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Super-Tree Random Measures

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We use supercritical branching processes with random walk steps of geometrically decreasing size to construct random measures. Special cases of our construction give close relatives of the super-(spherically symmetric stable) processes. However, other cases can produce measures with very smooth densities in any dimension.

KEY WORDS: Random measures; branching processes; random walks.

1. INTRODUCTION

Let $\{p_k: k \in \mathbf{N}\}$ be a probability distribution on $\{1, 2, ...\}$ with $p_1 < 1$. Let $\mu = \sum_k kp_k$ (>1) and suppose $\sum_k k^2 p_k < \infty$. In this paper we will use a branching process with offspring distribution $\{p_k\}$ to construct random measures using the following recipe:

(i) We start at time 0 with the "progenitor," one individual of generation 0, who immediately splits into k individuals (of generation 1) with probability p_k.

(ii) The jth individual of generation n is displaced from its parent by an amount $a_n X_{n,j}$ where the $X_{n,j}$ are i.i.d. random vectors in $\mathbf{R}^d$ and the a_n are positive numbers.

(iii) We assign mass μ^{-n} at the location of each individual in generation n to construct a random measure $\nu_n(\omega, dx)$ and let $n \to \infty$.

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Of course the first thing we have to shown is that the limit exists. Let S denote a generic random variable equal in law to $\sum_{n=1}^{\infty} a_n X_{n,1}$, and X denote a generic random variable equl in law to each $X_{n,j}$.

Theorem 1. Suppose $\sum_{n=1}^{\infty} a_n X_{n,1}$ converges a.s. Then with probability one the sequence of measures $\nu_n(\omega, dx)$ converges weakly to a random measure $\nu(\omega, x)$ on $\mathbf{R}^d$ which has $P(\nu(\mathbf{R}^d) > 0) = 1$ and $E\nu(A) = P(S \in A)$ for all Borel sets A.

Our main interest here will be in the case $a_n = \rho^n$ where $0 < \rho < 1$, but it will also be interesting to look at $a_n = n^{-q}$ where $q > 0$, and some of our results can be proved for general sequences. To explain our motivation for the construction and the title of the paper consider the special case

$$a_n = \rho^n, \qquad E\exp(it \cdot X_{n,j}) = e^{-|t|^\alpha} \qquad \text{where} \quad 0 < \alpha \leqslant 2$$

In this case the convergence hypothesis in Theorem 1 is trivial to check (see the comment following Theorem 7). We claim that when $\rho = \mu^{-1/\alpha}$ this is a close relative of a super-(spherically symmetric α-stable) process at time $\sigma = \sum_{n=1}^{\infty} \mu^{-n}$ when started from a pointmass at 0 at time 0. To see this, note that (i) $\mu^{-n/\alpha} X_{n,j}$ corresponds to running a symmetric stable process for time μ^{-n} and (i) if we go back to time $t = \sigma - \mu^{-n}$ then we find about $\mu^n = 1/(\sigma - t)$ individuals, which is the same as "the number of individuals" in a superprocess (finite variance branching) at time t who have offspring alive at time σ. To make the phrase in quotes precise we must either use the historical process (see Dawson and Perkins,[3] Prop. 3.5) or think in terms of approximating branching processes.

If $\rho = \mu^{-\beta/\alpha}$ then the number of ancestors at time $t = \sum_{k=1}^{n} \mu^{-k\beta} < \sigma = \sum_{k=1}^{\infty} \mu^{-k\beta}$ scales like $(\sigma - t)^{-1/\beta}$. This corresponds to the number of ancestors at time t of those individuals alive at time σ in a superprocess with critical branching laws in the domain of attraction of a one-sided stable law of index $1 + \beta$ for $0 < \beta \leqslant 1$ (see Dawson and Perkins,[3] Prop. 3.5). Note, however, we can now have any value of $\beta > 0$.

Until otherwise indicated, and this won't be until Theorem 7, we will assume

$$a_n = \rho^n \text{ for some } \rho \text{ in } (0, 1) \text{ and } S = \sum_{1}^{\infty} \rho^n X_{n,1} \text{ converges a.s.} \quad (1.1)$$

Our first results about the measures constructed above computes the Hausdorff dimension of their support. First, we recall the usual heuristic argument for computing such dimensions (see e.g., Mandelbrot[13]). At the time the progenitor splits we get a mean number μ of copies of the original measure scaled by ρ. Thus scaling space by $1/\rho$ gives mean μ copies of the original and we expect the Hausdorff dimension to be

$$D = (\log \mu)/\log(1/\rho) \tag{1.2}$$

providing of course that $D \leqslant d$. To check the heuristic, note that multiplying a d-dimensional cube by 2 results in 2^d copies and $d = (\log 2^d)/\log 2$. Similarly, multiplying the standard Cantor set by 3 results in 2 copies and its dimension is log 2/log 3.

The next result says that the heuristic answer is right as long as the distribution of S is not too singular. First we need some definitions. Let m_γ denote the x^γ-Hausdorff measure and let $\dim(A)$ denote the Hausdorff dimension of a set A in $\mathbf{R}^d$. The carrying dimension of a measure m on $\mathbf{R}^d$ is γ ($\text{cardim}(m) = \gamma$) if for any $\gamma' > \gamma$ there is a Borel set B such that $m(B^c) = 0$ but $m_{\gamma'}(B) = 0$, while for any $\gamma' < \gamma$ and any Borel set B, $m(B^c) = 0$ implies $m_{\gamma'}(B) = \infty$. If m is a nonzero finite measure, define the energy dimension of m to be

$$\text{endim}(m) = \sup\left\{\alpha \colon \iint |x-y|^{-\alpha}\, dm(x)\, dm(y) < \infty\right\} \qquad (\sup \varnothing = 0)$$

It follows easily from a well-known Density Theorem for Hausdorff measures that

$$m(B) > 0 \text{ implies } \dim(B) \geqslant \text{endim}(m) \tag{1.3}$$

(see Dawson,[1] 7.2.1 and 7.2.2), and therefore

$$\text{endim}(m) \leqslant \text{cardim}(m) \tag{1.4}$$

If $\text{cardim}(m) = \text{endim}(m)$, we call this common value the dimension of m ($\dim(m)$) and say that $\dim(m)$ exists. In this case, (1.3) shows that m can be supported by a Borel set of dimension $\dim(m)$ but will assign zero mass to any set of dimension less than $\dim(m)$. This terminology is not standard but is suitable for our purposes.

Theorem 2. Assume (1.1), let S' denote an independent copy of S, and let P_S denote the law of S.

(a) $\text{cardim}(\nu) \leqslant \min(D, \text{cardim}(P_S))$.

(b) $\text{endim}(\nu) \geqslant \min(D, \text{endim}(P_S))$.

(c) If $\dim(P_S)$ exists then so does $\dim(\nu)$ a.s. and $\dim(\nu) = \min(D, \dim(P_S))$ a.s.

