return to the original equation (1.1), change variables $u = u_1$, $v = u_2$, and let

$$f(u,v) = p(1-u-v) - ruv$$

$$g(u,v) = q(1-u-v)^2 - ruv$$

to make our notation match that of the proof of Theorem 2.2 on pp. 359–362 of Gardner (1982). After reading a page or two of our proof, the reader will sympathize with our desire to stay as close to our source as possible. We have to give the details of the proof for three reasons: (i) Gardner gives his proof only for the case $d = 1$, (ii) our system has $\partial g/\partial v = 0$ when $u = 1$ and $v = 0$, and (iii) Gardner mistakenly omitted two terms from his expression for the derivative in formula (16) on p. 360 of his paper. In one respect our proof is simpler. Since we are only proving half of

Gardner's result, we can ignore the function ϕ that enters into his proof and this simplifies the computations somewhat.

The idea behind the proof is to use the travelling wave of Proposition 2 to construct something that satisfies appropriate partial differential inequalities (see (3.9) below) to give an upper bound on u and a lower bound on v. Continuing to change our notation to match Gardner, we let

$$\hat{u} = U_1 \quad \hat{v} = U_2$$

be the two functions that comprise the travelling wave. Gardner was interested in proving that monotone functions converge to the travelling wave, but we are interested in a result for compactly supported initial data. So, following the approach in the Appendix of Durrett and Neuhauser (1993), we let

$$h(x) = \begin{cases} x^2 - (x^3/3) & \text{if } 0 \leqq x \leqq 1 \\ x - 1/3 & \text{if } x \geqq 1 \end{cases} \tag{3.2}$$

To see the reason for this choice note that

$$h'(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leqq x \leqq 1 \\ 1 & \text{if } x \geqq 1 \end{cases}$$

$$h''(x) = \begin{cases} 2 - 2x & \text{if } 0 \leqq x \leqq 1 \\ 0 & \text{if } x \geqq 1 \end{cases}$$

so h is C^2, that is, has two continuous derivatives. For the future, note that

$$h'(x) \leqq 1, \quad h'(x) \leqq 2x, \text{ and } h''(x) \leqq 2\,. \tag{3.3}$$

Let $\zeta = h(|\,x\,|) - \sigma t + s(t)$ where $s(t) = -A + B\log(1+t)$ and

$$\gamma(x,t) = \hat{u}(\zeta) + Q_1(\zeta)\psi(t) - u(x,t) \tag{3.4}$$

$$\delta(x,t) = v(x,t) - \hat{v}(\zeta) + Q_2(\zeta)\psi(t)$$

where $\psi(t) = \kappa e^{-\rho t}$, and there is a constant $K_0 \geqq 1$ so that

$$Q_i(z) = \begin{cases} Q_i^- & \text{for } z \leqq -K_0 \\ Q_i^+ & \text{for } z \geqq K_0 \end{cases} \tag{3.5}$$

Here Q_i^+ and Q_i^- are positive constants that will be chosen in Lemma 3.2. We assume that the Q_i are C^2 and satisfy

$$Q_i(z) \geqq Q_0 > 0 \tag{3.6}$$

$$|Q_1'(z)| \leqq \hat{u}'(z)/2 \quad |Q_2'(z)| \leqq -\hat{v}'(z)/2\,. \tag{3.7}$$

The last requirement can be satisfied if we multiply our original choices of the Q_i by small positive constants. Let Q_* be such that

$$|Q_i(z)|, |Q_i'(z)|, |Q_i''(z)| \leqq Q_*\,. \tag{3.8}$$

To prove our result we will choose our constants so that $\gamma(x,0)$, $\delta(x,0) \geqq 0$, and if $\mathcal{L} = \frac{\partial}{\partial t} - \Delta$ then

$$\mathcal{L}\gamma(x,t) \geqq 0 \text{ when } \gamma = 0,\ \delta \geqq 0 \tag{3.9}$$
$$\mathcal{L}\delta(x,t) \geqq 0 \text{ when } \gamma \geqq 0,\ \delta = 0\,.$$

Applying Theorem 4.1 in Chueh et al. (1977) we conclude that $\gamma(x,t),\ \delta(x,t) \geqq 0$ and it follows that

$$u(x,t) \leqq \hat{u}(\zeta) + Q_1(\zeta)\psi(t)$$
$$v(x,t) \geqq \hat{v}(\zeta) - Q_2(\zeta)\psi(t)\,.$$

Since $\zeta = h(|\,x\,|) - \sigma t - A + B\log(1+t)$, the Q_i are bounded and $\psi(t) = ke^{-\rho t}$ the desired result follows from this.

