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A SPATIAL MODEL FOR THE ABUNDANCE OF SPECIES

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The voter model, with mutations occurring at a positive rate α, has a unique equilibrium distribution. We investigate the logarithms of the relative abundance of species for these distributions in $d \geq 2$. We show that, as $\alpha \to 0$, the limiting distribution is right triangular in $d = 2$ and uniform in $d \geq 3$. We also obtain more detailed results for the histograms that biologists use to estimate the underlying density functions.

1. Introduction.

In the seminal paper of Fisher, Corbet and Williams (1943), field data collected at light traps on the number of individuals representing various butterfly and moth species was fitted to a log series distribution ($f_n = C_{\theta,n}\theta^n/n$). Later, other investigators fit various species abundance data in a wide variety of settings to other distributions, including the lognormal [Preston (1948)] and negative binomial [Brian (1953)]. More recently, various mathematical models have been proposed to derive these distributions. [See, e.g., May (1975), Engen and Lande (1996) and the accompanying references.]

Here, as in our paper [Bramson, Cox and Durrett (1996)] on species area curves, we employ a different approach to the abundance of species based on the voter model with mutation. Specifically, we analyze the limiting behavior of the size distributions, for the unique equilibrium, as the mutation rate α goes to 0. To put our ideas in perspective, we begin with a brief review of three of the traditional approaches.

The most popular species abundance distribution is the *lognormal distribution*, which has been fit to data from a wide variety of circumstances, including geographically diverse communities of birds, intertidal organisms, insects and plants. [See Preston (1948, 1962), Williams (1953, 1964), Whittaker (1965, 1970), (1972), Batzli (1969), Hubbell (1995).] The theoretical explanation for the lognormal given on pages 88–89 of May (1975) is typical. Define $r_i(t)$ to be the per capita instantaneous growth rate of the ith species at time t, that is,

$$r_i(t) = \frac{1}{N_i(t)}\frac{dN_i(t)}{dt} = \frac{d}{dt}\ln N_i(t).$$

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The last equation integrates to

$$\ln N_i(t) = \ln N_i(0) + \int_0^t r_i(s)\, ds.$$

If, as May says, "the ever-changing hazards of a randomly fluctuating environment are all important in determining populations," then, one might reason, the integral is a sum of random variables to which the central limit theorem can be applied, and the distribution of abundances should follow a lognormal law.

While the last argument is simple, and maybe persuasive, there are a number of data sets that do not fit the lognormal distribution very well. An alternative to the lognormal model is given by *MacArthur's broken stick distribution* [see MacAuthur (1957, 1960)]. Here, one imagines that the proportions $(p_1, p_2, \dots, p_n)$ of the volume occupied by n given species to be chosen at random from the set of all possible vectors, that is, those with nonnegative coordinates that sum to one. For this reason Webb (1974) calls this the *proportionality space model*. A simple way of generating such p_i's is to put $n-1$ independent uniform random variables on (0, 1) and look at the lengths of the intervals that result, hence, the name, "broken stick distribution." Quoting May's (1975) survey again, "This distribution of relative abundance is to be expected whenever an ecologically homogeneous group of species apportion randomly among themselves a fixed amount of some governing resource." Broken stick abundance patterns have been found in data for birds by MacArthur (1960), Tramer (1969), and Longuet-Higgins (1971).

One of the weaknesses of the "broken stick" approach is that it simply chooses a nice distribution based on symmetry, without a direct consideration of the underlying mechanisms. Engen and Lande (1996) have recently (see their pages 174–175) introduced a dynamic model in which new species enter the community at times given by a Poisson process, and where the log abundances of the species $Y_t = \log(X_t)$ evolve according to the independent diffusion processes

$$dY_t^i = (r - g(\exp(Y_t^i)))\, dt + \sigma(\exp(Y_t^i))\, dB_t^i.$$

Here, $r > 0$ is a fixed growth rate, $g(x)$ is a "density regulation function," and $\sigma(x) = \sigma_e^2 + \sigma_d^2 e^{-x}$, with σ_e being the environmental and σ_d the demographic stochasticity. Engen and Lande then showed that, if $g(x) = \gamma \ln(x + \nu)$, with $\nu = \sigma_e^2/\sigma_d^2$, the species abundances in equilibrium are given by the lognormal distribution. Although the last approach is dynamic, the reader should note that the sizes of the different species there (as well as in May's derivation of the lognormal) are independent. That is, there is no competition between the species, as there is, at least implicitly, in the broken stick model.

Our approach to modelling species abundances will be a combination of the last two approaches described above. We introduce a simple dynamic model in which species "apportion randomly among themselves a fixed amount of some governing resource," which we represent by a grid, and think of as space. In our model, known as the *multitype voter model with mutation*, the state of the system at time t is given by a random function ξ_t: $\mathbb{Z}^d \to (0,1)$, with $\xi_t(x)$ being the type, or species, of the individual at site x at time t. We index our species by values w in the interval $(0,1)$, so we can pick new species at random from the set of possibilities without duplicating an existing species. (One can substitute the term *allele* here for species, and so also interpret this as a spatial infinite alleles model.)

The model has two mechanisms, invasion and mutation, that are described by the following rules:

1. Each site x, at rate 1, invades one of its $2d$ nearest neighbors y, chosen at random, and changes the value at y to the value at x.
2. Each site x, at rate α, mutates, changing to a new type w', chosen uniformly on $(0,1)$.

It is not difficult to show the following asymptotic behavior for ξ_t.

PROPOSITION 1.1. *The multitype voter model with mutation has a unique stationary distribution ξ_∞. Furthermore, for any initial ξ_0, $\xi_t \Rightarrow \xi_\infty$ as $t \to \infty$.*

[See Bramson, Cox and Durrett (1996), hereafter abbreviated BCD.] Here, $\Rightarrow$ denotes weak convergence, which in this setting is just convergence of finite-dimensional distributions.

The rate at which species enter the system through mutation is α, which should be thought of as migration or genetic mutation. Consequently, we want α to be small and investigate the limiting behavior of the species abundance distribution as $\alpha \to 0$. This, of course, requires some notation. We define the *patch size in A* for the type at site x at time t to be the number of sites y in A with $\xi_t(y) = \xi_t(x)$, that is, the number of sites y in A that have the same type as x. Let $N(A,k)$ be the number of types in ξ_∞ with patch size in A equal to k, and, for $I \subset [0,\infty)$, let $N(A,I) = \sum_{k\in I} N(A,k)$.

In this paper, we only consider $A = B(L)$, the cube of side L centered at the origin intersected with $\mathbb{Z}^d$. It is convenient to divide by $|B(L)|$ to obtain the species abundance per unit volume,

$$N^L(I) = \frac{N(B(L),I)}{|B(L)|}.$$

One immediate advantage of this normalization is that, by invoking the ergodic theorem [as in Section I.4 of Liggett (1985)], we can conclude that

$$\lim_{L\to\infty} N^L(I) = N^\infty(I)$$

exists almost surely. Using results in Section 3, it is easy to see that $N^\infty(I)$ is constant almost surely; we refer to $N^\infty(I)$ as the underlying *theoretical abundance distribution*.

Our main results are given in Theorem 1 and its refinements, Theorems 2 and 3. In Theorem 1, we give estimates on $N^L([1, 1/\alpha^y])$, for $y > 0$, where $\alpha \to 0$ and $L \to \infty$ so that αL^2 is bounded away from 0. Before presenting our results, we review what is known in the "mean field" case. Consider the voter model with mutation on the complete graph with n sites, which is also referred to as the *infinite alleles model.* Each site invades one of the other $n-1$ sites chosen at random, at rate 1, and mutation occurs at each site, at rate $\alpha = \theta/(n-1)$. Here, $\theta > 0$ is fixed. The equilibrium distribution is given by the *Ewens sampling formula*. That is, if $k = (k_1, k_2, \dots, k_n)$, where each k_i is a nonnegative integer, and $\sum_i ik_i = n$, the probability that the voter model in equilibrium has exactly k_i species with patch size i, for $i = 1, 2, \dots, n$, is

$$\frac{n!}{\theta_{(n)}} \prod_{i=1}^{n} \frac{\theta^{k_i}}{i^{k_i} k_i!},$$

where $\theta_{(n)} = \theta(\theta+1)\cdots(\theta+n-1)$. [See Section 7.1 of Kelly (1979) for a derivation of the formula.]

Hansen (1990), motivated by the study of random permutations, used this framework to study the species abundance distribution. In analogy with our function $N(B(L), I)$, let $K_n(I)$ be the number of species with patch size in I for the mean field voter model in equilibrium. Hansen proved that, as $n \to \infty$,

$$\frac{K_n([1, n^u]) - \theta u \log n}{\sqrt{\theta \log n}}, \qquad 0 \le u \le 1,$$

converges weakly to a standard Brownian motion (with time parameter u). Donnelly, Kurtz and Tavaré (1991) gave a proof of this result by using a linear birth process with immigration. Arratia, Barbour and Tavaré (1992) introduced a different technique for studying related functionals of patch sizes distributed according to the Ewens sampling formula. Using $\approx$ to denote approximate equality, we note that Hansen's result implies that $K_n([1, n^u]) \approx \theta \log(n^u)$. Since $\theta = \alpha(n-1)$, if we define y by setting $n^u = 1/\alpha^y$, this becomes

$$\frac{K_n([1, 1/\alpha^y])}{n} \approx y\alpha \log(1/\alpha). \tag{1.1}$$

In Theorem 1, we will show that if $\alpha \to 0$, with αL^2 bounded away from zero, then for all $0 < y \le 1$,

$$N^L([1, 1/\alpha^y]) \approx y\,\alpha(\log(1/\alpha))/\gamma_d \quad \text{in } d \ge 3, \tag{1.2}$$

where γ_d is the probability that simple symmetric random walk in $\mathbb{Z}^d$ never returns to the origin. To compare this with the mean field model, it seems natural to set $n = L^d$, in which case (1.1) and (1.2) appear quite similar. However, it is interesting to note that the assumption that αL^2 is bounded away from zero forces $\theta = \alpha(L^d - 1)$ to tend to infinity, which is not consistent with the assumption before (1.1) that θ is constant. More important is the fact that the right side of (1.2) is not correct in $d = 2$. Indeed, the correct result is

$$N^L([1, 1/\alpha^y]) \approx y^2\alpha(\log(1/\alpha))^2/2\pi \quad \text{in } d = 2, \tag{1.3}$$

which is rather different than the form suggested by the mean field result.

We will shortly state precise versions of (1.2) and (1.3). First, we point out that (1.2) [along with (1.1)] is a type of "log-uniform" limit statement, which indicates that $N^L([1,(1/\alpha)^y])$, properly normalized, converges weakly to the uniform distribution in y. Similarly, (1.3) is a "log-triangular" limit statement. To be more precise, we introduce the following notation. Let $\bar{\alpha} = 1/\alpha$, and define, for $y \geq 0$,

$$F_\alpha^L(y) = \begin{cases} N^L([1,\bar{\alpha}^y])/(\alpha(\log\bar{\alpha})^2/2\pi), & \text{in } d = 2, \\ N^L([1,\bar{\alpha}^y])/(\alpha(\log\bar{\alpha})/\gamma_d), & \text{in } d \geq 3, \end{cases}$$

with $F_\alpha^L(y) = 0$ for $y < 0$. We denote by $U(y)$ the distribution function with uniform density on $(0,1)$, that is,

$$U(y) = \begin{cases} 0, & y \leq 0, \\ y, & 0 \leq y \leq 1, \\ 1, & y \geq 1. \end{cases}$$

We denote by $V(y)$ the distribution function with "right triangular" density $2y$ for $0 < y < 1$, that is,

$$V(y) = \begin{cases} 0, & y \leq 0, \\ y^2, & 0 \leq y \leq 1, \\ 1, & y \geq 1. \end{cases}$$

Also, define

$$G_d(y) = \begin{cases} V(y), & \text{in } d = 2, \\ U(y), & \text{in } d \geq 3. \end{cases}$$

THEOREM 1. *Suppose $d \geq 2$. Let $\beta > 0$ and assume that $L = L(\alpha) \geq \beta\bar{\alpha}^{1/2}$. Then, for any $\varepsilon > 0$,*

$$\sup_y P(|F_\alpha^L(y) - G_d(y)| > \varepsilon) \to 0 \quad \text{as } \alpha \to 0. \tag{1.4}$$

Theorem 1 does not apply directly to the histograms of abundance counts reported in the literature. In Preston (1948), for example, abundance counts are grouped into "octaves," 1–2, 2–4, 4–8, 8–16, 16–32, ..., splitting in half the observations that are exactly powers of 2. To avoid trouble with the boundaries, some later investigators [see, e.g., Chapter 3 of Whittaker (1972)] viewed the 1 cell as an interval [0.5,1.5], and then multiplied by 3 to get disjoint classes [1.5,4.5], [4.5,13.5], etc. The use of such histograms in the literature implicitly assumes that the underlying density functions are sufficiently regular to produce "smooth" data. In our setting, the behavior of the density functions is given by a local limit analog of (1.4). For this, we fix a ratio $r > 1$ to be the width of the cells, and look at the volume-normalized abundance of species, $N^L([r^k, r^{k+1}))$. Theorem 2 below provides the desired refinement of Theorem 1.