(d) If $S - S'$ has a bounded density then $\dim(\nu) = \min(D, d)$ a.s.

Of course the hypothesis of (d) is satisfied if $X - X'$ or X has a bounded density (X' is an independent copy of X) and, in particular, is satisfied if X has a symmetric stable distribution. As an example when (c) applies but (d) does not, suppose $d = 1$, $P(X = 1) = P(X = -1) = 1/2$, and $\rho \in (0, 1/2)$. It is then well-known that $\dim(P_S) = \log 2/\log(1/\rho)$ (in fact this is also the dimension of the closed support). To see the lower bound for $\text{endim}(P_S)$ note that $P(|S - S'| \leqslant \rho^N) 2^N$ is bounded and bounded away from 0 as N becomes large. The obvious covering arguments gives the upper bound for $\text{cardim}(P_S)$. Therefore (c) implies that $\dim(\nu) = \min(\log \mu, \log 2)/\log(1/\rho)$.

Note that the heuristic computation leading to (1.2) says nothing about the form of the distribution of X, except for the implicit assumption that the particles don't land on top of each other. If you find the fact "dimension is independent of the motion" surprising in (d), or think it is wrong, note that when we make our identification with super-(stable processes) $\rho = \mu^{-1/\alpha}$ (or $\mu^{-\beta/\alpha}$).

The previous Theorem does not assert the existence of a density when $\dim(\nu) = d$. However, a simple argument, see e.g., Kallenberg,[11] (Thm. 2.8, Sec. 2.3) shows:

Theorem 3. If (1.1) holds, $S - S'$ has a bounded density, and $D > d$, then almost surely ν is absolutely continuous with respect to Lebesgue measure.

Note that when $E \exp(it \cdot X_{n,j}) = \exp(-|t|^\alpha)$ and $\rho = \mu^{-\beta/\alpha}$, these conclusions agree with known results for super-(spherically symmetric α-stable) processes with $1 + \beta$-stable branching: the dimension of the support is α/β if $d \geqslant \alpha/\beta$ (see Dowson,[2] Chap. 7), while there is a density iff $d < \alpha/\beta$ (see Dawson,[3] Thm. 8.3.1).

We considered the Borel support in Theorem 2 because the closed support (i.e., the smallest closed set K with $\nu(K^c) = 0$) can be much larger. Let $B(z, r) = \{y \colon |y - z| < r\}$ be the open ball of radius r centered at z.

Theorem 4. Suppose that for some $\gamma > 0$ and each $\varepsilon > 0$ there are strictly positive constants $C(\varepsilon)$ and $r(\varepsilon)$ such that

$$P(X \in B(z, \varepsilon |z|)) \geqslant C(\varepsilon) |z|^{-\gamma} \qquad \text{for} \quad |z| > r(\varepsilon) \tag{1.5}$$

If $\mu\rho^\gamma \geqslant 1$, then the closed support of ν is $\mathbf{R}^d$ almost surely.

Note that (1.5) holds if the distribution of X is spherically symmetric and for some $0 < c_1 \leqslant c_2 < \infty$,

$$c_1 r^{-\gamma} \leqslant P(|X| > r) \leqslant c_2 r^{-\gamma} \qquad \text{for } r \text{ large} \tag{1.6}$$

This theorem again agrees with known results for super-(spherically symmetric α-stable) processes. If $\rho = \mu^{-\beta/\alpha}$ for some $\beta \in (0, 1]$ and $\alpha \in (0, 2]$, and X has a symmetric α-stable law, then (1.6) holds with $\gamma = \alpha$, $\mu\rho^{\alpha} = \mu^{1-\beta} \geqslant 1$ (with equality if $\beta = 1$). Therefore the closed support of ν is almost surely $\mathbf{R}^d$. The corresponding result for super-(spherically symmetric α-stable) processes may be found in Evans and Perkins[8] (Cor. 5.3 and the ensuing Example (i)).

Note that for any $\mu > 1 > \rho > 0$ we have $\mu\rho^{\gamma} \geqslant 1$ when γ is close enough to 0, so for any $D < d$ there are examples where the dimension of the Borel support is D but the closed support is $\mathbf{R}^d$. One can construct a nonrandom measure with these properties by taking a fixed nonzero measure ν_0 whose support has dimension D and then considering

$$\sum_{n=1}^{\infty} 2^{-n} \nu_0 \circ \theta_{q^n}$$

where θ_x denotes translation by x and $\{q_n\}$ is a sequence that is dense in $\mathbf{R}^d$. Since a particle of generation n gives rise to a copy of the original measure scaled by μ^{-n} this is probably a reasonable mental picture.

Our next result complements Theorem 4 and gives an upper bound on the dimension of the closed support.

Theorem 5. Suppose $P(|X| > r) \leqslant Cr^{-\gamma}$ for all $r > 0$ and $\mu\rho^{\gamma} < 1$. Then with probability one the closed support of ν is compact and has Hausdorff dimension at most

$$\bar{D} = \frac{\log \mu}{\log(1/\rho) - (1/\gamma) \log \mu}$$

Note that the assumption $\mu\rho^{\gamma} < 1$ is the opposite of the one in Theorem 4 and guarantees that the denominator in $\bar{D}$ is positive. Theorems 2 and 5 imply that when $P(|X| > r) \to 0$ faster than any polynomial and $\dim(P_S) = d$ (e.g., X Gaussian), then the dimension of the closed support is the same as that the Borel support. We believe that if $\mu\rho^{\gamma} < 1$, $S - S'$ has a bounded density, and (1.6) holds, then the closed support has dimension $\bar{D} \wedge d$ but proving this seems difficult.

Theorem 3 establishes the existence of a density but does not give much information about its properties. The next result shows that as the "dimension" D gets larger the measure gets smoother.

Theorem 6. Suppose that $\int_{\mathbf{R}^d} |t|^k \, |Ee^{it \cdot S}| \, dt < \infty$. If $k \in \mathbf{Z}_+$ satisfies $k < D/2 - d$ then almost surely ν has a density function that has a continuous kth derivative.

The condition in Theorem 6 may look difficult to check but noting

$$|Ee^{it\cdot S}| = \left| \prod_{n=1}^{\infty} Ee^{i\rho^n t\cdot X_{n,1}} \right| \leqslant |E\exp(i\rho t\cdot X_{1,1})|$$

one sees that it holds for the stable laws and a number of other examples with smooth densities.