Let $u_- = \alpha,\ v_- = \beta$ be the limits of $\hat{u}$ and $\hat{v}$ at $-\infty$ and $u_+ = 1,\ v_+ = 0$ be the limits at $+\infty$. The first step in the proof of (3.9) is to observe that the travelling wave converges exponentially fast to its values at $\pm\infty$. (See Lemma 3.2 on page 349 in Gardner (1982).) That is, there are constants C_0 and λ so that

$$|\hat{u}'(z)|, |\hat{v}'(z)|, |f(\hat{u}(z), \hat{v}(z))|, |g(\hat{u}(z), \hat{v}(z))| \leqq C_0 e^{-\lambda|z|}\,. \tag{3.10}$$

This can be proved by linearizing around the fixed points in question. The next step is

Lemma 3.2 *We can find* $Q_i^+, Q_i^- > 0$ *for* $i = 1,2$ *so that*

$$-\nabla f(u_-, v_-) \cdot (Q_1^-, -Q_2^-) > 0 \quad \nabla g(u_-, v_-) \cdot (Q_1^-, -Q_2^-) > 0$$
$$-\nabla f(u_+, v_+) \cdot (Q_1^+, -Q_2^+) > 0 \quad \nabla g(u_+, v_+) \cdot (Q_1^+, -Q_2^+) > 0\,.$$

Proof. Geometrically, this is possible because the zero set of f viewed as a function of u downcrosses the zero set of g at these fixed points. To check this algebraically we note that

$$\nabla f = (-p - rv, -p - ru) \tag{3.11}$$
$$\nabla g = (-2q(1-u-v) - rv, -2q(1-u-v) - ru)\,,$$

so we have

$$-\nabla f \cdot (Q_1, -Q_2) = (p + rv)Q_1 - (p + ru)Q_2$$
$$\nabla g \cdot (Q_1, -Q_2) = -(2q(1-u-v) + rv)Q_1 + (2q(1-u-v) + ru)Q_2\,.$$

At $(u_+, v_+) = (1,0)$ this says $pQ_1^+ - (p+r)Q_2^+ > 0$ and $rQ_2^+ > 0$ which is clearly possible. At (u_-, v_-) we want to pick Q_i^- so that

$$(p + rv_-)Q_1^- > (p + ru_-)Q_2^-$$
$$(2q(1-u_- - v_-) + rv_-)Q_1^- < (2q(1-u_- - v_-) + ru_-)Q_2^-$$

which is possible if (note that we can set $Q_2^- = 1$ without loss of generality)

$$\frac{p + ru_-}{p + rv_-} < \frac{2q(1-u_- - v_-) + ru_-}{2q(1-u_- - v_-) + rv_-}\,.$$

To show that this holds, we note that $u_- < v_-$ so

$$2pru_- + prv_- < 2prv_- + pru_- .$$

Using the equality $1 - u_- - v_- = p/q$ (see (1.4)) we can rewrite the last equation as

$$2q(1 - u_- - v_-)ru_- + prv_- < 2q(1 - u_- - v_-)rv_- + pru_- .$$

Adding $p2q(1 - u_- - v_-) + r^2u_-v_-$ to both sides of the previous inequality we have

$$(p + ru_-)(2q(1 - u_- - v_-) + rv_-) < (p + rv_-)(2q(1 - u_- - v_-) + ru_-) ,$$

which implies the desired inequality. ∎

The next step is to make our choices of constants. The reader will see the reasons for these choices in the proof. We have collected all the choices together here to make it clear that it is possible to make the ten choices we desire.
(i) Pick $\mu > 0$ so that

$$-\nabla f(u_-, v_-) \cdot (Q_1^-, -Q_2^-), \nabla g(u_-, v_-) \cdot (Q_1^-, -Q_2^-) \geqq 4\mu$$

$$-\nabla f(u_+, v_+) \cdot (Q_1^+, -Q_2^+), \nabla g(u_+, v_+) \cdot (Q_1^+, -Q_2^+) \geqq 4\mu .$$

(ii) Pick $\eta > 0$ so that $Q_*\eta \leqq \mu$.
(iii) Pick $\kappa \leqq 1$ so that $\kappa Q_* \leqq \min\{p/r, 1\}$ and if $x, y \geqq 0$ and $x + y \leqq 1 + Q_*\kappa$ then $g_x(x, y) \leqq \eta$. Here g_x denotes the partial derivative of g with respect to x. Note that our proof of quasi-monotonicity shows (after change of variables) that $g_x(x, y) \leqq 0$ when $x, y \geqq 0$ and $x + y \leqq 1$.
(iv) Pick $\alpha \in (0, Q_0\kappa)$ so that if $\|(a_-, b_-) - (u_-, v_-)\|_1 \leqq \alpha$, and $\|(a_+, b_+) - (u_+, v_+)\|_1 \leqq \alpha$ then

$$-\nabla f(a_-, b_-) \cdot (Q_1^-, -Q_2^-), \nabla g(a_-, b_-) \cdot (Q_1^-, -Q_2^-) \geqq 3\mu$$

$$-\nabla f(a_+, b_+) \cdot (Q_1^+, -Q_2^+), \nabla g(a_+, b_+) \cdot (Q_1^+, -Q_2^+) \geqq 3\mu .$$