If, in Theorem 1, convergence of the underlying density functions also were to hold, we would expect that, in $d = 2$,

$$\begin{aligned} N^L([r^k, r^{k+1})) &\approx \frac{\alpha(\log\bar{\alpha})^2}{2\pi}\int_{\log(r^k)/\log\bar{\alpha}}^{\log(r^{k+1})/\log\bar{\alpha}} 2y\,dy \\ &= \frac{\alpha}{2\pi}\big(\big(\log(r^{k+1})\big)^2 - \big(\log(r^k)\big)^2\big) \\ &= \alpha(2k+1)(\log r)^2/2\pi \\ &\approx \alpha k(\log r)^2/\pi \end{aligned}$$

for large k. A similar calculation shows that, in $d \geq 3$, we would expect

$$N^L([r^k, r^{k+1AQ})) \approx \alpha(\log r)/\gamma_d.$$

Theorem 2 shows that the abundance counts $N^L([r^k, r^{k+1}))$ are simultaneously well approximated by the formulas just derived over a wide range. Fix $r > 1$ and $\varepsilon > 0$, and let $E_L(k)$ be the event that our approximation in the kth cell, $[r^k, r^{k+1})$, is off by at least a small factor, that is, $E_L(k)$ is the event

$$\big|N^L\big([r^k, r^{k+1})\big) - \alpha k(\log r)^2/\pi\big| > \varepsilon\alpha k \quad \text{in } d = 2,$$
$$\big|N^L([r^k, r^{k+1})) - \alpha(\log r)/\gamma_d\big| > \varepsilon\alpha \quad \text{in } d \geq 3.$$

Also, set $\gamma_2 = \pi$, and

$$\hat{\alpha} = \begin{cases} \gamma_2\bar{\alpha}/(\log\bar{\alpha}), & \text{in } d = 2, \\ \gamma_d\bar{\alpha}, & \text{in } d \geq 3. \end{cases}$$

THEOREM 2. *Suppose $d \geq 2$. Let $r > 1$, $\beta > 0$, and assume that $L = L(\alpha) \geq \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$. Then, for any $\varepsilon > 0$,*

$$\lim_{\delta\to 0}\limsup_{\alpha\to 0} P\Big(\bigcup_k E_L(k)\colon r^k \in [\delta^{-1}, \delta\hat{\alpha})\Big) = 0. \tag{1.5}$$

The further restriction of L in this theorem, from that in Theorem 1, comes from the fact that we are considering abundance sizes rather than their logarithms. The conclusion of Theorem 2 is almost certainly true under some slightly weaker condition, but since it is an approximation that holds uniformly for about $\log\bar{\alpha}$ size classes, we suspect that $L/\bar{\alpha}^{1/2}$ must go to ∞ at some rate.

The largest patch size covered by Theorem 2 is $[r^k, r^{k+1})$, where r^k is small relative to $\hat{\alpha}$. The distribution of larger patch sizes, that is, those of the form $[a\hat{\alpha}, b\hat{\alpha})$, differs from the distribution of the smaller patch sizes because of their long formation time relative to $\bar{\alpha}$. The precise result is the following.

THEOREM 3. *Suppose* $d \geq 2$. *Let* $\beta > 0$ *and assume that* $L = L(\alpha) \geq \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$. *Then, for any* $\varepsilon > 0$, *and* a, b *with* $0 < a < b$,

$$\lim_{\alpha\to 0} P\left(\left|\hat{\alpha}N^L([a\hat{\alpha}, b\hat{\alpha}]) - \int_a^b z^{-1}e^{-z}\,dz\right| > \varepsilon\right) = 0. \tag{1.6}$$

The reader might wish to check to what extent Theorem 3 is consistent with Theorem 1 if one is allowed greater liberty on how to choose a and b. First, note that the above integral is infinite for $a = 0$, which is consistent with Theorem 1, since $N^L([1, \hat{\alpha}])$ is of order $(\log\bar{\alpha})/\hat{\alpha}$ rather than $1/\hat{\alpha}$. Now, fix $0 < y_1 < y_2 < 1$. We define a and b so that $\bar{\alpha}^{y_1} = a\hat{\alpha}$ and $\bar{\alpha}^{y_2} = b\hat{\alpha}$, and "apply" Theorem 3 to estimate $N^L([\bar{\alpha}^{y_1}, \bar{\alpha}^{y_2}]) = N^L([a\hat{\alpha}, b\hat{\alpha}])$. We might expect, since a and b tend to 0 as $\alpha \to 0$, that for small α,

$$N^L([\bar{\alpha}^{y_1}, \bar{\alpha}^{y_2}]) \approx \frac{1}{\hat{\alpha}}\int_a^b \frac{1}{z}e^{-z}\,dz \approx \frac{1}{\hat{\alpha}}\int_a^b \frac{1}{z}\,dz.$$

The last term equals

$$\frac{1}{\hat{\alpha}}\log(b/a) = \begin{cases} (y_2 - y_1)\alpha(\log\bar{\alpha})^2/\pi, & \text{in } d = 2, \\ (y_2 - y_1)\alpha(\log\bar{\alpha})/\gamma_d, & \text{in } d \geq 3. \end{cases}$$

This is again consistent with Theorem 1 in $d \geq 3$. In $d = 2$, however, these asymptotics fail, since the limit in Theorem 1 is log-triangular rather than log-uniform in y.

We conclude this section by providing a sketch of the reasoning used for Theorems 1–3, and then comment briefly on the behavior of ξ_t in $d = 1$. In Section 2, we will employ a *percolation substructure* to construct ξ_t. We will also construct two coalescing random walk systems. The first is denoted ζ_t^A, with $A \subset \mathbb{Z}^d$. This system starts from $\zeta_0^A = A$, and consists of particles which execute rate-one independent random walks, except that particles coalesce when a particle jumps onto a site occupied by another particle. The second system is denoted $\hat{\zeta}_t^A$ and has the same coalescing random walk dynamics, but, in addition, particles are killed (i.e., removed from the system) independently of the motion at rate α, with killed particles being removed from the system. Each particle in ζ_t^A, or $\hat{\zeta}_t^A$, has a certain *mass*, that is, the number of coalesced particles at that site. We will let $\hat{\zeta}_t^A(k)$ be the set of particles at time t with mass k, and let $\hat{\zeta}_t^A(I) = \bigcup_{k\in I}\hat{\zeta}_t^A(k)$.

We will make use of a *duality* relation on the percolation substructure connecting ξ_t and $\hat{\zeta}_t^A$ to explain, and later prove, Theorem 1. This relationship implies that $N(B(L), I)$ is equal in distribution to the total number of particles in $\hat{\zeta}_t^{B(L)}(I)$ that are killed over times $t \geq 0$. To estimate the latter quantity, we make use of several estimates and two well-known results. Let p_t be the density of particles in the coalescing random walk system $\zeta_t^{\mathbb{Z}^d}$; this can also be written as $p_t = P(0 \in \zeta_t^{\mathbb{Z}^d})$. Also, let n_t be the mass of the particle in $\zeta_t^{\mathbb{Z}^d}$ at the origin (set $n_t = 0$ if there is no such particle). Note that $p_t = P(n_t > 0)$ and is nonincreasing in t. As usual, for $t \to t_0$, $f(t) = o(g(t))$ means $f(t)/g(t) \to 0$, and $f(t) \sim g(t)$ means $f(t)/g(t) \to 1$. Our estimates are

$$\text{if } t = o(\bar{\alpha}) \quad \text{as } \alpha \to 0, \text{ then } |\hat{\zeta}_t^{B(L)}| \approx |\zeta_t^{B(L)}|, \tag{1.7}$$

$$\text{if } t = o(L^2) \quad \text{as } L \to \infty, \text{ then } |\zeta_t^{B(L)}| \approx |\zeta_t^{\mathbb{Z}^d} \cap B(L)|, \tag{1.8}$$

$$\text{if } t = o(L^2) \quad \text{as } L \to \infty, \text{ then } |\zeta_t^{\mathbb{Z}^d} \cap B(L)| \approx p_t|B(L)|. \tag{1.9}$$

Combining (1.7)–(1.9), we have that if $t = o(\bar{\alpha})$ and $L \geq \beta\bar{\alpha}^{1/2}$, then

$$|\hat{\zeta}_t^{B(L)}| \approx p_t|B(L)|. \tag{1.10}$$

To utilize this approximation, we need information on the asymptotic behavior of both p_t and n_t.

From Bramson and Griffeath (1980),

$$p_t \sim \begin{cases} (\log t)/(\pi t), & \text{in } d = 2, \\ 1/(\gamma_d t), & \text{in } d \geq 3, \end{cases} \tag{1.11}$$

and

$$P\big(p_t n_t \leq b \,\big|\, n_t > 0\big) \to \int_0^b e^{-u}\,du, \qquad b > 0, \tag{1.12}$$

as $t \to \infty$.

We now give a heuristic derivation of Theorem 1. Fix y, $0 < y < 1$, and let $I = [1, \bar{\alpha}^y]$. Based on the above connection between $N(B(L), I)$ and $\hat{\zeta}_t^{B(L)}(I)$, and the fact that individual particles in $\hat{\zeta}_t^{B(L)}$ are killed at rate α, we expect that

$$N(B(L), I)) \approx \alpha \int_0^\infty \big|\hat{\zeta}_t^{B(L)}(I)\big|\,dt. \tag{1.13}$$

According to (1.12), the typical particle in $\zeta_t^{\mathbb{Z}^d}$ should have mass size "about" $1/p_t$. (We will actually be working on a logarithmic scale.) In view of (1.11), this suggests that for times $t \leq \bar{\alpha}^y$, most particles in $\zeta_t^{\mathbb{Z}^d}$ will have mass size "smaller" than $\bar{\alpha}^y$, and at later times, few particles will have mass size "smaller" than $\bar{\alpha}^y$. On account of (1.7) and (1.8), this should also be true for the particles in $\hat{\zeta}_t^{B(L)}$. So,

$$\alpha \int_0^\infty \big|\hat{\zeta}_t^{B(L)}(I)\big|\,dt \approx \alpha \int_0^{\bar{\alpha}^y} \big|\hat{\zeta}_t^{B(L)}(I)\big|\,dt.$$

Now, by (1.10),

$$\alpha \int_0^{\bar{\alpha}^y} \big|\hat{\zeta}_t^{B(L)}\big|\,dt \approx \alpha|B(L)| \int_0^{\bar{\alpha}^y} p_t\,dt.$$

Using (1.11), it is easy to see that as $\alpha \to 0$,

$$\alpha \int_0^{\bar{\alpha}^y} p_t\, dt \sim \begin{cases} \dfrac{\alpha}{2\pi} y^2 (\log \bar{\alpha})^2, & \text{in } d = 2, \\ \dfrac{\alpha}{\gamma_d} y \log \bar{\alpha}, & \text{in } d \geq 3. \end{cases} \tag{1.14}$$

By combining the approximations from (1.13) through (1.14), we find that $N^L([1, \bar{\alpha}^y])$ should be, approximately, the right side of (1.14) for small α. This is the limit in (1.4) of Theorem 1 for $0 < y < 1$. For $y \geq 1$, one derives an upper bound on $N^L([1, \infty))$ by using reasoning similar to that for $0 < y < 1$. The approximations (1.8) and (1.9) are employed, although one needs to replace (1.14) and the term $\bar{\alpha}^y$ in the above integrands by suitable quantities.

To understand the restriction $L \geq \beta \bar{\alpha}^{1/2}$ in Theorem 1, we trace backwards in time the position of the type presently at a given site $x \in B(L)$. (This corresponds to the random walk $\zeta_t^{\{x\}}$.) It will typically take about time of order $\bar{\alpha}$ for this path to undergo a mutation, at which point its type is determined. During this time, the random walk will have moved a distance of order $\bar{\alpha}^{1/2}$; this suggests that the "radius" of the patch size for the type at x will be about that large. When $L = \beta \bar{\alpha}^{1/2}$, we may lose a proportion of the patch, because it sticks out of the box $B(L)$, affecting the approximation (1.8). However, since we will be working on a logarithmic scale anyway, this loss is not important. On the other hand, if L is of smaller order than $\bar{\alpha}^{1/2}$, "most" of the patch will lie outside $B(L)$, and we will not observe the underlying theoretical abundance distribution. For Theorem 2, where we do not use a logarithmic scale, this problem of part of a given patch not being contained in $B(L)$ is more serious. Thus, we require L to be larger.

Theorem 3 is closely related to a result of Sawyer (1979). To state his result, let $\nu(x)$ be the patch size at site x for a realization of the equilibrium state of the voter model with mutation, that is, $\nu(x) = |\{z\colon \xi_\infty(z) = \xi_\infty(x)\}|$. Sawyer proved that $E\nu(O) \sim \hat{\alpha}$ as $\alpha \to 0$, and also that

$$P\big(\nu(O)/E\nu(O) \leq b\big) \to \int_0^b e^{-u}\, du, \qquad b > 0. \tag{1.15}$$

The same result obviously holds for any other fixed site x, or for a site chosen at random from $B(L)$. Now, when a site is chosen at random, a patch has probability of being chosen that is proportional to its size. Removing this "size-bias" from Sawyer's result introduces the factor y^{-1} into the density in (1.6). Theorem 3 is thus a weak law of large numbers for $\hat{\alpha} N^L([a\hat{\alpha}, b\hat{\alpha}])$, with the limits in (1.15), after adjusting for the size-bias, giving the corresponding means.

So far, we have not considered the behavior of the voter model with mutation in $d = 1$. The asymptotics, in this case, are different than for higher dimensions. The analog of (1.11) is $p_t \sim 1/(\pi t)^{1/2}$, with the limiting distribution corresponding to (1.12) being given by a folded normal; the analogs of

(1.7)–(1.9) hold as before. In particular, one now has the more rapid growth

$$\int_0^u p_t\,dt \sim 2(u/\pi)^{1/2} \quad \text{as } u \to \infty.$$

Reasoning analogous to that through (1.14) therefore suggests that

$$N(B(L), [1, y(\bar{\alpha})^{1/2}]) \approx \alpha|B(L)| \int_0^{y^2\bar{\alpha}} p_t\,dt \approx \alpha^{1/2} L y \tag{1.16}$$

for small $y > 0$ as $\alpha \to 0$. Here, the upper limit of integration for the integral is less justified than for $d \geq 2$, since we are not operating on a logarithmic scale. According to (1.16), patch sizes, in $d = 1$, should typically be of order of magnitude $\bar{\alpha}^{1/2}$. This, of course, contrasts with the scaling required for our results in $d \geq 2$. A reasonable conjecture is that, in $d = 1$, if $\alpha^{1/2} L(\alpha) \to \infty$ as $\alpha \to 0$, then

$$\bar{\alpha}^{1/2} N^L([1, y\bar{\alpha}^{1/2}]) \to G_1(y), \tag{1.17}$$

for some nondegenerate distribution function G_1.