If we suppose, for example, that $\int |E\exp(it\cdot X_{n,1})|\,dt < \infty$ then the inversion formula implies that $X_{n,1}$ (and hence $X_{n,1} - X_{n,2}$) has a bounded continuous density. In this case Theorem 3 implies there is a density when $D > d$ while Theorem 6 implies there is a continuous density when $D > 2d$. This gap comes from the fact that we use the decay of the random characteristic function $\phi(t, \omega) = \int e^{it\cdot x} v(\omega, dx)$ as $|t| \to \infty$ to establish the smoothness of v and this is not sharp. For example in the case of one-dimensional super-Brownian motion ($d = 1$, $X_{n,j}$ are normal, $\rho = \mu^{-1/2}$, $D = 2$), Theorem 6 does not allow us to conclude there is a continuous density even though results of Konno and Shiga[12] (see also Reimers[14]) suggest that it is Hölder continuous of any index $\gamma < 1/2$. In this case, the weakness is in the harmonic analysis and not in the probability. Our estimate $E(|\phi(t)|^2) \leqslant C|t|^{-D}$ (which follows easily from (5.3) later) is sharp but this is not enough for the existence of a continuous density in this case. Roelly-Coppoletta,[15] (Thm. 1.12, p. 54), had similar troubles with this technique.

In the case of the stable laws, we can compute the characteristic function of $S = \sum_{n=1}^{\infty} a_n X_{n,j}$ exactly and this enables us to get results for general sequences a_n. We therefore now drop the assumption (1.1).

Theorem 7. Suppose $E\exp(it\cdot X_{n,j}) = e^{-|t|^\alpha}$ and $A_n = \sum_{m=n+1}^{\infty} a_m^\alpha$ is finite. If

$$\sum_{n=0}^{\infty} \mu^{-n/2} A_n^{-(k+d)/\alpha} < \infty$$

then almost surely v has a C^k density.

First note that $\sum_{m=1}^{\infty} a_m^\alpha < \infty$ is necessary and sufficient for the sum defining S to converge a.s. (Compute the characteristic function of the sum S and recall that for independent random variables, convergence of the infinite sum in distribution implies that it converges almost surely—see e.g. Durrett,[4] Exercise 3.24 in Chap. 1.)

If $a_n = \rho^n$ then $A_n \sim C\rho^{n\alpha}$ and we have convergence when $\mu^{-1/2}\rho^{-(k+d)} < 1$. That is, $(\log \mu)/2 > (k+d)\log(1/\rho)$ or $D/2 - d > k$ which is the same as the conclusion of Theorem 6. If $a_n = n^{-r\alpha}$ with $r > 1$ then $A_n \sim Cn^{1-r}$ so we have convergence for all k and the density is C^∞. It is easy to show

Theorem 8. Suppose the $X_{n,j}$ are independent standard normals and $a_n = n^{-r/2}$. If $r > 3$ then ν has compact support.

So we have situations in which the random measure ν has a C^∞ density with compact support.

Theorem 8 is a simple sufficient condition for compact support. By working harder it is possible (for standard normal $\{X_n\}$) to derive a necessary and sufficient condition on $\{a_n\}$ which, in the context of Theorem 8, shows ν has compact support if $r > 2$ and dense support if $r \leqslant 2$. This latter result, which may yet appear in a future paper of two of us, in fact shows that if the condition fails then the support is a.s. dense. The sufficiency as a simple consequence of Dudley's metric entropy condition for the continuity of Gaussian processes, applied to the positions of the particles as a Gaussian process indexed by a family tree. Steve Evans has pointed out to us that in certain related settings one can derive the necessity of the condition from his results on Gaussian processes indexed by local fields (Evans[6, 7]).

The last and most complicated thing to describe is the genesis of this paper. Allouba and Durrett "invented" this construction in the course of Allouba's dissertation research. They wrote to Perkins for some help with lower bounds on the dimension of the support, only to find that Hawkes and Perkins had extensive notes written in 1990 which contained proofs of Theorems 1 and 2; and the stronger form of Theorem 8 discussed earlier for the special case in which the $X_{n,i}$ are i.i.d. normal with mean 0 and variance 1. Further collaboration including a visit by Durrett to Vancouver in August 1995 led to the results presented here.

The authors would like to express their appreciation to Steve Evans for explaining his results on local fields and thus inspiring us to tackle the non-Gaussian case. They are also grateful for a NSERC Collaborative Projects Grant which supported an informal conference at which the conversations with Evans occurred and which supported Durrett's visit to Vancouver.

2. CONSTRUCTION OF THE RANDOM MEASURE

The notation and approach here follows that of Hawkes.[10] Let $\mathbf{Z}_+$ be the set of positive integers, let $F_n = \mathbf{Z}_+^{\{1,2,\ldots,n\}}$ for $n \geqslant 1$, $F_0 = \{0\}$, and

$F = \bigcup_{n=0}^{\infty} F_n$. F_n is the set of all possible individuals in generation n. If $f \in F_n$ we write $|f| = n$.

Let $I = \mathbf{Z}_+^{\{1,2,...\}}$ be the set of all possible infinite lines of descent. We equip $\mathbf{Z}_+$ with the discrete topology and I with the product topology. Let $\mathscr{B}(I)$ be the Borel subsets of I. Our approach will be to first construct a random measure on $(I, \mathscr{B}(I))$ then transfer the measure to $(\mathbf{R}^d, \mathscr{R}^d)$ where $\mathscr{R}^d =$ the Borel subsets of $\mathbf{R}^d$. Let $\{p_k, k \geqslant 0\}$ be a probability distribution with

$$(*) \qquad p_0 = 0, \qquad p_1 < 1, \qquad \mu = \sum_k k p_k, \qquad \sum_k k^2 p_k < \infty$$

To construct a branching process we let Z^f, $f \in F$, defined on some probability space $(\Omega_b, \mathscr{F}_b, P_b)$, be i.i.d. with $P(Z^f = k) = p_k$. Here, b is for branching.

Let $f \in F_m$ and $n \geqslant m$. If for $g \in F_n$ we let $g|m$ denote the first m coordinates of g (with $g|0 = 0$) then

$$K_n^f(\omega) = \{ g \in F_n : g|m = f,\ g_{j+1} \leqslant Z^{g|j} \text{ for } m \leqslant j < n \}$$

gives the descendants of f in generation n. For $J \subset F_n$ let

$$Y_n^f(\omega, J) = \mu^{-n} \, |J \cap K_n^f(\omega)|$$

where $|A|$ denotes the cardinality of A. In words we assign mass μ^{-n} to each point of K_n^f. Well-known results for branching processes imply that under assumption $(*)$ we have

Theorem 9. If

$$f \in F_m \text{ then } W_f = \lim_{n \to \infty} \mu^{-n} K_n^f \text{ exists a.s.} \tag{2.1}$$

and in L^2, and has $P(W_f > 0) = 1$, $EW_f = \mu^{-m}$.