(v) Pick $K_1 \geqq K_0$ so that if $z \leqq -K_1$ then $\|(\hat{u}, \hat{v}) - (u_-, v_-)\|_1 \leqq \alpha$ and if $z \geqq K_1$ then $\|(\hat{u}, \hat{v}) - (u_+, v_+)\|_1 \leqq \alpha$.
(vi) Pick ρ small enough so that $\rho Q_* \leqq \mu$, and $\rho \leqq \lambda\sigma/3$. Here σ is the wave speed and λ is the constant in (3.3).
(vii) Pick $K_2 \geqq K_1$ large enough so that $(2d + \sigma + 1)C_0 e^{-\lambda K_2} \leqq \kappa\mu$.
(viii) Pick $\beta > 0$ so that $\hat{u}'(z), -\hat{v}'(z) \geqq \beta$ when $|z| \leqq K_2$.
(ix) Let $R_* = (p + 4q + 2r + \eta)Q_*$ and pick B large enough so that for all $t \geqq 0$

$$\frac{B}{2(t+1)} \geqq \frac{d-1}{1 + \sigma t/3} \quad \text{and} \quad \frac{B\beta}{4(t+1)} \geqq (\mu + (1 + \sigma)Q_* + R_*)\kappa e^{-\rho t} .$$

(x) Choose $A \geqq 0$ large enough so that $s(t) = -A + B\log(t+1) \leqq -1 - K_2 + (\sigma t/3)$ for all $t \geqq 0$.

Lemma 3.3 $\gamma(x, 0) \geqq 0$ *and* $\delta(x, 0) \geqq 0$.

Proof. Recall that

$$\gamma(x,t) = \hat{u}(\zeta) + Q_1(\zeta)\psi(t) - u(x,t)$$

$$\delta(x,t) = v(x,t) - \hat{v}(\zeta) + Q_2(\zeta)\psi(t)\,,$$

where $\psi(t) = \kappa e^{-\rho t}$, $\zeta = h(|\,x\,|) - \sigma t + s(t)$ and $s(t) = -A + B\log(1+t)$. When $t = 0$ we have $\psi(0) = \kappa$ and $\zeta = h(|\,x\,|) - A$. Let

$$\bar{u}(x) = \hat{u}(h(|\,x\,|) - A) + Q_1(h(|\,x\,|) - A)\kappa$$

$$\underline{v}(x) = \hat{v}(h(|\,x\,|) - A) - Q_2(h(|\,x\,|) - A)\kappa\,.$$

We want to show that $\bar{u}(x) \geqq u(x,0)$ and $\underline{v}(x) \leqq v(x,0)$. When $|\,x\,| \geqq K_1 + A + 1/3$ which is $\geqq 1$ since $K_1 \geqq K_0 \geqq 1$ and $A \geqq 0$, we have

$$h(|\,x\,|) - A = |\,x\,| - 1/3 - A \geqq K_1 \geqq K_0\,,$$

so $Q_i(h(|\,x\,|) - A) = Q_i^+ \geqq Q_0$ by (3.5) and (3.6). The choice of K_1 in (v) implies $\hat{u}(h(|\,x\,|) - A) \geqq 1 - \alpha$ and $\hat{v}(h(|\,x\,|) - A) \leqq \alpha$ so the choice of $\alpha < Q_0\kappa$ in (iv) implies that $\bar{u}(x) > 1$ and $\underline{v}(x) < 0$ and the desired inequalities hold for $|\,x\,| \geqq K_1 + A + 1/3$. Since the travelling waves are monotone and $Q_i(z) \geqq Q_0$ by (3.6) we have

$$\bar{u} \geqq u_- + Q_0\kappa \quad \underline{v} \leqq v_- - Q_0\kappa$$

for any x. If we take $\varepsilon = Q_0\kappa$ and $L = (K_1 + A + 1/3)$ in Proposition 3 then our assumptions imply that when $\|x\|_\infty \leqq L, u(x,0) \leqq u_- + Q_0\kappa$ and $v(x,0) \geqq v_- - Q_0\kappa$. The two regions $\|x\|_\infty \leqq L$ and $|\,x\,| \geqq L$ cover all of space so the proof is complete. □

The first step in computing $\mathcal{L}\gamma$ and $\mathcal{L}\delta$ is to observe that

$$\frac{d}{dx_i} k(h(|\,x\,|)) = k'(h(|\,x\,|))h'(|\,x\,|)\left(\sum x_i^2\right)^{-1/2} x_i$$

$$\frac{d^2}{dx_i^2} k(h(|\,x\,|)) = k''(h(|\,x\,|))h'(|\,x\,|)^2\left(\sum x_i^2\right)^{-1} x_i^2$$
$$+ k'(h(|\,x\,|))h''(|\,x\,|)\left(\sum x_i^2\right)^{-1} x_i^2$$
$$+ k'(h(|\,x\,|))h'(|\,x\,|)\left\{-\left(\sum x_i^2\right)^{-3/2} x_i^2 + \left(\sum x_i^2\right)^{-1/2}\right\}$$

$$\Delta k(h(|\,x\,|)) = k''(h(|\,x\,|))h'(|\,x\,|)^2 + k'(h(|\,x\,|))\left\{h''(|\,x\,|) + h'(|\,x\,|)\frac{d-1}{|\,x\,|}\right\}.$$