The remainder of the paper is organized as follows. We describe duality between the voter model and coalescing random walks in Section 2 and introduce some further notation. In Section 3, we prove a random walk estimate that makes (1.8) precise. In Section 4, we prove a variance estimate that enables us to replace $|\zeta_t^{B(L)}(I)|$ and $|\hat{\zeta}_t^{B(L)}(I)|$ by their means. In Section 5, respectively, Section 6, we use the preliminaries developed in Sections 2–4 to prove Theorem 1, respectively, Theorem 2. The proof of Theorem 3 is given in Section 7, and employs estimates similar to those in Sections 5 and 6, together with (1.15).

2. Duality and notation.

The main goals of this section are to construct the voter model with mutation and related quantities from a percolation substructure, and to give the resulting *duality* with coalescing random walk systems. [The voter model and its duality with coalescing random walks were first studied by Clifford and Sudbury (1973) and Holley and Liggett (1975).] As in BCD, we follow the approach of Griffeath (1979) and Durrett (1988), and introduce a percolation substructure $\mathscr{P}$, which is the following collection of independent Poisson processes and random variables:

$\{T_n^x, n \geq 1\},\ x \in \mathbb{Z}^d,$	independent rate-one Poisson processes,
$\{Z_n^x, n \geq 1\},\ x \in \mathbb{Z}^d,$	i.i.d. random variables, $P(Z_n^x = z) = (2d)^{-1}$ if $\lvert z\rvert = 1$,
$\{S_n^x, n \geq 1\},\ x \in \mathbb{Z}^d,$	independent rate-α Poisson processes,
$\{U_n^x, n \geq 1\},\ x \in \mathbb{Z}^d,$	i.i.d. random variables, uniform on $(0, 1)$.

We use $\mathscr{P}$ to construct the voter model with mutation. Informally, the procedure is as follows: at the times T_n^x, site x chooses the site $y = x + Z_x^n$, which adopts the value at x; at times S_n^x, site x undergoes a mutation event, and adopts the value U_n^x.

More formally, we first define the basic voter model η_t by defining certain paths on $\mathrm{R}^d \times [0,\infty)$. At times T_n^x, if $y = x + Z_n^x$, we write a δ at the point (y, T_n^x), and draw an arrow from (x, T_n^x) to (y, T_n^x). We say that there is a path up from $(x, 0)$ to (y, t) if there is a sequence of times $0 = s_0 < s_1 < s_2 \cdots < s_n < s_{n+1} = t$, and spatial locations $x = x_0, x_1, \ldots, x_n = y$, so that we have the following:

1. For $1 \le i \le n$, there is an arrow from x_{i-1} to x_i at time s_i.
2. For $0 \le i \le n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any δ's.

For each set of sites A, we put $\eta_0^A = A$, and define, for $t > 0$,

$$\eta_t^A = \{y\text{: for some } x \in A \text{ there is a path up from } (x,0) \text{ to } (y,t)\}.$$

Here η_t^A is the basic voter model with possible opinions 0 and 1, with *occupied* sites corresponding to the opinion 1. If A denotes the set of sites occupied by 1's at time 0, then η_t^A is the set of sites occupied by 1's at time t.

One can define the multitype voter model analogously. Assume now that the types belong to the interval $(0, 1)$. Given the types of all sites at time 0, the type at site y at time t is the type of the unique site x such that there is a path up from $(x, 0)$ to (y, t) in the percolation substructure. We incorporate mutation into our model using the Poisson processes S_n^x and the uniform random variables U_n^x. Fix ξ_0, where $\xi_0(x) \in (0,1)$ is the type of the site x at time 0. To determine $\xi_t(y)$, choose the unique site x such that there is a path up from $(x, 0)$ to (y, t). If there is no mutation event on this path, put $\xi_t(y) = \xi_0(x)$. Otherwise, let (z, t') be the point on this path with the property that $S_n^z = t'$ for some n, and there are no other mutation events on the path from (z, t') up to (y, t). Then, set $\xi_t(y) = U_n^z$.

An important feature of this construction is that we can construct a dual process on the same probability space. We reverse the directions of the arrows, and define paths going down in the analogous way. For each set of sites B, for fixed t and $0 \le s \le t$, put

$$\zeta_s^{B,t} = \{x\text{: for some } y \in B, \text{ there is a path down from } (y,t) \text{ to } (x, t-s)\}.$$

Then, η_t^A and $\zeta_s^{B,t}$ are *dual* in the sense that

$$\{\eta_t^A \cap B \ne \varnothing\} = \{A \cap \zeta_t^{B,t} \ne \varnothing\}. \tag{2.1}$$

The finite-dimensional distributions of $\zeta_s^{B,t}$, for $s \le t$, do not depend on t, so we can let ζ_s^B denote a process defined for all $s \ge 0$ with these finite-dimensional distributions, and call ζ_s^B the dual of η_t^A. It follows from (2.1), that

$$P\big(\eta_t^A \cap B \ne \varnothing\big) = P\big(A \cap \zeta_t^B \ne \varnothing\big). \tag{2.2}$$

It is easy to see that the dual process ζ_s^B is a *coalescing random walk.* The individual particles in ζ_s^B perform independent rate-one random walks, with the collision rule that when two particles meet, they coalesce into a single particle. We note that (2.2), with $A = \{O\}$ and $B = \mathrm{Z}^d$, gives $p_t = P(\eta_t^{\{O\}} \ne \varnothing)$; this shows p_t is nonincreasing in t.

In a similar fashion, we can define coalescing random walk with killing, $\hat{\zeta}_s^{B,t}$, $s \le t$, by killing, or removing from the system, any particle which experiences a mutation. We also let $\hat{\zeta}_s^B$ denote a process defined for all $s \ge 0$ with the same finite-dimensional distributions as $\hat{\zeta}_s^{B,t}$. When the processes ζ_s^B and $\hat{\zeta}_s^B$ are constructed on a common percolation substructure, $\hat{\zeta}_s^B \subset \zeta_s^B$ always holds.

Connected with these processes, we introduce the following terminology. For any set of sites A, we define the mass of the particle in ζ_t^A at site x and at time t by

$$n_t^A(x) = \sum_{y \in A} 1\{\zeta_t^y = x\},$$

and let $\hat{n}_t^A(x)$ denote the analogous quantity for $\hat{\zeta}_t^A$; note that $n_t^A(x) = 0$ if $x \notin \zeta_t^A$. We keep track of the locations of walks with mass size in a given set I by

$$\zeta_t^A(I) = \{x \in \mathrm{Z}^d \colon n_t^A(x) \in I\},$$

and let $\hat{\zeta}_t^A(I)$ denote the analogous quantity for $\hat{\zeta}_t^A$. Define

$$Y_{t_1,t_2}^A(I) = \text{the number of mutation events occurring on } (\zeta_s^A(I), t_1 \le s < t_2),$$

and let $\hat{Y}_{t_1,t_2}^A$ denote the analogous quantity for $\hat{\zeta}_s^A$. Note that mutations occur at rate α at each site of ζ_s^A, although they do not affect ζ_s^A. One can check that

$$\big||\hat{\zeta}_t^A(I)| - |\zeta_t^A(I)|\big| \le \hat{Y}_{0,t}^A$$

always holds. We will make use of the weaker inequality

$$\big||\hat{\zeta}_t^A(I)| - |\zeta_t^A(I)|\big| \le Y_{0,t}^A. \tag{2.3}$$

When applying the above terminology to the case $A = \mathrm{Z}^d$, we will typically omit the superscript, for example, writing ζ_t for $\zeta_t^{\mathrm{Z}^d}$. Also, we will usually omit the set I when $I = [1, \infty)$.

As in Section 1, we use ξ_∞ to denote the unique equilibrium distribution for ξ_t and $N(A, I)$ to denote the number of species of ξ_∞ with patch size k in A satisfying $k \in I$. By inspecting the percolation substructure and the definitions of ξ_t and $\hat{\zeta}_t^A$, it is not difficult to see that

$$N(A, I) =_d \hat{Y}_{0,\infty}^A(I). \tag{2.4}$$

From this, it is immediate that, for any J given times $0 = t_0 < t_1 < \cdots < t_J = t$,

$$N(A, I) =_d \sum_{i=1}^{J} \hat{Y}_{t_{i-1},t_i}^A(I) + \hat{Y}_{t_J,\infty}^A(I). \tag{2.5}$$

One also has from elementary properties of the Poisson process that, for $0 \leq t_1 < t_2$,

$$(2.6)\qquad \begin{aligned} EY^A_{t_1,t_2}(I) &= \alpha E\left(\int_{t_1}^{t_2} |\zeta^A_s(I)|\,ds\right),\\ E\hat{Y}^A_{t_1,t_2}(I) &= \alpha E\left(\int_{t_1}^{t_2} |\hat{\zeta}^A_s(I)|\,ds\right).\end{aligned}$$

In order to employ (2.5), we need to relate information on the size of realizations of $\int_{t_1}^{t_2} |\hat{\zeta}^A_s(I)|\,ds$ and $\hat{Y}^A_{t_1,t_2}(I)$. A useful tool for doing this is the following comparison.

LEMMA 2.1 (Poisson domination estimate). *Suppose that $\int_{t_1}^{t_2} \hat{\zeta}^A_s(I)\,ds \geq \lambda$ (respectively, $\leq \lambda$) holds on some event G, where $\lambda \in (0,\infty)$. Then there is a Poisson random variable X with mean $\alpha\lambda$ so that $\hat{Y}^A_{t_1,t_2}(I) \geq X$ (respectively, $\leq X$) on G.*

A similar result was used in BCD. To prove the lemma, one constructs a rate-α Poisson process $\{W(t), t \geq 0\}$, such that $\hat{Y}^A_{t_1,t_2}(I) = W(J(t_2)) - W(J(t_1))$, where $J(t) = \int_0^t \hat{\zeta}^A_s(I)\,ds$. The random variable $X = W(J(t_1) + \lambda) - W(J(t_1))$ is Poisson with mean $\alpha\lambda$. By assumption, $J(t_2) \geq J(t_1) + \lambda$ on G, and so $\hat{Y} \geq X$ there.

The following elementary estimate for Poisson random variables will also be useful.

LEMMA 2.2. *Let X be a Poisson random variable, and, for $\lambda > 0$, let $c_\lambda = \lambda \log \lambda - \lambda + 1$. Then, $c_\lambda > 0$ for $\lambda \neq 1$, and*

$$P(X \geq \lambda EX) \leq \exp(-c_\lambda EX), \qquad \lambda > 1,$$
$$P(X \leq \lambda EX) \leq \exp(-c_\lambda EX), \qquad \lambda < 1.$$

PROOF. For any $\theta > 0$,

$$P(X \geq \lambda EX) \leq E(e^{\theta X})e^{-\theta\lambda EX} = \exp\big((e^\theta - 1 - \theta\lambda)EX\big).$$

For $\lambda > 1$ and $\theta = \log \lambda$, $e^\theta - 1 - \theta\lambda = \lambda - 1 - \lambda \log \lambda = -c_\lambda$. Similarly,

$$P(X \leq \lambda EX) \leq E(e^{-\theta X})e^{\theta\lambda EX} = \exp\big((e^{-\theta} - 1 + \theta\lambda)EX\big).$$

For $0 < \lambda < 1$ and $\theta = -\log \lambda$, $e^{-\theta} - 1 + \theta\lambda = \lambda - 1 - \lambda \log \lambda = -c_\lambda$. Also, $c_1 = 0$, $c_1' = 0$ and $c_\lambda'' = 1/\lambda > 0$ for $\lambda > 0$. Hence, $c_\lambda > 0$ for all $\lambda > 0$, $\lambda \neq 1$. □

3. The approximations $\boldsymbol{\zeta}_t^{B(L)} \approx \boldsymbol{\zeta}_t \cap B(L)$ and $\hat{\boldsymbol{\zeta}}_t^{B(L)} \approx \hat{\boldsymbol{\zeta}}_t \cap B(L)$.

The main goal of this section is to show that $\zeta_t^{B(L)}(I)$, respectively, $\hat{\zeta}_t^{B(L)}(I)$, can be approximated, within a tolerable error, by $\zeta_t(I) \cap B(L)$, respectively, $\hat{\zeta}_t(I) \cap B(L)$. This is needed to make precise the heuristic argument given in the introduction, especially the estimates (1.7) and (1.8). Moreover, the processes ζ_t and $\hat{\zeta}_t$ are translation invariant, and hence more tractable than $\zeta_t^{B(L)}$ and $\hat{\zeta}_t^{B(L)}$.

Our first step is a standard large deviations estimate. For $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, we let $\|x\| = \max\{x_1, \dots, x_d\}$.