If $f \in F_m$ we let $D(f)$ and $\bar{D}(f)$ be the possible descendants of f in F and I respectively. That is,

$$D(f) = \{ g \in F : g|m = f \}$$

$$\bar{D}(f) = \{ g \in I : g|m = f \}$$

To define a random measure on $(I, \mathscr{B}(I))$ we begin by fixing ω outside a null set, Γ^c, off which $W_f(\omega)$ exists for all f in F, and setting

$$Y(\omega, \bar{D}(f)) = \begin{cases} W_f & \text{if } f \in K_m^0 \\ 0 & \text{otherwise} \end{cases}$$

Results of Hawkes[10] imply that

Theorem 10. For $\omega \in \Gamma$, $Y(\omega, \cdot)$ extends to a finite measure supported by the branching set

$$K = \{ i \in I : i \mid n \in K_n^0 \text{ for all } n \geq 1 \} \tag{2.2}$$

Let $\{X^f, f \in F\}$ be i.i.d. random variables taking values in $\mathbf{R}^d$. As for the branching random variables it is convenient to give a name, $(\Omega_d, \mathscr{F}_d, P_d)$, to the space on which they are defined. Here d is for displacement. We will construct our random measure on the product space

$$(\Omega, \mathscr{F}, P) = (\Omega_b \times \Omega_d, \mathscr{F}_b \times \mathscr{F}_d, P_b \times P_d)$$

Let a_n be a fixed sequence of positive real numbers. For $f \in F_n$ we let

$$S^f = \sum_{m=1}^{n} a_m X^{f \mid m}$$

be the spatial location of f. Let $\mathbf{1} = (1, 1, ...)$ and suppose that

$$(**) \qquad S = \sum_{m=1}^{\infty} a_m X^{1 \mid m} \qquad \text{converges a.s.}$$

This, of course, implies that for each $i \in I$

$$S^i = \sum_{m=1}^{\infty} a_m X^{i \mid m}$$

converges for $\omega_d \in \Lambda_i$ with $P_d(\Lambda_i) = 1$. Here $\Lambda_i = \{\omega : (i, \omega) \in \Lambda\}$, where

$$\Lambda = \{(i, \omega): \text{the series defining } S \text{ converges}\} \in \mathscr{B}(I) \times \mathscr{F}_d$$

To have S^i defined everywhere we set $S^i = \infty$ on Λ_i^c. Here ∞ is a point not in $\mathbf{R}^d$, which we think of as the point "at infinity" usually added to compactify it. We let $V = \mathbf{R}^d \cup \{\infty\}$ and $\mathscr{V}$ be the σ-algebra generated by $\mathscr{R}^d$ and $\{\infty\}$. Note that $(i, \omega) \to S^i(\omega)$ is a $\mathscr{B}(I) \times \mathscr{F}_d$-measurable map from $I \times \Omega$

to V. We abuse the notation slightly and consider $\mathscr{F}_b$ and $\mathscr{F}_d$ as the obvious sub-sigma fields of $\mathscr{F}$.

For $\omega_b \in \Gamma$, $Y(\omega_b, \cdot)$ defines a measure on $(I, \mathscr{B}(I))$ so we can define a random measure on $(V, \mathscr{V})$ by setting

$$\nu(\omega, A) = Y(\omega_b, \{i: S^i(\omega_d) \in A\}) \tag{2.3}$$

for $A \in \mathscr{V}$ and $\omega \in \Gamma \times \Omega_g$, and setting $\nu(\omega, A) = 0$ if $\omega_b \notin \Gamma$. The displacement random variables are independent of the branching ones and we have

$$\int Y(\omega_b, di)\, P_d(A_i^c) = 0$$

so we have $\nu(\omega, \{\infty\}) = 0$ for $P_b \times P_d$ a.e. ω. Note that

$$\nu(\omega, V) = Y(\omega_b, I) = W_0$$

so $P(\nu(\omega, \mathbf{R}^d) > 0) = 1$ and $E\nu(\omega, \mathbf{R}^d) = 1$. Conditioning on $\mathscr{F}_b$, we have for $A \in \mathscr{R}^d$,

$$\begin{aligned} E(\nu(A) \mid \mathscr{F}_b)(\omega_b, \omega_d) &= E\left(\int 1_A(S^i)\, Y(di) \,\middle|\, \mathscr{F}_d\right)(\omega_b, \omega_d) \\ &= \iint 1_A(S^i(\omega'_d))\, Y(\omega_b, di)\, P_d(d\omega'_d) \\ &= P(S \in A)\, Y(\omega_b, I) = P(S \in A)\, \nu(\omega, \mathbf{R}^d) \end{aligned}$$

Now take means to conclude $E\nu(A) = P(S \in A)$.

Equation (2.3) defines our random measure. Our next task in this section is to show that ν is the measure defined in the Introduction. Let

$$\nu_n(\omega, A) = \mu^{-n}\, |\{f \in K_n^0(\omega_b): S^f(\omega_d) \in A\}|$$

We regard $\nu_n(\omega)$ as a sequence of random variables taking values in the space $M_F(\mathbf{R}^d)$ of finite measures on $(\mathbf{R}^d, \mathscr{R}^d)$ endowed with the topology of weak convergence. Let $\mathscr{G}_n$ be the σ-field generated by $\{Z^f, |f| < n\}$ and $\{X^f, |f| \leqslant n\}$, which are the random variables we need to construct the fist n generations and determine the locations of the particles.

Proof of Theorem 1. Let $T_n = \sum_{m=n+1}^{\infty} a_m X^{1|m}$, and let

$$\bar{\nu}_n(\omega, A) = \int \nu_n(\omega, dx)\, P(x + T_n \in A)$$

To explain this definition, note that

$$\bar{\nu}_n(\omega, A) = E(\nu(A) \mid \mathscr{G}_n)(\omega)$$

To see this, write $\nu(A)$ as an inegral with respect to Y, decompose the integral as a sum of integrals over $\bar{D}(f)$, $f \in K_n^0$, condition on $\mathscr{G}_n \vee \mathscr{F}_b$ and use Fubini's Theorem as in the calculation of the unconditional mean. A standard result from martingale theory implies that the right-hand side of these converges a.s. and in L^1 to $\nu(\omega, A)$.

To estimate the difference between $\bar{\nu}_n$ and ν_n let

$$\mathrm{Lip}_1 = \{\psi: \mathbf{R}^d \to \mathbf{R}: |\psi(x) - \psi(y)| \leqslant |x-y| \wedge 1\}$$

Recall that if

$$d(\nu, \nu') = \sup\left\{\left|\int \psi\, d\nu - \int \psi\, d\nu'\right| : \psi \in \mathrm{Lip}_1\right\}, \qquad \nu, \nu' \in M_F(\mathbf{R}^d)$$

then d is a complete metric inducing the weak topology on $M_F(\mathbf{R}^d)$. See Ethier and Kurtz,[5] (p. 150). If $\psi \in \mathrm{Lip}_1$, then

$$\begin{aligned}\left|\int \psi\, d\nu_n - \int \psi\, d\bar{\nu}_n\right| &= \left|\int \psi(x) - E(\psi(x+T_n))\, \nu_n(\omega, dx)\right| \\ &\leqslant \int E(|\psi(x) - \psi(x+T_n)|)\, \nu_n(\omega, dx) \\ &\leqslant E(T_n \wedge 1)\, \nu_n(\omega, \mathbf{R}^d)\end{aligned}$$

This shows $d(\nu_n, \bar{\nu}_n) \to 0$ a.s. Choose a countable set $\{\psi_i\}$ of bounded continuous functions such that $\eta_n \to \eta$ in $M_F(\mathbf{R}^d)$ if and only if $\eta_n(\psi_i) \to \eta(\psi_i)$ for all $i \in \mathbf{N}$. Since $\bar{\nu}_n(\psi_i) \to \nu(\psi_i)$ for all i a.s., we have $\bar{\nu}_n \to \nu$ a.s. It follows from before that $\nu_n \to \nu$ a.s. The other assertions in Theorem 1 have already been established. □

We end this Section by calculating the mean measure of $\nu \times \nu$. This will be important in the proofs of Theorems 3 and 6. Let

$$S_n = \sum_{m=1}^{n} a_m X^{1 \mid m}$$

T_n be as before and let T'_n be independent of (S_n, T_n) and have the same law as T_n.