Recalling that $\mathcal{L} = \frac{\partial}{\partial t} - \Delta$, $\gamma(x,t) = \hat{u}(\zeta) + Q_1(\zeta)\psi(t) - u(x,t)$, and taking $k(y) = \hat{u}(\zeta)$ and then $k(y) = Q_1(\zeta)\psi(t)$ where $\zeta = y - \sigma t + s(t)$ we have

$$\mathcal{L}\gamma = (-\sigma + s')\hat{u}' + (-\sigma + s')Q_1'\psi + Q_1\psi' - u_t$$
$$- (\hat{u}'' + Q_1''\psi)h'(|\,x\,|)^2 - (\hat{u}' + Q_1'\psi)\left(h''(|\,x\,|) + h'(|\,x\,|)\frac{d-1}{|\,x\,|}\right) + \Delta u\,.$$

Substituting $\psi' = -\rho\psi$, $-\hat{u}'' = \sigma\hat{u}' + f(\hat{u},\hat{v})$, and $-u_t + \Delta u = -f(u,v)$ we have

(3.12) $$\mathcal{L}\gamma = (-\sigma + s')\hat{u}' + (-\sigma + s')Q_1'\psi - Q_1\rho\psi + \{\sigma\hat{u}' + f(\hat{u},\hat{v}) - Q_1''\psi\}h'(|x|)^2 - (\hat{u}' + Q_1'\psi)\left(h''(|x|) + h'(|x|)\frac{d-1}{|x|}\right) - f(u,v)$$

$$= \left(-\sigma(1 - h'(|x|)^2) + s' - h''(|x|) - h'(|x|)\frac{d-1}{|x|}\right)\hat{u}' + \left(-\sigma + s' - h''(|x|) - h'(|x|)\frac{d-1}{|x|}\right)Q_1'\psi - Q_1''\psi h'(|x|)^2 - Q_1\rho\psi + f(\hat{u},\hat{v})(h'(|x|)^2 - 1) + \{f(\hat{u},\hat{v}) - f(u,v)\}\,.$$

Similarly differentiating $\delta(x,t) = v(x,t) - \hat{v}(\zeta) + Q_2(\zeta)\psi(t)$ gives

$$\mathcal{L}\delta = v_t - (-\sigma + s')\hat{v}' + (-\sigma + s')Q_2'\psi + Q_2\psi' - \Delta v + (\hat{v}'' - Q_2''\psi)h'(|x|)^2 + (\hat{v}' - Q_2'\psi)\left(h''(|x|) + h'(|x|)\frac{d-1}{|x|}\right)$$

and substituting $v_t - \Delta v = g(u,v)$, $\psi' = -\rho\psi$, and $\hat{v}'' = -\sigma\hat{v}' - g(\hat{u},\hat{v})$ we have

(3.13) $$\mathcal{L}\delta = g(u,v) - (-\sigma + s')\hat{v}' + (-\sigma + s')Q_2'\psi - Q_2\rho\psi - \{\sigma\hat{v}' + g(\hat{u},\hat{v}) + Q_2''\psi\}h'(|x|)^2 + (\hat{v}' - Q_2'\psi)\left(h''(|x|) + h'(|x|)\frac{d-1}{|x|}\right)$$

$$= \left(-\sigma(1 - h'(|x|)^2) + s' - h''(|x|) - h'(|x|)\frac{d-1}{|x|}\right)(-\hat{v}') + \left(-\sigma + s' - h''(|x|) - h'(|x|)\frac{d-1}{|x|}\right)Q_2'\psi - Q_2''\psi h'(|x|)^2 - Q_2\rho\psi - g(\hat{u},\hat{v})(h'(|x|)^2 - 1) + \{g(u,v) - g(\hat{u},\hat{v})\}\,.$$

The right-hand sides of (3.12) and (3.13) are lengthy but fortunately they are very similar and several of the terms are easy to deal with since $h'(|x|) = 1$ and $h''(|x|) = 0$ when $|x| \geqq 1$. The hard part of the proof is to deal with $|\zeta| \geqq K_2$ and in this case it is the differences $f(\hat{u},\hat{v}) - f(u,v)$ and $g(u,v) - g(\hat{u},\hat{v})$ that keep $\mathcal{L}\gamma$ and $\mathcal{L}\delta$ positive so we begin by investigating those terms. Since our aim is to prove (3.9),

our bounds on f assume $\gamma = 0 \quad \delta \geqq 0$

our bounds on g assume $\gamma \geqq 0 \quad \delta = 0$.