LEMMA 3.1. *Let S_t be a d-dimensional simple random walk starting at the origin that takes jumps at rate 1, and let $\psi(\theta) = (e^{\theta} + e^{-\theta})/2$ and $I(a) = \sup_{\theta>0}[a\theta - (\psi(\theta) - 1)/d]$. For all $a > 0$,*

$$P\Big(\max_{t \le u} \|S_t\| > au\Big) \le 4d \exp(-uI(a)). \tag{3.1}$$

REMARK. The function $I(a)$ can be computed explicitly from the information given in the statement, but for our purposes, it will be enough to recall that (i) general theory implies $I(a)$ is increasing and convex on $(0, \infty)$, with $I(0) = 0$, and (ii) from the formula, it follows easily that $I(a) \sim a^2d/2$ as $a \to 0$. Thus, from (ii), for some $a_0 > 0$,

$$I(a) \ge a^2d/3 \quad \text{for } 0 \le a \le a_0. \tag{3.2}$$

PROOF. By considering the coordinates S_t^i separately, it is enough to prove that

$$P\Big(\max_{t \le u} S_t^1 > au\Big) \le 2 \exp\big(-uI(a)\big).$$

Since S_t^1 is symmetric, by the reflection principle, it is enough to show

$$P(S_u^1 > au) \le \exp\big(-uI(a)\big).$$

The moment generating function of S_u^1 is given by

$$E \exp(\theta S_u^1) = \exp\big(u\{\psi(\theta) - 1\}/d\big),$$

so for $\theta > 0$, it follows from Chebyshev's inequality that

$$P(S_u^1 > au) \le \exp\big(-\theta au + u\{\psi(\theta) - 1\}/d\big).$$

Optimizing over θ now gives the desired result. $\square$

The next step is to use the estimate just derived to show that, with high probability, the coalescing random walk system started from $B(L)$ does not stray "too far" from $B(L)$ up to time L^2 and also that random walks started outside $B(L)$ do not penetrate "too far" into $B(L)$ by time L^2. To state the precise result, we introduce the following notation. For a given $c > 0$, define

$$w_L(t) = c(\log L)^{1/2}\sqrt{t + (\log L)^2}. \tag{3.3}$$

We note that (a) $w_L(t)$ is "considerably" larger than the displacement we expect from a random walk by time t, and (b) $w_L(0) = c(\log L)^{3/2}$. Now, define

$$\begin{aligned} H_{\text{out}} &= \sum_{x\in B(L)} 1\{\zeta_t^x \notin B(L + w_L(t)) \text{ for some } 0 \le t \le L^2\}, \\ H_{\text{in}} &= \sum_{x\notin B(L)} 1\{\zeta_t^x \in B(L - w_L(t)) \text{ for some } 0 \le t \le L^2\}. \end{aligned} \tag{3.4}$$

Here, H_{out} is the number of particles that start in the box $B(L)$ and escape from $B(L + w_L(t))$ by time L^2. Likewise, H_{in} gives the number of particles that start outside the box $B(L)$ and enter $B(L - w_L(t))$ by time L^2. We consider $\Omega_0 = \{H_{\text{out}} = 0, H_{\text{in}} = 0\}$ to be a "good" event, since on this set, we have adequate control over the movement of random walks in our percolation substructure. The following result shows that we may choose c in (3.3) large enough to make Ω_0 very likely.

LEMMA 3.2. *There exists $c > 0$ such that for large L,*

$$P(\Omega_0) \ge 1 - 1/L^{d+1}. \tag{3.5}$$

PROOF. Using the notation of Lemma 3.1, we have

$$EH_{\text{out}} \le |B(L)|\, P(\|S_t\| > w_L(t)/2 \text{ for some } 0 \le t \le L^2).$$

Since $|B(L)| \le (L+1)^d$, we can prove $P(H_{\text{out}} \ge 1) \le 1/2L^{d+1}$ by showing that, for large L,

$$P(\|S_t\| > w_L(t)/2 \text{ for some } t \le L^2) \le 1/4L^{2d+1}. \tag{3.6}$$

To estimate the left side above, we first note that, for $m \ge 0$,

$$w_L(t) \ge c2^m(\log L)^{3/2} \quad \text{if } t \in [(4^m - 1)(\log L)^2, (4^{m+1} - 1)(\log L)^2).$$

Let m^* be the largest m such that $(4^{m+1} - 1)(\log L)^2 \le L^2$. The probability in (3.6) is bounded above by

$$\sum_{m=0}^{m^*} P(\|S_t\| > c2^{m-1}(\log L)^{3/2} \text{ for some } t \le 4^{m+1}(\log L)^2). \tag{3.7}$$

Lemma 3.1 implies that

$$P(\|S_t\| > a_m u_m \text{ for some } 0 \le t \le u_m) \le 4d \exp(-u_m I(a_m)). \tag{3.8}$$

Taking $u_m = 4^{m+1}(\log L)^2$ and $a_m = c2^{m-1}(\log L)^{3/2}/u_m$, it follows from (3.2) that

$$u_m I(a_m) \ge \frac{c^2 4^{m-1}(\log L)^3}{4^{m+1}(\log L)^2}\,\frac{d}{3} = \frac{dc^2 \log L}{48}.$$

For c large enough, it follows that for each m, the right side of (3.8) is at most $1/L^{2d+2}$. Since m^* is at most a constant multiple of $\log L$, (3.7) is at most a constant multiple of $(\log L)/L^{2d+2}$. For large L, this gives (3.6).

The estimation of H_{in} might at first seem more difficult because of the infinite sum over $x \notin B(L)$. However, any random walk which enters $B(L)$ after starting outside $B(L)$ must pass through the boundary of $B(L)$. Using the percolation substructure, it is easy to see that not more than a Poisson mean-L^2 number of random walks may leave a given site during the time interval $[0, L^2]$. Since the boundary of $B(L)$ has at most CL^{d-1} sites for some finite C, it is not difficult to see that $P(H_{\text{in}} \geq 1)$ is bounded above by

$$CL^{d-1}L^2 P\big(\|S_t\| > w_L(t)/2 \text{ for some } t \leq L^2\big)$$
$$\leq CL^{d+1} \sum_{m=0}^{m^*} P\big(\|S_t\| > c2^{m-1}(\log L)^{3/2} \text{ for some } t \leq 4^{m+1}(\log L)^2\big).$$

Arguing as in the first part of the proof, using (3.1), we find that for large enough c, $P(H_{\text{in}} \geq 1) \leq 1/2L^{d+1}$ for large L. Combined with the corresponding estimate for H_{out}, this proves (3.5). □

Let $\zeta_t^L(I) = \zeta_t(I) \cap B(L)$, and let $\hat{\zeta}_t^L(I)$ denote the analogous quantity for $\hat{\zeta}_t(I)$. On the good event Ω_0, we expect that $\zeta_t^{B(L)}(I) \approx \zeta_t^L(I)$ and $\hat{\zeta}_t^{B(L)}(I) \approx \hat{\zeta}_t^L(I)$. To state the precise meaning of our approximation, let $A(t)$ be the annular region

$$A(t) = B\big(L + w_L(t)\big) - B\big(L - w_L(t)\big), \tag{3.9}$$

with $w_L(t)$ being given by (3.3).

LEMMA 3.3. *Let $\chi = \zeta$ or $\hat{\zeta}$.* (i) *For all $I \subset [1, \infty)$ and $t \leq L^2$,*

$$\Big| \big|\chi_t^{B(L)}(I)\big| - \big|\chi_t^L(I)\big| \Big| \leq \big|\zeta_t \cap A(t)\big| \quad \text{on } \Omega_0.$$

(ii) *For large L, all $I \subset [1, \infty)$ and all $t \leq L^2$,*

$$E\Big| \big|\chi_t^{B(L)}(I)\big| - \big|\chi_t^L(I)\big| \Big| \leq 2|A(t)|p_t.$$

PROOF. It is easy to check from the definition of Ω_0 that for $t \leq L^2$,

$$n_t^{B(L)}(x) = \begin{cases} n_t(x), & \text{for } x \in B(L - w_L(t)), \\ 0, & \text{for } x \notin B(L + w_L(t)). \end{cases}$$

So, on Ω_0, for all $I \subset [1, \infty)$ and $t \leq L^2$,

$$\big|\zeta_t^{B(L)}(I)\big| = \big|\zeta_t^{L - w_L(t)}(I)\big| + \sum_{x \in A(t)} 1\big\{n_t^{B(L)}(x) \in I\big\}. \tag{3.10}$$

Since $1\{n_t^{B(L)}(x) \in I\} \leq 1\{n_t(x) \geq 1\}$, it follows that on Ω_0,

$$\Big|\big|\zeta_t^{B(L)}(I)\big| - \big|\zeta_t^L(I)\big|\Big| \leq |\zeta_t \cap A(t)| \quad \text{for all } I \subset [1, \infty) \text{ and } t \leq L^2. \tag{3.11}$$

This proves (i) for $\chi = \zeta$. The reasoning which led to (3.10) applies equally well to $\hat{\zeta}_t$; moreover, $1\{\hat{n}_t^{B(L)}(x) \in I\} \le 1\{n_t(x) \ge 1\}$ clearly holds. Thus, (3.11), and hence (i), follows for $\chi = \hat{\zeta}$.

To derive (ii) from (i), we note that (i) and the definition of p_t imply

$$E\left(\big||\chi_t^{B(L)}(I)| - |\chi_t^L(I)|\big|; \Omega_0\right) \le |A(t)| p_t.$$

To bound the expectation over Ω_0^c, we note that both $|\zeta_t^{B(L)}|$ and $|\zeta_t^L|$ are bounded above by $|B(L)|$. Thus, by Lemma 3.2,

$$E\left(\big||\zeta_t^{B(L)}(I)| - |\zeta_t^L(I)|\big|; \Omega_0^c\right) \le |B(L)|/L^{d+1} \le 2/L.$$

To complete the proof of (ii) for $\chi = \zeta$, it suffices to show that the right side above is of smaller order than $|A(t)| p_t$. This is trivial, since for $t \le L^2$, using the asymptotics (1.11) for p_t,

$$|A(t)|\, p_t \ge CL^{d-1} w_L(0)/L^2 \ge C w_L(0)/L$$

for an appropriate positive constant C, and since $w_L(0) \to \infty$ as $L \to \infty$. Finally, this argument also holds for $\chi = \hat{\zeta}$. □

In the proof of Theorem 2 we will use the decomposition (2.5), with $t_J = \bar{\alpha} \log \bar{\alpha}$, since it will turn out that mutations after that time can be ignored (i.e., we will see that $\hat{Y}^{B(L)}_{\bar{\alpha} \log \bar{\alpha}, \infty}$ is negligible). To prepare for handling one of the more technical steps in the other terms in (2.5), we give an estimate here that will allow us to adequately control the number of mutations that occur in a suitable space-time region. (This estimate is not needed for the proof of Theorem 1.) Here and later on in the paper, C will stand for a positive constant whose exact value does not concern us and will be allowed to vary from line to line.

LEMMA 3.4. *Fix $c > 0$ as in (3.3), and let $A(t)$ be as in (3.9). Let $T = \bar{\alpha} \log \bar{\alpha}$ and $\beta > 0$, and suppose $L \ge \beta \bar{\alpha}^{1/2} (\log \bar{\alpha})^2$. For small $\alpha > 0$ and appropriate $C > 0$,*

$$E\left(\int_0^T |\zeta_t \cap A(t)|\, dt\right) \le \begin{cases} C|B(L)|, & \text{in } d = 2, \\ C|B(L)|/\log L, & \text{in } d \ge 3. \end{cases} \tag{3.12}$$

PROOF. By translation invariance, the left side of (3.12) equals $\int_0^T |A(t)|\, p_t\, dt$. The proof of (3.12) is simply a straightforward estimation of this integral. Recall the definitions of $w_L(t)$ and $A(t)$. For some constant C, $|A(t)| \le CL^{d-1} w_L(t)$, and therefore

$$\begin{aligned} \int_0^T |A(t)| p_t\, dt &\le \int_0^T CL^{d-1} w_L(t) p_t\, dt \\ &\le CL^{d-1}\left((\log L)^{7/2} + (\log L)^{1/2} \int_{(\log L)^2}^T t^{1/2} p_t\, dt\right). \end{aligned} \tag{3.13}$$

The asymptotics for p_t in (1.11) imply that, for large L,

$$\int_{(\log L)^2}^{T} t^{1/2}\, p_t\, dt \le \begin{cases} CT^{1/2}\log T, & \text{in } d=2,\\ CT^{1/2}, & \text{in } d\ge 3.\end{cases} \tag{3.14}$$

Together, (3.13) and (3.14) give

$$\begin{aligned}&\int_0^T |A(t)|\, p_t\, dt\\ &\quad\le \begin{cases} CL^{d-1}\big((\log L)^{7/2}+(\log L)^{1/2}T^{1/2}\log T\big), & \text{in } d=2,\\ CL^{d-1}\big((\log L)^{7/2}+(\log L)^{1/2}T^{1/2}\big), & \text{in } d\ge 3,\end{cases}\end{aligned} \tag{3.15}$$

for a new choice of C.

It follows from the assumption $L\ge \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$, $\beta>0$, and a little algebra, that $\bar{\alpha}\le L^2/\beta^2(\log L)^4$ for small α. So, for small α,

$$T=\bar{\alpha}\,\log\,\bar{\alpha}\le \frac{2L^2}{\beta^2(\log L)^3}. \tag{3.16}$$

Substitution of (3.16) into (3.15) then gives (3.12) for an appropriate constant C. □

4. Variance estimates and weak laws. The goal of this section is to show that when t is not too close to L^2, $|\zeta_t^{B(L)}(I)|$ and $|\hat{\zeta}_t^{B(L)}(I)|$ can be approximated by their expected values within tolerable errors. Proposition 4.1, which extends Proposition 2 of BCD, makes this precise. For $a>0$, $r>0$ and $I\subset[1,\infty)$, define

$$\Gamma^L_{a,r}(t)=\Big\{\Big||\zeta_t^{B(L)}(I)|-E|\zeta_t^{B(L)}(I)|\Big|>ap_t|B(2L)|/(\log L)^r\Big\}$$

and let $\hat{\Gamma}^L_{a,r}(t)$ denote the analogous quantity for $\hat{\zeta}_t^{B(L)}$.