Lemma 1. Let ϕ: $\mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ be bounded and measurable. Then

$$E\left(\iint \phi(x, y)\, dv(x)\, dv(y)\right)$$
$$= (1-\mu^{-1})\, E(W_0^2) \sum_{n=0}^{\infty} \mu^{-n} E(\phi(S_n + T_n, S_n + T'_n)) \qquad (2.4)$$

Proof. If suffices to consider $\phi \geqslant 0$. Let $\kappa(i, j) = \sup\{n: i_m = j_m \text{ for all } m \leqslant n\}$ be the generation number of the last common ancestor of i and j, with $\kappa(i, j) = 0$ if $i_1 \neq j_1$, and let

$$p_n(\omega) = \iint 1_{\{\kappa(i,j) \geqslant n\}}\; Y(\omega, di)\; Y(\omega, dj)$$

be the "probability" that $\kappa(i, j) \geqslant n$. It is easy to see that

$$p_n(\omega) = \sum_{f \in K_n^0} W_f^2 \qquad \text{and} \qquad Ep_n(\omega) = \mu^{-n} EW_0^2 \qquad (2.5)$$

The expected value we wish to compute equals

$$E\left(\iint \phi(S^i, S^j)\; Y(di)\; Y(dj)\right)$$
$$= \sum_{n=0}^{\infty} E\left(\iint 1(\kappa(i, j) = n)\, \phi(S^i, S^j)\; Y(di)\; Y(dj)\right)$$

Use Fubini's theorem so see the nth summand equals

$$\iiiint \phi(S^i(\omega_d), S^j(\omega_d))\, dP_d(\omega_d)\, 1(\kappa(i, j) = n)\; Y(\omega_b, di)\; Y(\omega_b, d_j)\, dP_b(\omega_b)$$
$$= E(\phi(S_n + T_n, S_n + T'_n))\, E(p_n - p_{n+1})$$

We now use (2.5) to complete the proof. □

3. DIMENSION OF THE BOREL SUPPORT

Assume (1.1). By the converse to the Borel-Cantelli Lemma this implies $\sum_{m=1}^{\infty} P(|X| > \rho^{-m})$ is finite, which in turn implies that $E(\log^+ |X|) < \infty$. Markov's inequality now gives

$$\lim_{r \to \infty} (\log r)\, P(|X| > r) = 0 \qquad (3.1)$$

Let $\delta \in (0, 1)$ and let $r_n(\delta) \uparrow \infty$ be defined by

$$r_n(\delta) = \inf\{r: P(|X| > r) \leqslant \delta/n^2\}$$

Note that (3.1) implies that for fixed δ,

$$r_n(\delta) = \exp(n\varepsilon_n) \qquad \text{with} \quad \varepsilon_n \to 0 \tag{3.2}$$

Imitating definitions from the previous section, we let

$$\hat{K}_n^f(\omega) = \{g \in K_n^f: |X^{g|m}| \leqslant r_m \text{ for all } m \geqslant n\}$$
$$\hat{Y}_n^f(\omega, J) = \mu^{-n} |J \cap \hat{K}_n^f(\omega)|$$

The reader should note that these quantities and the other ones wearing hats next depend on δ even though we have not recorded this dependence in the notation.

A straightforward generalization of well-known results for branching processes (see e.g., Harris,[9] (p. 13) or Durrett,[4] (pp. 218, 219)) implies that under assumption (*).

Theorem 11. If

$$f \in F_m \text{ then } \hat{W}_f = \lim_{n \to \infty} \mu^{-n} \hat{K}_n^f \text{ exists a.s. and in } L^2 \tag{3.3}$$

and has

$$E\hat{W}_f = \mu^{-m} \prod_{n=m+1}^{\infty} P(|X| \leqslant r_n(\delta))$$

As before, we can define a random measure of $(I, \mathscr{B}(I))$ by setting

$$\hat{Y}(\omega, \bar{D}(f)) = \begin{cases} \hat{W}_f & \text{if } f \in \hat{K}_m^0 \\ 0 & \text{otherwise} \end{cases}$$

for $\omega \in \hat{\Gamma}$, a set of probability 1 on which $\hat{W}_f$ exists for all $f \in F$, and results of Hawkes[10] imply that

Theorem 12. For $\omega \in \hat{\Gamma}$ with $P_b(\hat{\Gamma}) = 1$, $\hat{Y}(\omega, \cdot)$ extends to a finite measure supported by the branching set

$$\hat{K} = \{i \in I: i \,|\, n \in \hat{K}_n^0 \text{ for all } n \geqslant 1\} \tag{3.4}$$

Again, we can define a random measure on $(V, \mathscr{V})$ by setting

$$v^{\delta}(\omega, A) = \hat{Y}(\omega_b, \{i: S^i(\omega_d) \in A\}) \tag{3.5}$$

for $A \in \mathscr{V}$ and $\omega \in \hat{\Gamma} \times \Omega_d$, and setting $\nu^\delta(\omega, A) = 0$ if $\omega_b \notin \hat{\Gamma}$. We have $\nu^\delta(\omega, A) \leq \nu(\omega, A)$ and

$$E(\nu - \nu^\delta)(\omega, \mathbf{R}^d) = 1 - \prod_{n=1}^{\infty} P(|X| \leq r_n(\delta)) \leq \delta \sum_{n=1}^{\infty} n^{-2} \tag{3.6}$$

Let $\hat{K}^0_\infty = \bigcup_{n=1}^\infty \hat{K}^0_n$, $\hat{\Delta}_n = \sup\{|S^g - S^f|: f \in \hat{K}^0_\infty, g \in \hat{K}^0_\infty, g \in D(f)\}$ and

$$b_n = \sum_{m=n+1}^{\infty} \rho^m r_m(\delta)$$

It follows from our definitions that $\hat{\Delta}_n \leq b_n$ and from (3.2) that

$$\text{If } \rho < \sigma < 1 \qquad \text{then} \quad \hat{\Delta}_n \leq \sigma^n \quad \text{for large } n \tag{3.7}$$