Lemma 3.4 *Let* $R_* = (p + 4q + 2r + \eta)Q_*$ *be the constant in* (ix).

$$f(\hat{u},\hat{v}) - f(u,v),\, g(u,v) - g(\hat{u},\hat{v}) \geqq \begin{cases} 2\mu\psi & \text{when } |\zeta| \geqq K_2 \\ -R_*\psi & \text{otherwise} \end{cases}\,.$$

Proof. Now, from (3.11) it is easy to see that $f_v(u,v) = -p - ru \leqq 0$ whenever $u \geqq -p/r$. If $\gamma = 0$ and $\delta \geqq 0$ then using (3.4) and (3.8) $u = \hat{u} + Q_1\psi$ and $v \geqq \hat{v} - Q_2\psi \geqq -Q_*\kappa \geqq -p/r$ by the choice of κ in (iii) so since f is decreasing in its second argument

$$f(u,v) = f(\hat{u} + Q_1\psi, v) \leqq f(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi)\,.$$

Subtracting both sides from $f(\hat{u},\hat{v})$ gives

$$f(\hat{u},\hat{v}) - f(u,v) \geqq f(\hat{u},\hat{v}) - f(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi)\,. \tag{3.14}$$

To deal with the difference of the g's we note that if $\gamma \geqq 0$ and $\delta = 0$ then $u \leqq \hat{u} + Q_1\psi$ and $v = \hat{v} - Q_2\psi$. Inside the triangle $x, y \geqq 0$, and $x + y \leqq 1$ we have $g_x(x,y) \leqq 0$. To control the contribution to the difference of the g's from moving outside the triangle we note that

$$u + v \leqq \hat{u} + Q_1\psi + \hat{v} - Q_2\psi \leqq 1 + Q_*\psi \tag{3.15}$$

since $\hat{u} + \hat{v} \leqq 1$ and $0 \leqq Q_i \leqq Q_*$. So our choice of κ in (iii) implies that for the points we encounter outside the triangle $g_u(u,v) \leqq \eta$ and hence

$$g(u,v) = g(u, \hat{v} - Q_2\psi) \geqq g(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi) - Q_*\psi\eta\,. \tag{3.16}$$

Subtracting $g(\hat{u},\hat{v})$ from both sides and recalling the choice of η in (ii) we have

$$\begin{aligned} g(u,v) - g(\hat{u},\hat{v}) &\geqq g(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi) - g(\hat{u},\hat{v}) - Q_*\psi\eta \\ &\geqq g(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi) - g(\hat{u},\hat{v}) - \mu\psi\,. \end{aligned} \tag{3.17}$$

To estimate the differences of f and g that appear in the lower bounds in (3.14) and (3.17) we observe that

$$\begin{aligned} f(\hat{u},\hat{v}) - f(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi) &= -\nabla f(w)\cdot(Q_1, -Q_2)\psi \\ g(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi) - g(\hat{u},\hat{v}) &= \nabla g(z)\cdot(Q_1, -Q_2)\psi \end{aligned} \tag{3.18}$$

where w and z lie on the line segment that connects $(\hat{u},\hat{v})$ to $(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi)$. Up to this point all of our calculations have been valid for any ζ and t. Now the choice of K_1, which is $\leqq K_2$, implies that $\|(\hat{u},\hat{v}) - (u_-, v_-)\|_1 \leqq \alpha$ for $\zeta \leqq -K_2$ and $\|(\hat{u},\hat{v}) - (u_+, v_+)\|_1 \leqq \alpha$ for $\zeta \geqq K_2$. So it follows from the choice of α in (iv) that if $|\zeta| \geqq K_2$ then

$$\begin{aligned} f(\hat{u},\hat{v}) - f(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi) &\geqq 3\mu\psi \\ g(\hat{u} + Q_1\psi, \hat{v} - Q_2\psi) - g(\hat{u},\hat{v}) &\geqq 3\mu\psi\,. \end{aligned} \tag{3.19}$$

Combining (3.19) with (3.14), and (3.17) gives the result for $|\zeta| \geqq K_2$.
For the other result, we observe that by (3.11) and (3.15) the right hand side of (3.18) can be estimated by

$$\begin{aligned} -\nabla f(w)\cdot(Q_1, -Q_2)\psi &= ((p + rv)Q_1 - (p + ru)Q_2)\psi \\ &\geqq -(p + r(1 + Q_*\psi))Q_*\psi \end{aligned}$$

since $Q_1 \geqq 0, -Q_2 \geqq -Q_*$, and $u \leqq 1 + Q_*\psi$; and

$$\nabla g(z) \cdot (Q_1, -Q_2)\psi = (-(2q(1-u-v)+rv)Q_1 + (2q(1-u-v)+ru)Q_2)\psi$$
$$\geqq -(2q+r)(1+Q_*\psi)Q_*\psi$$

since $1 \geqq 1-u-v \geqq -Q_*\psi$ and $v \leqq 1+Q_*\psi$. Now $\psi(t) = \kappa e^{-\rho t}$ and our choice of κ in (iii) implies $Q_*\psi \leqq 1$ so combining the last result with (3.18), (3.14), and the first inequality in (3.17) it follows that for all ζ