PROPOSITION 4.1. *Fix $r>0$ and $c_0>0$. There exists a constant C such that for large enough L and $I=[m_1,m_2)$, any $m_1\le m_2$ with $m_1\ge 1$ and $m_2\le\infty$,*

$$\begin{aligned}&P\Big(\bigcup_t \Gamma^L_{8,r}(t)\colon t\in\big[0,c_0L^2/(\log L)^3\big]\Big)\\ &\quad\le \begin{cases} C(\log\log L)(\log L)^{3r-2}, & \text{in } d=2,\\ C(\log L)^{1+3r-d}, & \text{in } d\ge 3,\end{cases}\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&P\Big(\bigcup_t \hat{\Gamma}^L_{8,r}(t)\colon t\in\big[0,c_0L^2/(\log L)^3\big]\Big)\\ &\quad\le \begin{cases} C(\log\log L)(\log L)^{3r-2}, & \text{in } d=2,\\ C(\log L)^{1+3r-d}, & \text{in } d\ge 3.\end{cases}\end{aligned} \tag{4.2}$$

REMARK. We have set $a = 8$ above solely as a matter of convenience. In Sections 5–7, we will set $r = 1/6$. For this choice, the above exponent is $3r-2 = -3/2$ in $d = 2$ and $1 + 3r - d \le -3/2$ in $d \ge 3$; the important point is that in both cases, this exponent is strictly less than -1. For the proof of Theorem 2, we need to consider times up to $c_0 L^2/(\log L)^3$, as in the left sides of (4.1) and (4.2). For the proofs of Theorems 1 and 3, we need consider only times up to $L^2/(\log L)^4$ (in which case the proof of the $d = 2$ estimate would simplify somewhat).

The proof of Proposition 4.1 requires a variance estimate that is closely related to Lemma 4.3 of BCD. Recall from Section 2 the basic voter model η_t^A, the coalescing random walk $\zeta_s^{A,t}$, $s \le t$ and the coalescing random walk with killing $\hat{\zeta}_s^{A,t}$, all of which are defined on the percolation substructure $\mathscr{P}$. Our variance estimate applies to the number of walks, starting from some set A, that, at time t, are of a given mass size and are in $B(L)$.

LEMMA 4.1. *There exists a finite constant C such that for large L, all $A \subset \mathbb{Z}^d$, $I \subset [1,\infty)$ and $t \in [0, L^3]$,*

$$\begin{aligned}
&\operatorname{var}\Biggl(\sum_{x\in B(L)} 1\{n_t^A(x) \in I\}\Biggr) \le CL^d p_t^2 (\log L)^{d/2} (t \vee \log L)^{d/2},\\
&\operatorname{var}\Biggl(\sum_{x\in B(L)} 1\{\hat{n}_t^A(x) \in I\}\Biggr) \le CL^d p_t^2 (\log L)^{d/2} (t \vee \log L)^{d/2}.
\end{aligned} \tag{4.3}$$

PROOF. The arguments for the above two inequalities are identical. Writing j_x for either $1\{n_t^A(x) \in I\}$ or $1\{\hat{n}_t^A(x) \in I\}$, one can expand the left side of (4.3), in either case, as

$$\sum_{x,\,y\in B(L)} \bigl(E(j_x j_y) - E(j_x)E(j_y)\bigr).$$

Our approach will be to specify $l > 0$ (depending on L and t), splitting the above quantity into

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|\le l}} \bigl(E(j_x j_y) - E(j_x)E(j_y)\bigr) + \sum_{\substack{x,\,y\in B(L)\\ \|x-y\|> l}} \bigl(E(j_x j_y) - E(j_x)E(j_y)\bigr). \tag{4.4}$$

The upper bounds for these sums will depend on whether t is "small" or "large," meaning $t \le A_0 \log L$ or $A_0 \log L < t \le L^3$, where

$$A_0 = \frac{12}{d a_0^2}\left(\frac{5d}{2} + 5\right) \tag{4.5}$$

[a_0 is the constant from (3.2)]. This gives us four quantities to compute. The reason for the particular choice of A_0 will become clear later.

Let us begin with a general inequality, which we will need for "small" distances $\|x - y\|$. It follows from the definition of the j_x that for any $l > 0$, $E(j_x j_y) \le P(x \in \zeta_t,\ y \in \zeta_t)$. By Lemma 1 of Arratia (1981),

$$P\bigl(x \in \zeta_t,\ y \in \zeta_t\bigr) \le P(x \in \zeta_t)P(y \in \zeta_t) = p_t^2.$$

Thus, for some constant C,

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|\le l}} E(j_x j_y) \le CL^d l^d p_t^2. \tag{4.6}$$

We consider first the "large" t case, $A_0 \log L < t \le L^3$, where

$$l = \left[\frac{12}{d}\left(\frac{5d}{2}+5\right)\right]^{1/2} (t \log L)^{1/2}. \tag{4.7}$$

Plugging (4.7) into (4.6) implies that, for an appropriate C,

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|\le l}} E(j_x j_y) \le CL^d p_t^2 (t \log L)^{d/2}. \tag{4.8}$$

This gives us a bound for the first sum in (4.4) for large t as needed in (4.3).

To estimate the second sum in (4.4) for $A_0 \log L < t \le L^3$, we let $G_{x,\,y}$ denote the event that η_s^x and η_s^y intersect at some time $s \le t$. Then,

$$E(j_x j_y) - E(j_x)E(j_y) \le P(G_{x,y}). \tag{4.9}$$

A proof of this fact can be given by using two independent graphical substructures to construct versions of η_s^x and η_s^y until the first time they intersect, at which point one switches to a common graphical substructure. See the proof of (2.6) in Griffeath (1979) for more details.

To estimate $P(G_{x,\,y})$, we note that for $x,\, y \in \mathrm{Z}^2$, with $\|x-y\| > l$,

$$G_{x,\,y} \subset \{\eta_s^x \not\subset x + B(l) \text{ or } \eta_s^y \not\subset y + B(l) \text{ for some } s \le t\}. \tag{4.10}$$

Using duality again, as in the estimate of H_{in} in the proof of Lemma 3.2, we see that

$$P(\eta_s^O \not\subset B(l) \text{ for some } s \le t) \le Cl^{d-1} t P\Big(\max_{s\le t} \|S_s\| > l/2\Big) \tag{4.11}$$

for an appropriate constant C. It is straightforward to check from (4.5) and (4.7) that $l/2 \le a_0 t$ for $t \ge A_0 \log L$. Therefore, we may apply the inequality (3.2) and obtain

$$P\Big(\max_{s\le t} \|S_s\| > l/2\Big) \le C \exp(-dl^2/12t),$$

where C depends on d. Plugging in (4.7) gives

$$P\Big(\max_{s\le t} \|S_s\| > l/2\Big) \le C \exp\big(-((5d/2)+5)\log L\big) = CL^{-((5d/2)+5)}. \tag{4.12}$$

Combining (4.9)–(4.12), we obtain, for an appropriate constant C and all $t > A_0 \log L$,

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|> l}} P(G_{x,\,y}) \le CL^{2d} l^{d-1} t L^{-((5d/2)+5)}. \tag{4.13}$$

By substituting in the value of l and rearranging the right side of (4.13), we find that this equals, for a new constant C,

$$CL^d(t\log L)^{d/2}\big[(t/\log L)^{1/2}L^{-(3d/2)-5}\big]. \tag{4.14}$$

Taking into account the restriction $t \le L^3$ and that $d \ge 2$, (4.13) and (4.14) yield

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|>l}} P(G_{x,\,y}) \le CL^d(t\log L)^{d/2}L^{-13/2}. \tag{4.15}$$

In order to show that the right side of (4.15) is bounded above by the right side of (4.3), it suffices to show that $p_t^2 \ge L^{-13/2}$ for $A_0\log L \le t \le L^3$. But this follows immediately from the asymptotics for p_t given in (1.11),

$$p_t^2 \ge C/t^2 \ge C/L^6 \quad \text{for large } t \le L^3.$$

Therefore, by (4.9) and (4.15), we have, for $A_0\log L \le t \le L^3$,

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|>l}} \big(E(j_x j_y) - Ej_x Ej_y\big) \le CL^d p_t^2(t\log L)^{d/2}, \tag{4.16}$$

as needed for (4.3).

We turn to the case of "small" t estimates, $t \le A_0\log L$. Here, we take $l = 2bA_0\log L$, where $b > 1$ will be chosen later. Substituting this value of l into (4.6) gives, for an appropriate constant C,

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|\le l}} E(j_x j_y) \le CL^d p_t^2(\log L)^d. \tag{4.17}$$

This again gives the bound in (4.3) for the first sum in (4.4).

We now estimate the second sum in (4.4) for $t \le A_0\log L$. As remarked after Lemma 3.1, $I(t)$ is convex, with $I(0) = 0$. Also, one has $l/2t \ge b$ for $t \le A_0\log L$. Thus, $I(l/2t) \ge (l/2bt)I(b)$. Using first Lemma 3.1 and then this inequality, we have, for appropriate C,

$$\begin{aligned} P\Big(\max_{s\le t}\|S_s\| > l/2\Big) &\le C\exp(-tI(l/2t)) \le C\exp(-lI(b)/2b) \\ &= C\exp(-A_0 I(b)\log L). \end{aligned}$$

Since $I(b)\to\infty$ as $b\to\infty$, we may choose b sufficiently large so that $A_0 I(b) \ge d+1$. It follows that for such b,

$$P\Big(\max_{s\le t}\|S_s\| > l/2\Big) \le CL^{-d-1}.$$

Therefore, using (4.10) and (4.11), there is a constant C such that for $t \le A_0\log L$,

$$\sum_{\substack{x,\,y\in B(L)\\ \|x-y\|>l}} P(G_{x,\,y}) \le CL^{2d}l^{d-1}tL^{-d-1}.$$

Plugging in $l = 2bA_0 \log L$ and $t \le A_0 \log L$ gives, for new C,

$$\sum_{\substack{x,\, y \in B(L) \\ \|x-y\| > l}} P(G_{x,\, y}) \le CL^{d-1} (\log L)^d.$$

In order to see that the right side of the above inequality is bounded above by the right side of (4.3) for $t \le A_0 \log L$, it suffices to check that $p_t^2 \ge L^{-1}$. But this is easily verified using monotonicity and the asymptotics (1.11), which imply that, for some positive constant C,

$$p_t^2 \ge p_{A_0 \log L}^2 \ge C/(\log L)^2, \qquad t \le A_0 \log L.$$

Therefore, in view of (4.9), we have proved, for $t \le A_0 \log L$,

$$\sum_{\substack{x,\, y \in B(L) \\ \|x-y\| > l}} \bigl(E(j_x j_y) - E j_x E j_y\bigr) \le CL^d p_t^2 (\log L)^d. \tag{4.18}$$

Together with (4.8), (4.16) and (4.17), this proves (4.3). □

We are now ready to prove Proposition 4.1. Let us consider $\zeta_t^{B(L)}$ and outline our approach to the proof of (4.1); the argument for (4.2) is the same. Using differences, it is clearly enough to consider just intervals of the form $I = [1, m)$, $m \le \infty$, with $\Gamma_{4,\, r}^L$ replacing $\Gamma_{8,\, r}^L$ in (4.1). For this, we will define a sequence of times $t(k)$, $k = 0, 1, \dots, K$, with $t(0) = 0$ and $t(K) = c_0 L^2/(\log L)^3$ and show that the sum of the probabilities of the events $\Gamma_{2,\, r}^L(t(k))$ is no larger than the right side of (4.1). We will then argue that "nothing goes wrong" at times in between the times $t(k)$. To estimate $P(\Gamma_{2,\, r}^L(t(k)))$, we would like to obtain a variance estimate from Lemma 4.1, and then apply Chebyshev's inequality; unfortunately, the lemma cannot be used directly on $|\zeta_t^{B(L)}(I)|$. To remedy the situation, we restrict $\zeta_t^{B(L)}$ to $x \in B(2L)$, and introduce the approximating process

$$\check{\zeta}_t^{B(L)}(I) = \bigl\{x \in B(2L) \colon n_t^{B(L)}(x) \in I\bigr\}, \tag{4.19}$$

to which Lemma 4.1 applies. We will then use Lemma 3.2 to show that $\zeta_t^{B(L)}(I)$ and $\check{\zeta}_t^{B(L)}(I)$ are equal with high probability. By combining these arguments, we then prove (4.1). This approach works well in $d \ge 3$, but to obtain the bounds needed in $d = 2$, it must be slightly modified by separately considering the time intervals $[0, c_0 L^2/(\log L)^4]$ and $[c_0 L^2/(\log L)^4, c_0 L^2/(\log L)^3]$.

Proof of Proposition 4.1. We will prove (4.1) for $I = [1, m)$, $m \le \infty$, and $\Gamma_{4,\, r}^L$ replacing $\Gamma_{8,\, r}^L$. The proof of (4.2) is identical, and so we will only briefly comment on it. The argument for (4.1) consists of three parts: (i) the upper bounds on $P(\Gamma_{2,\, r}^L(t))$, $t \le c_0 L^2/(\log L)^q$, $q \ge 1$, given in (4.24), (ii) the upper bounds on $P(\bigcup_t \Gamma_{4,\, r}^L(t) \colon t \le c_0 L^2/(\log L)^q)$, given in (4.30), which imply the desired bounds for $d \ge 3$, and (iii) the refinement of these last bounds needed for $d = 2$.