3.1. Upper Bound on the Carrying Dimension

Let $c > \mu$ and $1 > \sigma > \rho$. From (3.3) and (3.7) it follows that if $n \geq N_0(\omega)$ then $|\hat{K}^0_n| \leq c^n$ and $\hat{\Delta}_n \leq \sigma^n$. When this occurs the support of ν^δ can be covered by c^n balls of radius σ^n. If

$$\alpha = -(1 + \varepsilon)(\log c)/\log \sigma$$

with $\varepsilon > 0$ then $c^n(\sigma^n)^\alpha = c^{-n\varepsilon} \to 0$. This shows that ν^δ is supported by a set of zero α-dimensional Hausdorff measure. Letting $\delta = 1/k^2$, $k \to \infty$, and using (3.6) and Borel-Cantelli we see that the same is true of ν. Since $\varepsilon > 0$, $c > \mu$, and $\sigma > \rho$ are arbitrary, we conclude that the carrying dimension of ν is less than or equal to $(\log \mu)/\log(1/\rho)$.

Let $\bar{d} > \text{cardim}(P_S)$ and let B support P_S and satisfy $\dim(B) \leq \bar{d}$. Then $\nu(B^c) = 0$ a.s. because its mean is zero by Theorem 1. This shows that $\text{cardim}(\nu) \leq \text{cardim}(P_S)$ a.s. and completes the proof of Theorem 2(a). □

3.2. Lower Bound on the Energy Dimension

Let $0 < \alpha < \min(D, \text{endim}(P_S))$. If S and S' are as in Theorem 2 and

$$\mathscr{E}_\alpha(\nu) = \iint |x - y|^{-\alpha} \, d\nu(x) \, d\nu(y)$$

is the Riesz α-energy of ν, then (2.4) implies

$$E(\mathscr{E}_\alpha(\nu)) = (1-\mu^{-1})\, E(W_0^2) \sum_{n=0}^{\infty} \mu^{-n} E(|T_n - T_n'|^{-\alpha})$$

$$= C \sum_{n=0}^{\infty} (\mu\rho^{\alpha})^{-n}\, E(|S-S'|^{-\alpha})$$

and this is finite by the choice of α. This shows $\mathscr{E}_\alpha(\nu) < \infty$ a.s. and we can conclude $\text{endim}(\nu) \geqslant \alpha$. This proves Theorem 2(b), and (c) is then immediate from this and (a).

To prove (d) it suffices to show that if $S-S'$ has a bounded density f, then we have $\text{endim}(P_S) \geqslant d$. Let $\alpha < d$ and note that

$$E(|S-S'|^{-\alpha}) = \int f(x)\, |x|^{-\alpha}\, dx \leqslant c\, \|f\|_\infty \int_0^1 r^{-\alpha+d-1}\, dr + 1 < \infty$$

This completes the proof of Theorem 2, so we turn to the proof of Theorem 3.

Proof of Theorem 3. The first step is to let $Q(x, r)$ be the cube of side $2r$ centered at x and show

$$\sup_n E\left(\int \nu(dx)\, \nu(Q(x, \rho^n))\, \rho^{-nd}\right) < \infty \tag{3.8}$$

The quantity in question is

$$\sup_n E\left(\iint 1(y - x \in Q(0, \rho^n))\, d\nu(y)\, d\nu(x)\right) \rho^{-nd}$$

By (2.4), this equals

$$C \sup_n \sum_{m=0}^{\infty} \mu^{-m}((T_m - T_m') \in Q(0, \rho^n))\, \rho^{-nd}$$

$$= C \sup_n \sum_{m=0}^{\infty} \mu^{-m} P((S-S') \in Q(0, \rho^{n-m}))\, \rho^{-nd}$$

As $S-S'$ has a bounded density, this is bounded by

$$C \sup_n \sum_{m=0}^{\infty} \mu^{-m} \rho^{nd-md} \rho^{-nd} = C \sum_{m=0}^{\infty} (\mu\rho^d)^{-m}$$

which is finite because $d < D = (\log \mu)/\log(1/\rho)$.

To get from (3.8) to the desired conclusion, recall the martingale proof of the Radon-Nikodym theorem. Let η be a measure with support in the unit cube. Let λ be Lebesgue measure on $[0, 1]^d$. Let η_r be the part of η absolutely continuous with respect to λ, and let $\eta_s = \eta - \eta_r$ be the singular part. Subdivide the unit cube into 2^{nd} cubes I_i^n with side 2^{-n} in the obvious way. Then we have (see Durrett,[4] p. 209).

Theorem 13. $M_n = \eta(I_i^n)/\lambda(I_i^n)$ on I_i^n defines a martingale with

$$M_n \to \begin{cases} d\eta_r/d\lambda & \lambda\text{-a.s.} \\ \infty & \eta_s\text{-a.s.} \end{cases} \tag{3.13}$$

Thus if $\eta_s \neq 0$, Fatou's lemma implies

$$\liminf_{n \to \infty} \int M_n \eta(x) = \infty$$

Applying this reasoning to the random measure ν restricted to a translate of the unit cube, and noting that if $\rho^m \geqslant 2^{-n} > \rho^{m+1}$, then the cube $Q(x, \rho^m)$ contains the cube of side 2^{-n} to which x belongs, we see that if ν has a singular part then the quantity inside the expectation in (3.8) converges to ∞. This contradicts (3.8) (use Fatou's lemma) and Theorem 3 is proved.

4. PROPERTIES OF THE CLOSED SUPPORT

To prove Theorem 4, we recall $B(x, r) = \{y: |x-y| < r\}$ take $r \leqslant 1$, and prove

Lemma 2. Suppose $\mu\rho^\gamma \geqslant 1$. If $x \in \mathbf{R}^d$ and $\varepsilon > 0$, then

$$P(\nu(B(x, r)) > 0) > 1 - \varepsilon \tag{4.1}$$

Proof. Consider the truncated process defined in Section 3, let $\hat{Z}_n = |\hat{K}_n^0|$ be the associated nonhomogeneous branching process and $\Omega_\infty = \{\hat{Z}_n > 0 \text{ for all } n\}$ be the survival event. The first step is to pick $\delta > 0$ small enough so that the survival probability, $P(\Omega_\infty) \geqslant 1 - \varepsilon/4$. To see that this is possible note that since $p_0 = 0$ we always have $(1, 1,\ldots 1) \in K_n^0$, so $P(\hat{Z}_n > 0) \geqslant 1 - \delta \sum_{m=1}^\infty m^{-2}$.