$$f(\hat{u},\hat{v}) - f(u,v) \geqq -(p+2r)Q_*\psi \geqq -R_*\psi \tag{3.20}$$
$$g(u,v) - g(\hat{u},\hat{v}) \geqq -(4q+2r+\eta)Q_*\psi \geqq -R_*\psi$$

and the proof is complete. □

To deal with the rest of (3.12) and (3.13), we observe that if we write $\hat{w}$ for $\hat{u}$ or $-\hat{v}$, $e(x,y)$ for $f(x,y)$ or $-g(x,y)$ and i for 1 or 2, then two expressions look the same and can be divided into four pieces

first $\quad \left(-\sigma(1-h'(|x|)^2) + s' - h''(|x|) - h'(|x|)\dfrac{d-1}{|x|}\right)(\hat{w}')$

second $\quad \left(-\sigma + s' - h''(|x|) - h'(|x|)\dfrac{d-1}{|x|}\right) Q_i'\psi - Q_i''\psi h'(|x|)^2$

third $\quad -Q_i\rho\psi$

fourth $\quad e(\hat{u},\hat{v})(h'(|x|)^2 - 1)$.

To estimate these terms we divide the argument into three cases.

Case 1. $|x| \leqq 1 + \sigma t/3$. Now $h(|x|) \leqq |x|$ and A was chosen so that $s(t) \leqq -1 - K_2 + (\sigma t)/3$ for all $t \geqq 0$. Thus

$$\zeta = h(|x|) - \sigma t + s(t) \leqq -K_2 - (\sigma t)/3 .$$

The last inequality implies $\zeta \leqq -K_0$, so $Q_i'(\zeta)$, $Q_i''(\zeta) = 0$, and the second term vanishes. To estimate the first term we observe that $\sigma \geqq 0$, s is increasing, $h''(x) \leqq 2$, and $h'(x) \leqq 2x$ so we have

$$-\sigma(1-h'(|x|)^2) + s' - h''(|x|) - h'(|x|)\frac{d-1}{|x|} \geqq -\sigma - 2 - 2(d-1) = -2d - \sigma$$

and it follows from (3.10) that the first term is larger than

$$-C_0 e^{-\lambda K_2 - \lambda\sigma t/3}(2d+\sigma) .$$

The third term is larger than $-Q_*\rho\psi \geqq -\mu\psi$ by the choice of ρ in (vi). To deal with the fourth term we observe that $0 \geqq h'(|x|)^2 - 1 \geqq -1$, and $\zeta \leqq -K_2 - (\sigma t)/3$ so (3.10) implies that the fourth term is larger than

$$-C_0 e^{-\lambda K_2 - \lambda\sigma t/3} .$$

Combining the four estimates from above with Lemma 3.4 and recalling that $\psi(t) = \kappa e^{-\rho t}$ it follows that

$$\mathcal{L}\gamma, \mathcal{L}\delta \geqq -(2d+\sigma+1)C_0 e^{-\lambda K_2 - \lambda\sigma t/3} + \mu\kappa e^{-\rho t} .$$

which is positive by the fact that $(2d+\sigma+1)C_0e^{-\lambda K_2} \leqq \kappa\mu$ by the choice of K_2 in (vii) and $\rho \leqq \lambda\sigma/3$ by the choice of ρ in (vi).

Case 2. $|\zeta| \geqq K_2$, $|x| \geqq 1+\sigma t/3$. In this case $Q_i'(\zeta)=0, Q_i''(\zeta)=0, h'(|x|)=1$ and $h''(|x|)=0$ so the second and fourth terms vanish. To bound the first we note that

$$-\sigma(1-h'(|x|)^2)+s'-h''(|x|)-h'(|x|)\frac{d-1}{|x|}$$

$$=s'-\frac{d-1}{|x|} \geqq \frac{B}{t+1}-\frac{d-1}{1+\sigma t/3} \geqq 0$$

by the choice of B. The third term is larger than $-Q_*\rho\psi \geqq -\mu\psi$ by the choice of ρ so using (3.19) to bound the fifth term

$$\mathcal{L}\gamma, \mathcal{L}\delta \geqq 2\mu\psi > 0\,.$$

Case 3. $|\zeta| \leqq K_2$. To handle the bounded interval that remains, we note that by the argument at the beginning of Case 1, $\zeta \geqq -K_2$ implies $|x| \geqq 1+\sigma t/3$ so $h'(|x|)=1$, $h''(|x|)=0$ and the sum of the first and second terms becomes

$$\left(s'-\frac{d-1}{|x|}\right)(\hat{w}'+Q_i'\psi)-Q_i''\psi-\sigma Q_i'\psi\,.$$