Upper bounds for $P(\Gamma^L_{2,r}(t))$. We begin by defining the following analog of $\Gamma^L_{a,r}(t)$. For $a>0$, $r>0$ and $I=[1,m)$, set

$$\check{\Gamma}^L_{a,r}(t)=\big\{\big||\check{\zeta}^{B(L)}_t(I)|-E|\check{\zeta}^{B(L)}_t(I)|\big|>ap_t|B(2L)|/(\log L)^r\big\}.$$

Lemma 4.1 [replacing $B(L)$ there with $B(2L)$ and A with $B(L)$] implies that there exists a constant C such that, for large L and $t\le L^3$,

$$\text{var}\big(\big|\check{\zeta}^{B(L)}_t(I)\big|\big)\le CL^d p_t^2(\log L)^{d/2}(t\vee\log L)^{d/2}. \tag{4.20}$$

Given $q\ge 1$, Chebyshev's inequality and (4.20) imply that there exists a constant C such that for large L,

$$P(\check{\Gamma}^L_{1,r}(t))\le C(\log L)^{2r-d(q-1)/2}\quad\text{for } t\le c_0L^2/(\log L)^q. \tag{4.21}$$

From the definition of $w_L(t)$ in (3.3), it is easy to see that $B(L+w_L(t))\subset B(2L)$ for large L and $t=o(L^2/\log L)$. Thus, $\{\zeta^{B(L)}_t(I)\ne\check{\zeta}^{B(L)}_t(I)\}\subset\{H_{\text{out}}\ge 1\}$, and Lemma 3.2 therefore implies

$$P\big(\big|\zeta^{B(L)}_t(I)\big|\ne\big|\check{\zeta}^{B(L)}_t(I)\big|\big)\le 1/L^{d+1}. \tag{4.22}$$

On account of this,

$$E\big|\check{\zeta}^{B(L)}_t(I)\big|\le E\big|\zeta^{B(L)}_t(I)\big|\le E\big|\check{\zeta}^{B(L)}_t(I)\big|+1/L. \tag{4.23}$$

For $t\le L^2$, monotonicity and the asymptotics (1.11) imply that for some positive constant C,

$$|B(2L)|p_t/(\log L)^r\ge CL^{d-2}/(\log L)^r,$$

which is, of course, of larger order than $1/L$ for large L. Moreover, $1/L^{d+1}$ is of smaller order than $(\log L)^{2r-d(q-1)/2}$. Therefore, on account of (4.21), (4.22) and the triangle inequality, there is a constant C such that for large L,

$$P(\Gamma^L_{2,r}(t))\le C(\log L)^{2r-d(q-1)/2}\quad\text{for } t\le c_0L^2/(\log L)^q. \tag{4.24}$$

Later, we will set $q=3$ and then $q=4$.

Upper bounds for $P(\bigcup_t\Gamma^L_{4,r}(t)\colon t\le c_0L^2/(\log L)^q)$. For the sequence of times $t(k)$, $k=0,1,\dots,K$, mentioned before the proof, we set $\lambda=1-(\log L)^{-r}$, $t(0)=0$, and let

$$t(k)=\inf\big\{t\colon E|\zeta^{B(L)}_t(I)|\le\lambda^k|B(2L)|\big\}$$

for $k\ge 1$, until the first value of k where $E|\zeta^{B(L)}_{t(k)}(I)|\le 1$ or $t(k)\ge c_0L^2/(\log L)^q$ would hold; we denote this value by K and set $t(K)=c_0L^2/(\log L)^q$. Automatically, $K\le C(\log L)^{1+r}$ for large enough C. This bound and (4.24) easily give, for a new constant C,

$$P\big(\Gamma^L_{2,r}(t(k))\text{ for some } k\le K\big)\le C(\log L)^{1+3r-d(q-1)/2}. \tag{4.25}$$

We now estimate the probability that $|\zeta_t^{B(L)}(I)|$ deviates excessively from its mean when t lies between the times $t(k)$. Clearly, $E|\zeta_t^{B(L)}(I)|$ is continuous in t. Setting $\mu_k = E|\zeta_{t(k)}^{B(L)}(I)|$, it follows from the definition of K that

$$\mu_k = \lambda^k|B(2L)|, \qquad k < K,$$
$$\mu_k \geq \lambda^k|B(2L)| - 1, \qquad k = K. \tag{4.26}$$

(The -1 above is for the possibility that $1 \geq \lambda^K|B(2L)| > E|\zeta_{c_0L^2/(\log L)^q}^{B(L)}(I)|$.) Since I is of the form $[1, m)$, both $|\zeta_t^{B(L)}(I)|$ and $E|\zeta_t^{B(L)}(I)|$ are nonincreasing in t. Therefore, for $k < K$ and $t \in [t_k, t_{k+1}]$,

$$\big|\zeta_t^{B(L)}(I)\big| - E\big|\zeta_t^{B(L)}(I)\big| \leq \big|\zeta_{t(k)}^{B(L)}(I)\big| - E\big|\zeta_{t(k+1)}^{B(L)}(I)\big|.$$

Adding and subtracting μ_k gives

$$\big|\zeta_t^{B(L)}(I)\big| - E\big|\zeta_t^{B(L)}(I)\big| \leq \Big(\big|\zeta_{t(k)}^{B(L)}(I)\big| - \mu_k\Big) + (\mu_k - \mu_{k+1}). \tag{4.27}$$

A similar argument gives the inequality

$$\big|\zeta_t^{B(L)}(I)\big| - E\big|\zeta_t^{B(L)}(I)\big| \geq \Big(\big|\zeta_{t(k+1)}^{B(L)}(I)\big| - \mu_{k+1}\Big) - (\mu_k - \mu_{k+1}). \tag{4.28}$$

The difference of the means $\mu_k - \mu_{k+1}$ is easily estimated. First, by (4.26), for $k < K$,

$$\mu_k - \mu_{k+1} \leq \left(\frac{1}{\lambda} - 1\right)(\mu_{k+1} + 1) + 1 = \frac{(\log L)^{-r}}{1 - (\log L)^{-r}}(\mu_{k+1} + 1) + 1.$$

Next, using inequality (4.23), one can check that

$$\mu_{k+1} \leq |B(2L)|p_{t(k+1)} + 1/L.$$

Monotonicity and the asymptotics (1.11) easily imply that, for an appropriate constant C and all $t \leq c_0L^2$, $|B(2L)|p_t \geq C\log L$ for large L. Thus, for large L, all $k < K$ and $t \in [t(k), t(k+1)]$,

$$\mu_k - \mu_{k+1} \leq 2(\log L)^{-r}|B(2L)|p_{t(k+1)} \leq 2(\log L)^{-r}|B(2L)|p_t. \tag{4.29}$$

By combining (4.25), (4.27), (4.28) and (4.29), we obtain

$$P\bigg(\bigcup_t \Gamma_{4,r}^L(t)\colon t \in \big[0, c_0L^2/(\log L)^q\big]\bigg) \leq C(\log L)^{1+3r-d(q-1)/2}. \tag{4.30}$$

Setting $q = 3$ in (4.30), we obtain (4.1) for $d \geq 3$.

Refinement for $d = 2$. Setting $q = 4$ in (4.30) for $d = 2$ yields a bound which is of smaller order than the right side of (4.1), but covers only the time period $[0, c_0L^2/(\log L)^4]$. That is, we have

$$P\bigg(\bigcup_t \Gamma_{4,r}^L(t)\colon t \in \big[0, c_0L^2/(\log L)^4\big]\bigg) \leq C(\log L)^{3r-2}. \tag{4.31}$$

We must now treat the time period $[c_0L^2/(\log L)^4, c_0L^2/(\log L)^3]$.

We proceed essentially as before. We set $t(0) = c_0 L^2/(\log L)^4$, and

$$t(k) = \inf\left\{t\colon E\left|\zeta_t^{B(L)}(I)\right| \le \lambda^k E\left|\zeta_{t(0)}^{B(L)}(I)\right|\right\},$$

for $k \ge 1$, until the first value of k where $E|\zeta_{t(k)}^{B(L)}| \le 1$ or $t(k) \ge c_0 L^2/(\log L)^3$ would hold; we denote this value by K, and set $t(K) = c_0 L^2/(\log L)^3$. As before, $\lambda = 1-(\log L)^{-r}$. To bound K, we note that (4.23) and the asymptotics (1.11) imply that

$$E\left|\zeta_{t(0)}^{B(L)}(I)\right| \le |B(2L)| p_{t(0)} + 1/L \le C(\log L)^5$$

for an appropriate constant C. It follows from this and the definition of λ, that there exists a constant C such that for large L, $K \le C(\log\log L)(\log L)^r$.

Plugging $d = 2$ and $q = 3$ into (4.24) gives

$$P(\Gamma_{2,r}^L(t(k))) \le C(\log L)^{2r-2}, \quad k \le K.$$

Given our bound on K, this implies

$$P\big(\Gamma_{2,r}^L(t(k)) \text{ for some } k \le K\big) \le C(\log\log L)(\log L)^{3r-2}. \tag{4.32}$$

Here, the analog of (4.26),

$$\mu_k = \lambda^k E|\zeta_{t(0)}^{B(L)}(I)|, \qquad k < K,$$

$$\mu_k \ge \lambda^k E|\zeta_{t(0)}^{B(L)}(I)| - 1, \qquad k = K,$$

holds, where μ_k is defined as before. The same reasoning as in (4.26)–(4.30) then shows that

$$\begin{aligned} &P\Big(\bigcup_t \Gamma_{4,r}^L(t)\colon t \in \big[c_0 L^2/(\log L)^4, c_0 L^2/(\log L)^3\big]\Big) \\ &\qquad \le C(\log\log L)(\log L)^{3r-2}. \end{aligned} \tag{4.33}$$

This inequality and (4.31) demonstrate (4.1) for $d = 2$.

The proof of (4.2) is identical to the proof we have given of (4.1). One replaces $\zeta_t^{B(L)}$ with $\hat\zeta_t^{B(L)}$ throughout, and uses $\hat n_t^{B(L)}$ in (4.19). The one significant point to note is that, as with $\zeta_t^{B(L)}$, both $|\hat\zeta_t^{B(L)}(I)|$ and $E|\hat\zeta_t^{B(L)}(I)|$ are nonincreasing in t. □

5. Proof of Theorem 1. The proof of Theorem 1 is based on (2.5) and the estimates (5.1)–(5.4) below. These estimates allow us to make rigorous the heuristic argument given in (1.7)–(1.14). We first state the estimates, next use them to establish Theorem 1 and then prove them at the end of the section.

The following statements hold for fixed ε, $\beta > 0$ uniformly over $L \ge \beta\bar\alpha^{1/2}$. First,

$$P\left(\frac{\left|\hat\zeta_t^{B(L)}\right|}{|B(L)| p_t} \in [1-\varepsilon, 1+\varepsilon] \text{ for all } t \le \bar\alpha/(\log\bar\alpha)^4\right) \to 1 \quad \text{as } \alpha \to 0. \tag{5.1}$$

Second, for $y \in (0,1)$, there exists $\delta = \delta(y,\varepsilon) > 0$ such that

$$P\biggl(\frac{|\hat{\zeta}_t^{B(L)}([1,\bar{\alpha}^y])|}{|B(L)|p_t} \in [1-\varepsilon, 1+\varepsilon] \text{ for all } t \le \delta\bar{\alpha}^y\biggr) \to 1 \quad \text{as } \alpha \to 0. \tag{5.2}$$

Third, if $\delta > 0$ and $0 < y < u < 1$, then for small α,

$$E\bigl|\hat{\zeta}_t^{B(L)}([1,\bar{\alpha}^y])\bigr| \le \delta|B(L)|p_t \quad \text{for all } t \in \bigl[\bar{\alpha}^u, \bar{\alpha}/(\log\bar{\alpha})^4\bigr]. \tag{5.3}$$

Fourth, for given $\delta > 0$, C and large L,

$$E\bigl|\zeta_t^{B(L)}\bigr| \le (1+\delta)|B(L)|p_t \quad \text{for all } t \le CL^2/(\log\log L)^2. \tag{5.4}$$

Since $E|\hat{\zeta}_t^{B(L)}| \le E|\zeta_t^{B(L)}|$, the right side of (5.4) provides an upper bound for $E|\hat{\zeta}_t^{B(L)}|$ as well.

To unify as much as possible our presentation of the $d = 2$ and $d \ge 3$ cases, we introduce the notation

$$q(t) = \int_0^t p_s\,ds. \tag{5.5}$$

From the asymptotics for p_s given in (1.11), it is easy to see that

$$q(t) \sim \begin{cases} (\log t)^2/2\pi, & \text{in } d = 2,\\ (\log t)/\gamma_d, & \text{in } d \ge 3, \end{cases} \tag{5.6}$$

as $t \to \infty$. For fixed positive y, it follows from (5.6) that

$$q(\bar{\alpha}^y) \sim \begin{cases} y^2(\log\bar{\alpha})^2/2\pi, & \text{in } d = 2,\\ y(\log\bar{\alpha})/\gamma_d, & \text{in } d \ge 3, \end{cases} \tag{5.7}$$

as $\alpha \to 0$.

Using (5.1)–(5.4), and the above asymptotics, we will prove

PROPOSITION 5.1. *Let $\beta > 0$ and $\varepsilon_0 > 0$ be fixed, and $L = L(\alpha) \ge \beta\bar{\alpha}^{1/2}$. For any given $y \in (0,1)$,*

$$P\biggl(\biggl|\frac{N^L([1,\bar{\alpha}^y])}{\alpha q(\bar{\alpha}^y)} - 1\biggr| > \varepsilon_0\biggr) \to 0 \quad \text{as } \alpha \to 0. \tag{5.8}$$

Furthermore,

$$P\bigl(N^L([1,\infty)) \ge (1+\varepsilon_0)\alpha q(\bar{\alpha})\bigr) \to 0 \quad \text{as } \alpha \to 0. \tag{5.9}$$

It is easy to see that Theorem 1 follows directly from Proposition 5.1, monotonicity, and (5.7). We therefore proceed to the proof of the proposition, deriving lower and upper bounds for (5.8) and upper bounds for (5.9).