Next we note that $\hat{Z}_n/\mu^n \to \hat{W}$ a.s. with $P(\hat{W} > 0) = P(\Omega_\infty)$ so we can pick an $a > 0$ and a larger integer N_1 so that

$$P(\hat{Z}_n \geqslant a\mu^n \text{ for all } n \geqslant N_1) \geqslant 1 - \varepsilon/2$$

Choose $N_2 \geqslant N_1$ such that

$$\sum_{N_2+1}^{\infty} \rho^m r_m(\delta) < r/2 \tag{4.2}$$

Let $n_0(\omega)$ be the first $n \geqslant N_2$ such that $\hat{Z}_n \geqslant a\mu^n$ and there is an f in $\hat{K}_n^0$ such that $S^f \in B(x, r/2)$ ($n_0 = \infty$ if no such n exists). All particles in the nth generation of the truncated process lie in the ball of radius

$$R_\delta = \sum_{m=1}^{\infty} \rho^m r_m(\delta) < \infty$$

So under our assumption (1.4) on the tail of the distribution of $|X|$, for $n \geqslant N_2$ (make N_2 larger if necessary) each particle in the nth generation of the truncated process has probability at least $C(R_\delta, x, r)\, \rho^{n\gamma}$ of having a child (in the nontruncated process) in $B(x, r/2)$. On $\{\hat{Z}_n \geqslant a\mu^n \; \forall n \geqslant N_1\} \equiv \Lambda$, for $n > N_2$ we have

$$\begin{aligned} P(n_0 > n+1 \mid \mathscr{G}_n \vee \mathscr{F}_b)(\omega) &\leqslant 1(n_0(\omega) > n) \prod_{f \in \hat{K}_n^0} (1 - P(S^f(\omega) + \rho^{n+1} X \in B(x, r/2))) \\ &\leqslant 1(n_0(\omega) > n)(1 - C\rho^{n\gamma})^{\hat{Z}_n} \\ &\leqslant 1(n_0(\omega) > n)\, e^{-aC} \end{aligned}$$

It follows that $n_0 < \infty$ a.s. on Λ. Clearly n_0 is a $(\mathscr{G}_n)$-stopping time and we may choose a $\mathscr{G}_{n_0}$-measurable random index f such that on $\{n_0 < \infty\}$, $f \in \hat{K}_{n_0}^0$ and $S^f \in B(x, r/2)$. We claim now that the descendants of f in the truncated population will produce positive mass in $B(x, r)$. Note that on $\{n_0 < \infty\}$, (4.2) shows that

$$|S^i - S^f| < r/2 \qquad \text{for} \quad \hat{Y} \text{ almost every } i \text{ in } \bar{D}(f)$$

and so $\nu^\delta(B(x, r)) \geqslant \nu^\delta(\bar{D}(f)) = \hat{W}_f$. Therefore

$$\begin{aligned} P(\nu(B(x, r)) > 0) &\geqslant P(\nu^\delta(B(x, r)) > 0, n_0 < \infty) \\ &\geqslant P(P(\hat{W}_f > 0 \mid \mathscr{G}_{n_0})\, 1(n_0 < \infty)) \\ &\geqslant (1 - \varepsilon/4)(1 - \varepsilon/2) > (1 - \varepsilon) \end{aligned}$$

This completes the proof. □

Proof of Theorem 5. We begin with a result about the maxima of the displacements for generation n, (4.3), which leads in turn to a bound on the maximum diameter of the set of descendents of individuals in generation n, (4.4), and in turn to an upper bound on the dimension of the closed support.

Lemma 3.

$$\limsup_{n\to\infty} \frac{1}{n} \log \max_{f\in K_n^0} |X^f| \leqslant \log(\mu^{1/\gamma}) \qquad \text{a.s.} \tag{4.3}$$

Proof. It follows from (2.1) that if $c>\mu$ then $|K_n^0|/c^n \to 0$ a.s. Letting $\sigma > c^{1/\gamma}$, using a basic inequality from measure theory and our assumption about the tail of $|X|$

$$P\left(\max_{f\in K_n^0} |X^f| > \sigma^n,\ |K_n^0| \leqslant c^n\right) \leqslant c^n P(|X| > \sigma^n) \leqslant c^n \cdot C\sigma^{-\gamma n}$$

Our choice of σ implies $c\sigma^{-\beta} < 1$ so the right hand side is summable. Since for any $\varepsilon > 0$ we can choose $\sigma < \mu^{1/\beta} + \varepsilon$ the desired result follows from the Borel-Cantelli lemma. □

Let $K_\infty^0 = \bigcup_{n=1}^\infty K_n^0$, $\Delta_n = \sup\{|S^g - S^f| : f\in K_n^0,\ g\in K_\infty^0,\ g\in D(f)\}$. From (4.3) it follows easily that we have

Lemma 4. Suppose $\rho\mu^{1/\gamma} < 1$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \Delta_n \leqslant \log(\rho\mu^{1/\gamma}) \qquad \text{a.s.} \tag{4.4}$$

Proof. Let $\sigma > \mu^{1/\gamma}$ with $\sigma\rho < 1$. (4.3) implies that for $n \geqslant N(\omega)$

$$\max_{f\in K_n^0} |X^f| \leqslant \sigma^n$$

and hence if $n \geqslant N(\omega)$ then $\Delta_n \leqslant \sum_{m=n}^\infty \rho^m \sigma^m$. This implies

$$\limsup_{n\to\infty} \frac{1}{n} \log \Delta_n \leqslant \log(\rho\sigma) \qquad \text{a.s.}$$

and the desired result follows. □

To prove Theorem 5 now, let $c>\mu$ and $\sigma > \rho\mu^{1/\gamma}$. From (2.1) and (4.4) it follows that if $n \geqslant N_0(\omega)$ then $|K_n^0| \leqslant c^n$ and $\Delta_n \leqslant \sigma^n$. When this occurs the closed support of ν can be covered by c^n balls of radius σ^n and, in particular, is compact. If

$$\alpha = -(1+\varepsilon)(\log c)/\log \sigma$$

with $\varepsilon > 0$ then $c^n(\sigma^n)^\alpha = c^{-n\varepsilon} \to 0$. This shows that the α-dimensional Hausdorff measure of the closed support of ν is 0. Since $\varepsilon > 0$, $c > \mu$, and $\sigma > \rho\mu^{1/\gamma}$ are arbitrary, the Hausdorff dimension is less than or equal to $-(\log \mu)/\log(\rho\mu^{1/\gamma})$. □

By repeating the last proof with minor modifications we can prove Theorem 8, as we now show:

Proof of Theorem 8. Our first step is to bound the displacements on level n.