To bound this expression, recall that $s(t)=-A+B\log(1+t)$, and in this case $|x| \geqq 1+\sigma t/3$ so our first choice of B in (ix) implies

$$s'-\frac{d-1}{|x|} \geqq \frac{B}{t+1}-\frac{d-1}{1+\sigma t/3} \geqq \frac{B}{2(t+1)}\,.$$

We have assumed in (3.7) that $|Q_i'| \leqq \hat{w}'/2$ so using (viii) and have $\psi \leqq \kappa \leqq 1$ by (iii)

$$\left(s'-\frac{d-1}{|x|}\right)(\hat{w}'+Q_i'\psi) \geqq \frac{B}{2(t+1)}\cdot\frac{\beta}{2}\,.$$

For the remaining pieces we note that $-Q_i''\psi \geqq -Q_*\psi, -\sigma Q_i'\psi \geqq -\sigma Q_*\psi$, the third term is larger than $-Q_*\rho\psi \geqq -\mu\psi$ as usual, and the fourth term vanishes since $h'(|x|)=1$. Using (3.20) to bound the fifth term and combining our estimates we have

$$\mathcal{L}\gamma, \mathcal{L}\delta \geqq \frac{B\beta}{4(t+1)}-(\mu+(1+\sigma)Q_*+R_*)\psi \geqq 0\,,$$

by the second choice of B in (ix).

4 Proof of Proposition 4

In this section we will prove that for fixed r, if p is small and q is large then $\sigma>0$. Now if $U_i(x-\sigma t)$ is a solution of (1.1) then

$$-\sigma U_i' = U_i'' + f_i(U_1,U_2) \tag{4.1}$$

where $f_1(u_1,u_2) = p(1-u_1-u_2) - ru_1u_2$ and $f_2(u_1,u_2) = q(1-u_1-u_2)^2 - ru_1u_2$. To prove that $\sigma > 0$ we will suppose that $\sigma \leqq 0$ and get a contradiction. Imitating an integration by parts trick well known in the p.d.e. literature (see for example, the derivation of (1.7) in Durrett and Neuhauser (1993)), we can multiply by U_i', and integrate to get

$$(4.2) \qquad 0 \leqq -\sigma \int (U_i')^2 dx = \int U_i'' U_i'(x) dx + \int f_i(U_1,U_2) U_i' dx$$
$$= \int f_i(U_1,U_2) U_i' dx \, .$$

Since the antiderivative of $U_i''U_i'$ is $(U_i')^2/2$ which vanishes at $\pm\infty$. To justify this step the estimate in (3.10) is useful.

We want to contradict (4.2) for fixed r if p is small and q is large. To do this we note that $U_1' \geqq 0$ and $f_1(u_1,u_2) \leqq 0$ if (u_1,u_2) lies in $\mathcal{A}$, the region above the curve $p(1-u_1-u_2) - ru_1u_2 = 0$, while $U_2' \leqq 0$ and $f_2(u_1,u_2) \geqq 0$ if (u_1,u_2) lies in $\mathcal{B}$ the region below the curve $q(1-u_1-u_2)^2 - ru_1u_2 = 0$. $\mathcal{A} \cup \mathcal{B}$ covers most of $\{(u_1,u_2) : u_i \geqq 0, u_1 + u_2 \leqq 1\}$. We will obtain our contradiction by showing that when p is small and q is large it is impossible to get from (α,β) to $(1,0)$ without making one of the integrals negative. To carry out this idea, we pick

$$(4.3) \qquad a > \alpha, \quad \beta > 1 - a > b$$

and let

$$S = \inf\{x : U_1(x) > a\} \text{ and } T = \inf\{x : U_2(x) < b\}$$

and divide into two cases:

Case 1. $S \leqq T$. In this case $U_2(x) \geqq b$ for $x \leqq S$ so using the fact that f_1 is decreasing in u_2 for fixed u_1 and changing variables $y = U_1(x)$ we have (see Fig. 4)

$$\int f_1(U_1,U_2)U_1' dx \leqq \int_{-\infty}^{S} f_1(U_1,b)U_1' dx + \int_{S}^{\infty} f_1(U_1,0)U_1' dx$$
$$\leqq \int_{\alpha}^{a} \{p(1-y-b) - ryb\} dy + \int_{a}^{1} p(1-y) dy$$
$$\leqq \int_{0}^{1} p(1-y) dy - \int_{\alpha}^{a} ryb \, dy$$
$$= \frac{p}{2} - \frac{rb}{2}(a^2 - \alpha^2)$$

so the integral is negative if

$$(4.4) \qquad p < rb(a^2 - \alpha^2) \, .$$

Case 2. $S \geqq T$. In this case $U_1(x) \leqq a$ for $x \leqq T$ and $U_1(x) \leqq 1 - U_2(x)$ for all x. The first bound is worse than the second when $U_2 \geqq 1 - a$, so we introduce