PROOF OF THE LOWER BOUND OF $N^L([1,\bar{\alpha}^y])$ IN (5.8). Fix $y \in (0,1)$ and $\varepsilon > 0$. By (5.2), there exists $\delta > 0$ such that, with probability tending to 1 as $\alpha \to 0$,

$$\int_0^{\delta\bar{\alpha}^y} \left|\hat{\zeta}_t^{B(L)}([1,\bar{\alpha}^y])\right| dt \geq (1-\varepsilon)\int_0^{\delta\bar{\alpha}^y} |B(L)| p_t\, dt = (1-\varepsilon)|B(L)|q(\delta\bar{\alpha}^y).$$

On account of (5.6), this is at least $(1-\varepsilon)^2|B(L)|q(\bar{\alpha}^y)$ for small α. So, by Lemma 2.1, there is a Poisson random variable X, with mean $EX = (1-\varepsilon)^2\alpha|B(L)|q(\bar{\alpha}^y)$, such that $P(\hat{Y}^{B(L)}_{0,\delta\bar{\alpha}^y}([1,\bar{\alpha}^y]) \geq X) \to 1$ as $\alpha \to 0$. Since $L = L(\alpha) \geq \beta\bar{\alpha}^{1/2}$, (5.7) shows that $EX \to \infty$. Consequently, by Lemma 2.2, $P(X \geq (1-\varepsilon)EX) \to 1$. Therefore,

$$P\big(\hat{Y}^{B(L)}_{0,\delta\bar{\alpha}^y}\big([1,\bar{\alpha}^y]\big) \leq (1-\varepsilon)^3\alpha|B(L)|q(\bar{\alpha}^y)\big) \to 0 \quad \text{as } \alpha \to 0. \tag{5.10}$$

By (2.4),

$$N\big(B(L),[1,\bar{\alpha}^y]\big) =_d \hat{Y}^{B(L)}_{0,\delta\bar{\alpha}^y}\big([1,\bar{\alpha}^y]\big) + \hat{Y}^{B(L)}_{\delta\bar{\alpha}^y,\infty}\big([1,\bar{\alpha}^y]\big).$$

Together with (5.10), this implies that

$$P\big(N^L\big([1,\bar{\alpha}^y]\big) \leq (1-\varepsilon)^3\alpha q\big(\bar{\alpha}^y\big)\big) \to 0 \quad \text{as } \alpha \to 0. \tag{5.11}$$

Choosing $\varepsilon_0 = 3\varepsilon$ gives us the desired lower bound for $N^L([1,\bar{\alpha}^y])$ in (5.8). □

PROOF OF THE UPPER BOUND OF $N^L([1,\bar{\alpha}^y])$ IN (5.8). The argument here is more involved, since we must estimate the number of killed particles for $\hat{\zeta}_t^{B(L)}$ over all time. Fix $y \in (0,1)$ and let $u \in (y,1]$, where u will be chosen close to y. Define the times

$$T_0 = 0, \quad T_1 = \bar{\alpha}^u, \quad T_2 = \bar{\alpha}/(\log\bar{\alpha})^4, \quad T_3 = \bar{\alpha}/(\log\log\bar{\alpha})^2, \quad T_4 = \infty.$$

Let $\check{Y}_i = \hat{Y}^{B(L)}_{T_{i-1},T_i}([1,\infty))$ for $i = 1,3,4$ and let $\check{Y}_2 = \hat{Y}^{B(L)}_{T_1,T_2}([1,\bar{\alpha}^y])$. By (2.5),

$$\begin{aligned} N(B(L),[1,\bar{\alpha}^y]) \;&=_d\; \sum_{i=1}^{4} \hat{Y}^{B(L)}_{T_{i-1},T_i}([1,\bar{\alpha}^y]) \\ &\leq\; \check{Y}_1 + \check{Y}_2 + \check{Y}_3 + \check{Y}_4. \end{aligned} \tag{5.12}$$

We will see that the main term in (5.12) is $\check{Y}_1$, and that the other terms are negligible. [Here, we could instead estimate the terms on the first line of (5.12); our choice will facilitate showing (5.9).] In what follows, $\varepsilon_1 > 0$ will be a fixed multiple of the value ε_0 appearing in (5.8).

The term $\check{Y}_1$. By (5.1), if $\varepsilon > 0$, then with probability tending to 1 as $\alpha \to 0$,

$$\int_0^{T_1}\left|\hat{\zeta}_t^{B(L)}\right| dt \leq (1+\varepsilon)|B(L)|\int_0^{T_1} p_t\, dt = (1+\varepsilon)|B(L)|q(T_1).$$

Using Lemma 2.1 again, we see that there is a Poisson random variable X with $EX = (1+\varepsilon)\alpha|B(L)|q(T_1)$ such that $P(\check{Y}_1 \leq X) \to 1$ as $\alpha \to 0$. Since

$L = L(\bar{\alpha}) \geq \beta\bar{\alpha}^{1/2}$, (5.7) implies $EX \to \infty$. Consequently, Lemma 2.2 implies that $P(X \geq (1+\varepsilon)EX) \to 0$, so we have

$$P\big(\check{Y}_1 \geq (1+\varepsilon)^2\alpha|B(L)|q(T_1)\big) \to 0 \quad \text{as } \alpha \to 0. \tag{5.13}$$

In view of (5.7), we may choose u close enough to y, and an appropriate ε, to obtain

$$P\big(\check{Y}_1 \geq (1+\varepsilon_1)\alpha|B(L)|q(\bar{\alpha}^y)\big) \to 0 \quad \text{as } \alpha \to 0. \tag{5.14}$$

The term $\check{Y}_2$. By (2.6) and (5.3), we have, for small α and $\varepsilon > 0$, with $\delta = \varepsilon^2$,

$$E\check{Y}_2 = \alpha \int_{T_1}^{T_2} E\big|\hat{\zeta}_t^{B(L)}([1, \bar{\alpha}^y])\big|\,dt \leq \varepsilon^2\alpha|B(L)| \int_{T_1}^{T_2} p_t\,dt \leq \varepsilon^2\alpha|B(L)|q(T_2).$$

It follows from Markov's inequality that

$$P\big(\check{Y}_2 \geq \varepsilon\alpha|B(L)|q(T_1)\big) \leq \varepsilon\frac{q(T_2)}{q(T_1)}. \tag{5.15}$$

Since (5.6) implies $q(T_2)/q(T_1)$ stays bounded as $\alpha \to 0$, it follows from (5.15) that

$$P\big(\check{Y}_2 \geq \varepsilon_1\alpha|B(L)|q(\bar{\alpha}^y)\big) \to 0 \quad \text{as } \alpha \to 0. \tag{5.16}$$

The term $\check{Y}_3$. We use (2.6) and the expectation estimate (5.4), with $\delta = 1$, to obtain

$$E\check{Y}_3 \leq 2\alpha|B(L)| \int_{T_2}^{T_3} p_t\,dt = 2\alpha|B(L)|\big(q(T_3) - q(T_2)\big).$$

By Markov's inequality, this implies

$$P\big(\check{Y}_3 \geq \varepsilon\alpha|B(L)|q(\bar{\alpha}^y)\big) \leq \frac{2\big(q(T_3) - q(T_2)\big)}{\varepsilon q(\bar{\alpha}^y)}. \tag{5.17}$$

Using the asymptotics in (5.6), one can check that for some finite constant C,

$$q(T_3) - q(T_2) \leq \begin{cases} C(\log\bar{\alpha})(\log\log\bar{\alpha}), & \text{in } d = 2, \\ C(\log\log\bar{\alpha}), & \text{in } d \geq 3. \end{cases}$$

In either case, the right side above is $o(q(\bar{\alpha}^y))$ as $\alpha \to 0$, so we have, setting $\varepsilon = \varepsilon_1$,

$$P\big(\check{Y}_3 \geq \varepsilon_1\alpha|B(L)|q(\bar{\alpha}^y)\big) \to 0 \quad \text{as } \alpha \to 0. \tag{5.18}$$

The term $\check{Y}_4$. We again use (5.4), with $\delta = 1$, and obtain $E|\hat{\zeta}_{T_3}^{B(L)}| \leq 2|B(L)|p(T_3)$. Together with the trivial bound $E\check{Y}_4 \leq E|\hat{\zeta}_{T_3}^{B(L)}|$ and Markov's inequality, this implies that

$$P(\check{Y}_4 > \varepsilon\alpha|B(L)|q(\bar{\alpha}^y)) \le \frac{2p(T_3)}{\varepsilon\alpha q(\bar{\alpha}^y)}. \tag{5.19}$$

It follows from the asymptotics for p_t in (1.11) that, for small α,

$$p_{T_3} \le \begin{cases} C\alpha(\log\bar{\alpha})(\log\log\bar{\alpha})^2, & \text{in } d=2,\\ C\alpha(\log\log\bar{\alpha})^2, & \text{in } d\ge 3.\end{cases}$$

In both cases, by (5.7), the right side above is $o(\alpha q(\bar{\alpha}^y))$ as $\alpha\to 0$, and thus, setting $\varepsilon=\varepsilon_1$,

$$P(\check{Y}_4 \ge \varepsilon_1\alpha|B(L)|q(\bar{\alpha}^y)) \to 0 \quad \text{as } \alpha\to 0. \tag{5.20}$$

Combining (5.12), (5.14), (5.16), (5.18) and (5.20), we obtain

$$P(N^L([1,\bar{\alpha}^y]) \ge (1+4\varepsilon_1)\alpha q(\bar{\alpha}^y)) \to 0 \quad \text{as } \alpha\to 0. \tag{5.21}$$

Setting $\varepsilon_0 = 4\varepsilon_1$, this gives the desired upper bound on $N^L([1,\bar{\alpha}^y])$ in (5.8). □

PROOF OF (5.9). The argument is analogous to that for the upper bound in (5.8). Define

$$T_0 = 0,\quad T_1 = \bar{\alpha}/(\log\bar{\alpha})^4,\quad T_2 = \bar{\alpha}/(\log\log\bar{\alpha})^2,\quad T_3=\infty,$$

and set $Y_i = \hat{Y}^{B(L)}_{T_{i-1},T_i}([1,\infty))$ for $i=1,2,3$. The analog of (5.12),

$$N(B(L),[1,\infty)) =_d Y_1+Y_2+Y_3, \tag{5.22}$$

now holds. Of course, $Y_2=\check{Y}_3$ and $Y_3=\check{Y}_4$ for $\check{Y}_3$ and $\check{Y}_4$ as in the proof of (5.8).

We first observe that the argument leading to (5.13) works equally well for our new choice of T_1, and yields

$$P(Y_1 \ge (1+\varepsilon_1)\alpha|B(L)|q(\bar{\alpha})) \to 0 \quad \text{as } \alpha\to 0.$$

Furthermore, on account of the monotonicity of $q(\bar{\alpha}^y)$ in y, it follows from (5.18) and (5.20), with $y=1$, that

$$P(Y_i \ge \varepsilon_1\alpha|B(L)|q(\bar{\alpha})) \to 0 \quad \text{as } \alpha\to 0$$

for $i=2,3$. Substituting the above bounds for Y_1, Y_2 and Y_3 into (5.22) implies (5.9) for $\varepsilon_0 = 3\varepsilon_1$. □

In order to complete the proof of Theorem 1, we still need to verify the bounds in (5.1)–(5.4). Since (5.4) is needed for the other parts, we consider it first.

PROOF OF (5.4). For $t\le L^2/(\log L)^2$, it is easy to derive the inequality in (5.4). By Lemma 3.3(ii), for $t\le L^2$,

$$E|\zeta_t^{B(L)}| \le E|\zeta_t^L| + 2|A(t)|p_t = (|B(L)|+2|A(t)|)p_t.$$

Recalling (3.3) and (3.9), it is easy to see that there is a constant C such that for $t \le L^2/(\log L)^2$,

$$\begin{aligned} |A(t)| &\le \left|A\left(L^2/(\log L)^2\right)\right| \le CL^{d-1}w_L\left(L^2/(\log L)^2\right) \\ &\le C\left|B(L)\right|/(\log L)^{1/2}. \end{aligned} \tag{5.23}$$

Thus, given $\delta > 0$ and large enough L,

$$E\left|\zeta_t^{B(L)}\right| \le (1+\delta)|B(L)|p_t \quad \text{for all } t \le L^2/(\log L)^2. \tag{5.24}$$

The treatment of (5.4) over $t \in (L^2/(\log L)^2, CL^2/(\log\log L)^2]$ requires more careful estimation. For $d = 2$, the result follows immediately from Proposition 3 of BCD and (1.11). The extension of the proposition to $d \ge 3$ is routine, so we omit the details here. □

PROOF OF (5.1). Since $L \ge \beta\bar{\alpha}^{1/2}$, $\bar{\alpha}/(\log\bar{\alpha})^4 = o(L^2/(\log L)^3)$ for small α. So by Proposition 4.1, with $r = 1/6$ and $m = \infty$, with probability tending to 1 as $\alpha \to 0$,

$$\left|\left|\zeta_t^{B(L)}\right| - E\left|\zeta_t^{B(L)}\right|\right| \le C|B(L)|p_t/(\log L)^{1/6} \quad \text{for all } t \le \bar{\alpha}/(\log\bar{\alpha})^4 \tag{5.25}$$

for $d \ge 2$. Also, by (3.3) and (3.9), for $t \le L^2/(\log L)^3$,

$$|A(t)| \le \left|A(L^2/(\log L)^3)\right| \le CL^{d-1}w_L\left(L^2/(\log L)^3\right) \le C|B(L)|/\log L \tag{5.26}$$

for some constant C. Now, using Lemma 3.3(ii) and $E|\zeta_t^L| = |B(L)|p_t$,

$$\left|E\left|\zeta_t^{B(L)}\right| - |B(L)|p_t\right| \le 2|A(t)|p_t \le C|B(L)|p_t/\log L$$

for all $t \le \bar{\alpha}/(\log\bar{\alpha})^4$. Combining this estimate with (5.25) yields

$$P\left(\frac{\left|\zeta_t^{B(L)}\right|}{|B(L)|p_t} \in [1-\varepsilon, 1+\varepsilon] \text{ for all } t \le \bar{\alpha}/(\log\bar{\alpha})^4\right) \to 1 \quad \text{as } \alpha \to 0. \tag{5.27}$$

To obtain (5.1), we need to replace $\zeta_t^{B(L)}$ in (5.27) with $\hat{\zeta}_t^{B(L)}$. To do this, we will show that the number of killed particles up to time $\bar{\alpha}/(\log\bar{\alpha})^4$ for $\zeta_t^{B(L)}$ is of smaller order than $|B(L)|p_{\bar{\alpha}/(\log\bar{\alpha})^4}$. To do this, we employ (2.3), from which it follows that

$$\left|\hat{\zeta}_t^{B(L)}\right| \le \left|\zeta_t^{B(L)}\right| \le \left|\hat{\zeta}_t^{B(L)}\right| + Y_{0,\bar{\alpha}/(\log\bar{\alpha})^4}^{B(L)}, \qquad t \le \bar{\alpha}/(\log\bar{\alpha})^4. \tag{5.28}$$

We will show that

$$P\left(Y_{0,\bar{\alpha}/(\log\bar{\alpha})^4}^{B(L)} \ge \varepsilon|B(L)|p_{\bar{\alpha}/(\log\bar{\alpha})^4}\right) \to 1 \quad \text{as } \alpha \to 0. \tag{5.29}$$

The limit (5.1) will then follow from (5.27)–(5.29) and the monotonicity of p_t.