Lemma 5. Suppose the X^f are i.i.d. standard normals. Then there is a constant $A = A(\mu)$ such that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \max_{f\in K_n^0} |X^f| \leqslant A \qquad \text{a.s.} \tag{4.5}$$

Proof. It follows from (2.1) that if $c > \mu$ then $|K_n^0|/c^n \to 0$ a.s. Using a basic inequality from measure theory and a standard result about the tail of the normal distribution (see e.g., Durrett,[4] (Thm. 1.3, Chap. 1)

$$P\left(\max_{f\in K_n^0} |X^f| > A\sqrt{n},\ |K_n^0| \leqslant c^n\right) \leqslant c^n P(|X| > A\sqrt{n}) \leqslant \frac{c^n}{A\sqrt{n}} e^{-A^2 n/2}$$

If we choose A so that $ce^{-A^2/2} < 1$ then the right-hand side is summable and the desired result follows from the Borel-Cantelli lemma. □

To prove Theorem 8 now, we note that (4.5) implies

Lemma 6. If $\sum_{n=1}^{\infty} a_n\sqrt{n} < \infty$ then ν has compact support. □

5. SMOOTHNESS RESULTS

The key to the developments here is the random characteristic function

$$\phi(t, \omega) = \int e^{it\cdot x}\nu(\omega, dx), \qquad t \in \mathbf{R}^d$$

Specifically our aim will be to show that

$$(*) \qquad E\left(\int |t|^k \, |\phi(t,\omega)| \, dt\right) < \infty$$

Once this is done the desired conclusions can be obtained from the following well-known result (see Rudin,[16] Thm. 7.25).

Lemma 7. Let $\psi(t) = \int e^{it \cdot x} \rho(dx)$ where ρ is a finite measure. If $\int |t|^k \, |\psi(t)| \, dt < \infty$ Then $\rho(dx) = f(x) \, dx$ where f has continuous kth order partial derivatives given by

$$f_{x_{i_1} \cdots x_{i_k}}(x) = \frac{1}{(2\pi)^d} \int \prod_{j=1}^{k} (-it_{i_j}) \, e^{-it \cdot x} \phi(t) \, dt \tag{5.1}$$

Note that $\phi(t) = \int \cos(t \cdot x) \, \nu(\omega, dx) + i \int \sin(t \cdot x) \, \nu(\omega, dx)$ and $|a+bi|^2 = a^2 + b^2$, so using (2.4) and the accompanying notation, we have

$$\begin{aligned} E\,|\phi(t)|^2 &= E\left\{\left(\int \cos(t \cdot x) \, \nu(\omega, dx)\right)^2 + \left(\int \sin(t \cdot x) \, \nu(\omega, dx)\right)^2\right\} \\ &= C \sum_{n=0}^{\infty} \mu^{-n} E\{\cos(t \cdot (S_n + T_n)) \cos(t \cdot (S_n + T_n')) \\ &\quad + \sin(t \cdot (S_n + T_n)) \sin(t \cdot (S_n + T_n'))\} \end{aligned}$$

Introducing the notation

$$c_n(x) = E\cos(t \cdot (x + T_n)), \quad s_n(x) = E\sin(t \cdot (x + T_n)), \quad e_n(x) = Ee^{it \cdot (x + T_n)}$$

we can condition on the value of S_n to write

$$\begin{aligned} &E\{\cos(t \cdot (S_n + T_n)) \cos(t \cdot (S_n + T_n')) + \sin(t \cdot (S_n + T_n)) \sin(t \cdot (S_n + T_n'))\} \\ &\quad = E(c_n(S_n)^2) + E(s_n(S_n)^2) = E\,|e_n(S_n)|^2 \end{aligned}$$

since $Ee^{it \cdot (x + T_n)} = E\cos(t \cdot (x + T_n)) + iE\sin(t \cdot (x + T_n))$ and hence $c_n(x)^2 + s_n(x)^2 = |e_n(x)|^2$. Since $e_n(x) = e^{it \cdot x} Ee^{it \cdot T_n}$, we have $|e_n(x)|^2 = |Ee^{it \cdot T_n}|^2$, i.e., $|e_n(x)|^2$ is constant. Combining this with the previous computations we have

$$E\,|\phi(t)|^2 = C \sum_{n=0}^{\infty} \mu^{-n} \, |Ee^{it \cdot T_n}|^2 \tag{5.3}$$

Using the fact that for positive numbers c_m we have

$$\left(\sum_m c_m\right)^{1/2} \leqslant \sum_m c_m^{1/2}$$

(to prove this square both sides) and Jensen's inequality we have

$$E\,|\phi(t)| \leqslant (E\,|\phi(t)|^2)^{1/2} \leqslant C^{1/2} \sum_{n=0}^{\infty} \mu^{-n/2}\,|Ee^{it\cdot T_n}|$$

Integrating and using Fubini's theorem we have

$$E\int_{\mathbf{R}^d} |t|^k\,|\phi(t)|\,dt \leqslant C^{1/2} \sum_{n=0}^{\infty} \mu^{-n/2} \int_{\mathbf{R}^d} |t|^k\,|Ee^{it\cdot T_n}|\,dt \tag{5.4}$$

Proof of Theorem 6. When $a_n = \rho^n$, clearly T_n/ρ^n is equal in law to T_0. Since we have supposed

$$I = \int_{\mathbf{R}^d} |t|^k\,|Ee^{it\cdot T_0}|\,dt < \infty$$

changing variables $t = s/\rho^n$, $dt = ds/\rho^{nd}$ we have

$$\int_{\mathbf{R}^d} |t|^k\,|Ee^{it\cdot T_n}|\,dt = \int_{\mathbf{R}^d} |s|^k\,|Ee^{is\cdot T_0}|\,ds\cdot\rho^{-n(k+d)} = I\cdot\rho^{-n}(k+d)$$

So (5.4) becomes

$$E\int_{\mathbf{R}^d} |t|^k\,|\phi(t)|\,dt \leqslant C^{1/2} \sum_{n=0}^{\infty} \mu^{-n/2}\rho^{-n(k+d)}\cdot I < \infty$$

when $\mu^{-1/2}\rho^{-(k+d)} < 1$. Taking logs and rearranging gives $(k+d)\log(1/\rho) < \log\mu)/2$ or $D/2 - d > k$. □

Proof of Theorem 7. Suppose $Ee^{it\cdot X} = e^{-|t|^\alpha}$.

$$|Ee^{it\cdot T_n}| = \exp(-A_n\,|t|^\alpha) \qquad \text{where} \quad A_n = \sum_{m=n+1}^{\infty} a_m^\alpha$$

Writing things in polar coordinates and then changing variables $r = s/A_n^{1/\alpha}$, $dt = ds/A_n^{1/\alpha}$

$$\int_{\mathbf{R}^d} |t|^k\,|Ee^{it\cdot T_n}|\,dt = C\int_0^\infty r^{k+d-1}e^{-A_n r^\alpha}\,dr = A_n^{-(k+d)/\alpha}\int_0^\infty s^{k+d-1}e^{-s^\alpha/2}\,ds$$

The last integral is finite since we always have $k+d \geqslant 1$. Plugging this result into (5.4) we have

$$E \int_{\mathbf{R}^d} |t|^k \, |\phi(t)| \, dt \leqslant C^{1/2} \sum_{n=0}^{\infty} \mu^{-n/2} A_n^{-(k+d)/\alpha}$$

which proves the desired result. □

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