$$R = \inf\{x \colon U_2(x) < 1 - a\} \, .$$

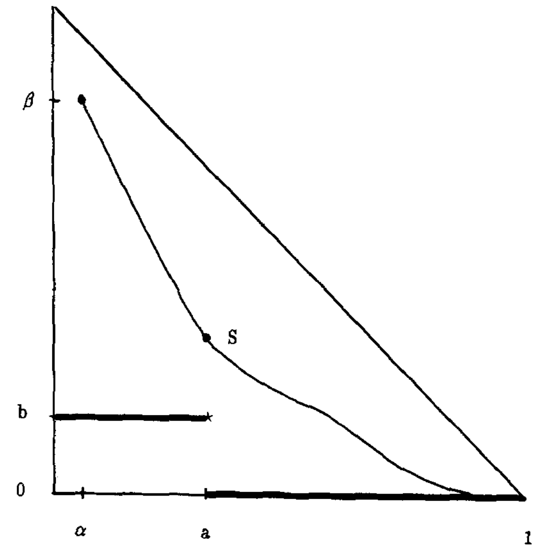

Fig. 4.

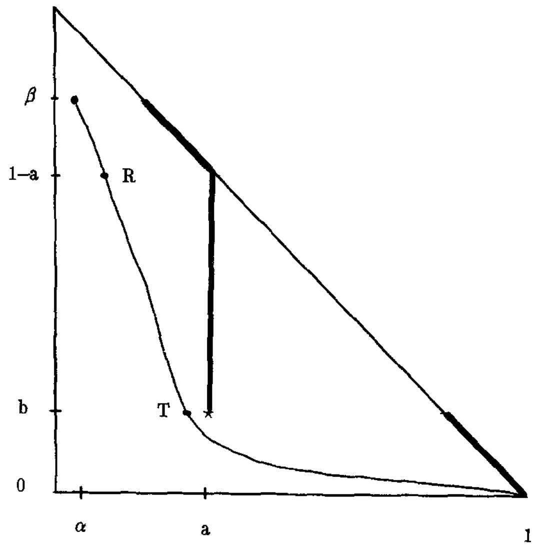

Fig. 5.

Using the fact that f_2 is decreasing in u_1 for fixed u_2 and $U_2' \leqq 0$, then changing variables $y = U_2(x)$, and reversing the limits which introduces a minus sign, we can bound $\int f_2(U_1, U_2)U_2' dx$ (see Fig. 5) by

$$\leqq \int_{-\infty}^{R} f_2(1-U_2,U_2)U_2' dx + \int_{R}^{T} f_2(a,U_2)U_2' dx$$

$$+ \int_{T}^{\infty} f_2(1-U_2,U_2)U_2' dx$$

$$\leqq -\int_{1-a}^{\beta} -ry(1-y)dy - \int_{b}^{1-a} q(1-a-y)^2 - ray\,dy - \int_{0}^{b} -r(1-y)dy$$

$$\leqq \int_{0}^{1} ry(1-y)dy - \int_{b}^{1-a} q(1-a-y)^2 dy$$

$$= r/6 - q(1-a-b)^3/3\,.$$

So the integral is negative if

$$q > r/\{2(1-a-b)^3\}\,. \tag{4.5}$$

Taking $a = 0.48, b = 0.225$ and assuming $\alpha \leqq 0.001$ converts (4.4) and (4.5) into the bounds

$$p < 0.05184r \quad q > r/2(0.295)^3 = 19.48r\,. \tag{4.6}$$

To complete the proof we have to show that when (4.6) holds we have what we assumed in the proof (i.e., (4.3))

$$\alpha \leqq 0.001 < 0.48 = a \quad \beta > 0.52 = 1-a\,. \tag{4.7}$$

To check the first inequality we note that if $p = xr$ and $q = yr$ then

$$\alpha = \frac{(y-x) - \sqrt{(y-x)^2 - 4x^2 y}}{2y}$$

$$= \frac{1}{2}\left\{\left(\frac{y-x}{y}\right) - \sqrt{\left(\frac{y-x}{y}\right)^2 - \frac{4x^2}{y}}\right\}$$

$$= \frac{1}{2}\left(1 - \frac{x}{y}\right)\left\{1 - \sqrt{1 - \frac{4x^2 y}{(y-x)^2}}\right\}.$$

Taking partial derivatives of $4x^2y/(y-x)^2$ and noting that (4.6) implies $x < y$ we find

$$\frac{\partial}{\partial y} = \frac{4x^2}{(y-x)^2} - 2\cdot\frac{4x^2 y}{(y-x)^3} = \frac{-4x^2(x+y)}{(y-x)^3} < 0$$

$$\frac{\partial}{\partial x} = \frac{8xy}{(y-x)^2} + 2\cdot\frac{4x^2 y}{(y-x)^3} = \frac{8xy^2}{(y-x)^3} > 0\,.$$

So decreasing x or increasing y decreases α. When $x = 0.05184$ and $y = 19.48$, $\alpha < 0.001 < 0.48 = a$ so the first bound in (4.7) holds. For these values of x and

$y, \beta = 0.99720$ and a similar monotonicity argument applies to β, so the second bound in (4.7) holds with room to spare.

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