The proof of (5.29) is straightforward. Using (2.6), and (5.4) with $\delta = 1$,

$$EY^{B(L)}_{0,\,\bar\alpha/(\log\bar\alpha)^4} = \alpha\int_0^{\bar\alpha/(\log\bar\alpha)^4} E\big|\zeta_t^{B(L)}\big|\,dt \tag{5.30}$$
$$\le 2\alpha|B(L)|\int_0^{\bar\alpha/(\log\bar\alpha)^4} p_t\,dt$$
$$= 2\alpha|B(L)|q\big(\bar\alpha/(\log\bar\alpha)^4\big).$$

By the asympotics in (5.6), as $\alpha\to 0$,

$$2\alpha|B(L)|q(\bar\alpha/(\log\bar\alpha)^4) \sim \begin{cases} 2|B(L)|\alpha(\log\bar\alpha)^2/(2\pi), & \text{in } d=2,\\ 2|B(L)|\alpha(\log\bar\alpha)/\gamma_d, & \text{in } d\ge 3.\end{cases} \tag{5.31}$$

On the other hand, the asymptotics (1.11) imply that

$$|B(L)|p_{\bar\alpha/(\log\bar\alpha)^4} \sim \begin{cases} C|B(L)|\alpha(\log\bar\alpha)^5, & \text{in } d=2,\\ C|B(L)|\alpha(\log\bar\alpha)^4, & \text{in } d\ge 3,\end{cases} \tag{5.32}$$

which dominates the quantity in (5.31). Together, (5.30)–(5.32) imply that

$$\frac{EY^{B(L)}_{0,\,\bar\alpha/(\log\bar\alpha)^4}}{|B(L)|p_{\bar\alpha/(\log\bar\alpha)^4}} \to 0 \quad \text{as } \alpha\to 0. \tag{5.33}$$

Markov's inequality and (5.33) imply (5.29). □

PROOF OF (5.2). In order to obtain (5.2) from (5.1), we must show that relatively few random walks have mass larger than $\bar\alpha^y$ at times $t\le\delta\bar\alpha^y$. The key to doing so is a "conservation of mass" argument. Since the total mass of the walks in $\hat\zeta_t^{B(L)}$ is at most $|B(L)|$, it is certainly the case that

$$\bar\alpha^y\big|\hat\zeta_t^{B(L)}\big((\bar\alpha^y,\infty)\big)\big| \le |B(L)|. \tag{5.34}$$

But

$$\big|\hat\zeta_t^{B(L)}\big([1,\bar\alpha^y]\big)\big| = \big|\hat\zeta_t^{B(L)}\big| - \big|\hat\zeta_t^{B(L)}((\bar\alpha^y,\,\infty))\big|,$$

so, using (5.34) and (5.1), it follows that, with probability tending to 1 as $\alpha\to 0$,

$$\big|\hat\zeta_t^{B(L)}\big([1,\bar\alpha^y]\big)\big| \ge \big|\hat\zeta_t^{B(L)}\big| - |B(L)|/\bar\alpha^y \ge (1-\varepsilon/2)|B(L)|p_t - |B(L)|/\bar\alpha^y \tag{5.35}$$

for all $t\le\bar\alpha/(\log\bar\alpha)^4$ and a given choice of $\varepsilon>0$. By monotonicity and the asymptotics (1.11), there is a constant C such that

$$p_t\bar\alpha^y \ge p_{\delta\bar\alpha^y}\bar\alpha^y \ge \begin{cases} C(\log(\delta\bar\alpha^y))/\delta, & \text{in } d=2,\\ C/\delta, & \text{in } d\ge 3,\end{cases}$$

for $t\le\delta\bar\alpha^y$. Consequently, we may choose $\delta>0$ small enough so that

$$1/\bar\alpha^y \le (\varepsilon/2)p_t \quad \text{for } t\le\delta\bar\alpha^y. \tag{5.36}$$

Combining (5.35) and (5.36) shows that with probability tending to 1 as $\alpha \to 0$,

$$\big|\hat{\zeta}_t^{B(L)}\big([1, \bar{\alpha}^y]\big)\big| \ge (1-\varepsilon)|B(L)|p_t \quad \text{for all } t \le \delta\bar{\alpha}^y.$$

Since the upper bound is immediate from (5.1), this implies (5.2). □

PROOF OF (5.3). On account of (2.3),

$$E\big|\hat{\zeta}_t^{B(L)}\big([1, \bar{\alpha}^y]\big)\big| \le E\big|\zeta_t^{B(L)}\big([1, \bar{\alpha}^y]\big)\big| + EY^{B(L)}_{0,\, \bar{\alpha}/(\log\bar{\alpha})^4} \tag{5.37}$$

for all $t \le \bar{\alpha}/(\log\bar{\alpha})^4$. The second term on the right side was estimated in (5.30)–(5.33). For the first term, we apply Lemma 3.3(ii), which implies that

$$E\big|\zeta_t^{B(L)}\big([1, \bar{\alpha}^y]\big)\big| \le E\big|\zeta_t^{L}\big([1, \bar{\alpha}^y]\big)\big| + 2|A(t)|p_t. \tag{5.38}$$

The second term on the right side of (5.38) is easy to handle. The assumption $L \ge \beta\bar{\alpha}^{1/2}$ implies that $\bar{\alpha}/(\log\bar{\alpha})^4 \le L^2/(\log L)^3$ for small α. By (5.26), there is a constant C such that for large L, $|A(t)| \le C|B(L)|/\log L$ for all $t \le L^2/(\log L)^3$. It follows that for given $\delta > 0$ and small α,

$$2|A(t)|p_t \le (\delta/3)|B(L)|p_t \quad \text{for } t \le \bar{\alpha}/(\log\bar{\alpha})^4. \tag{5.39}$$

The key to bounding the first term on the right side of (5.38) is the exponential limit law (1.12). First, we note that

$$\begin{aligned} E\big|\zeta_t^{L}([1, \bar{\alpha}^y])\big| &= E\bigg(\sum_{x\in B(L)} 1\big\{1 \le n_t(x) \le \bar{\alpha}^y\big\}\bigg) \\ &= |B(L)|P(1 \le n_t \le \bar{\alpha}^y). \end{aligned} \tag{5.40}$$

Next, for small α and $t \ge \bar{\alpha}^u$, monotonicity and the asymptotics (1.11) imply that for an appropriate constant C,

$$p_t\bar{\alpha}^y \le p_{\bar{\alpha}^u}\bar{\alpha}^y \le C(\log\bar{\alpha})\bar{\alpha}^{(y-u)}.$$

This last quantity tends to 0 as $\alpha \to 0$, since $y < u$. Therefore, by (5.40),

$$E\big|\zeta_t^{L}\big([1, \bar{\alpha}^y]\big)\big| \le |B(L)|P(1 \le n_t \le \delta/6p_t) \tag{5.41}$$

for small enough α. The limit (1.12) implies that, for large t,

$$P(1 \le n_t \le \delta/6p_t) \sim p_t\int_0^{\delta/6} e^{-u}\,du \le (\delta/6)\,p_t.$$

Using this fact in (5.41) gives, for small α and t in the indicated range,

$$E\big|\zeta_t^{L}\big([1, \bar{\alpha}^y]\big)\big| \le (\delta/3)|B(L)|p_t. \tag{5.42}$$

By (5.38), (5.39) and (5.42), we have

$$E\big|\zeta_t^{B(L)}\big([1,\bar{\alpha}^y]\big)\big| \le (2\delta/3)|B(L)|p_t \quad \text{for } t \le \bar{\alpha}/(\log\bar{\alpha})^4.$$

Together with (5.37) and (5.33), this implies (5.3). □

6. Proof of Theorem 2. To prove Proposition 5.1, and hence Theorem 1, for values $y \in (0,1)$, it was sufficient to show that $N^L([1,\bar{\alpha}^y]) \sim \alpha q(\bar{\alpha}^y)$ with high probability for given y and small α. To prove Theorem 2, we must show that $N^L([r^k, r^{k+1})) \sim \alpha k(\log r)^2/\pi$ in $d = 2$, and $N^L([r^k, r^{k+1})) \sim \alpha(\log r)/\gamma_d$ in $d \ge 3$, and that this holds with high probability for on the order of $\log\bar{\alpha}$ values of k simultaneously.

To state this concisely, we employ the notation

$$l(M) = \begin{cases} \log M, & \text{in } d = 2,\\ 1, & \text{in } d \ge 3. \end{cases} \tag{6.1}$$

Also, the reader should recall the definition of $\hat{\alpha}$ stated immediately before Theorem 2. Our main result in this section is the following analog of Proposition 5.1.

PROPOSITION 6.1. *Let $r > 1$, $\beta > 0$ and $\varepsilon_0 > 0$ be fixed. There exist $\delta > 0$ and C, such that for small $\alpha > 0$ and all $L \ge \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$, one has*

$$P\left(\left|\frac{N^L([M,Mr))}{\alpha l(M)} - \frac{\log r}{\gamma_d}\right| > \varepsilon_0, \Omega_1\right) \le \frac{C\log\log M}{(\log M)^{3/2}} \tag{6.2}$$

for all $M \in [\delta^{-1}, \delta\hat{\alpha}]$ and a suitable event Ω_1 (not depending on M), with $P(\Omega_1) > 1 - \varepsilon_0$.

Once one has Proposition 6.1, the proof of Theorem 2 is immediate. Summation of the right side of (6.2) over the values $M = r^k$ in $[\delta^{-1}, \delta\hat{\alpha}]$ gives an upper bound of the form

$$C \sum_{k \ge \frac{\log(\delta^{-1})}{\log r}} \frac{\log k}{k^{3/2}} \le \frac{C\log\log(\delta^{-1})}{\big(\log(\delta^{-1})\big)^{1/2}}$$

for new choices of the constant C. Rephrasing (6.2) using the events $E_L(k)$ given before Theorem 2, with $\varepsilon \ge \varepsilon_0$, one therefore gets

$$P\left(\bigcup_k E_L(k)\colon r^k \in [\delta^{-1}, \delta\hat{\alpha})\right) \le \varepsilon_0 + \frac{C\log\log(1/\delta)}{(\log(1/\delta))^{1/2}}. \tag{6.3}$$

Letting $\alpha \to 0$, $\delta \to 0$, and then $\varepsilon_0 \to 0$ implies (1.5). Thus, our goal is to prove Proposition 6.1. The quantities $r > 1$, $\beta > 0$ and $\varepsilon_0 > 0$ appearing in Proposition 6.1 are assumed to be fixed throughout this section; for convenience, we set $\varepsilon_0' = \varepsilon_0/12$. Also, note that for small $\delta > 0$, $M \ge \delta^{-1}$ will be large.

The strategy we adopt to show Proposition 6.1 is similar, in general terms, to that used for Proposition 5.1 in the proof of Theorem 1. We let $I = [M, Mr)$, and for $0 < \varepsilon_1 < K_1 < \infty$, which we shall shortly choose, we set $T_0^M = 0$, $T_5^M = \infty$ and

$$T_1^M = \varepsilon_1 M l(M),$$
$$T_2^M = K_1 M l(M),$$
$$T_3^M = \big((\log M)^2 M l(M)\big) \wedge T_4^M,$$
$$T_4^M = \bar{\alpha} \log \bar{\alpha}.$$

With these times, we employ (2.5) in the form

$$N(B(L), I) =_d \sum_{i=1}^{5} \hat{Y}_i, \tag{6.4}$$

where we have written $\hat{Y}_i$ for $\hat{Y}^{B(L)}_{T_M^{i-1}, T_M^i}(I)$. We will find it convenient to think of the times in $[T_0^M, T_1^M]$ as *small times*, the times in $[T_1^M, T_2^M]$ as *moderate times*, and the times in $[T_2^M, T_5^M]$ as *large times*. It will turn out that, for small ε_1 and large K_1, only the term $\hat{Y}_2$, representing the period of moderate times, will make a substantial contribution to (6.4).

Before specifying ε_1 and K_1, we first define, for $0 < a < b$,

$$g_{a,b}(s) = \frac{1}{\gamma_d s}\left(\exp\left(\frac{-a}{\gamma_d s}\right) - \exp\left(\frac{-b}{\gamma_d s}\right)\right), \qquad s > 0. \tag{6.5}$$

One can check that

$$\int_0^\infty g_{1,r}(s)\,ds = \frac{\log r}{\gamma_d},$$

since by a change of variables,

$$\int_0^t \frac{1}{s}\left[\exp\left(\frac{-1}{\gamma_d s}\right) - \exp\left(\frac{-r}{\gamma_d s}\right)\right] ds = \int_{t/r}^t \frac{1}{s} \exp\left(\frac{-1}{\gamma_d s}\right) ds$$
$$= \int_{1/r}^1 \frac{1}{s} \exp\left(\frac{-1}{t\gamma_d s}\right) ds,$$

which converges to $\log r$ as $t \to \infty$. We may therefore choose ε_1 and K_1, with $0 < \varepsilon_1 < K_1 \wedge (\varepsilon_0'/2) < \infty$, so that

$$0 < \frac{\log r}{\gamma_d} - \int_{\varepsilon_1}^{K_1} g_{1,r}(s)\,ds < \varepsilon_0'. \tag{6.6}$$

We assume that K_1 is chosen large enough so that $K_1 > C_1/\varepsilon_0'$, where C_1 is the constant in Lemma 6.1.

We now specify the event Ω_1 appearing in Proposition 6.1. With Ω_0 being the good event of Lemma 3.2, and $A(t)$ defined as in (3.9), we set

$$\Omega_1 = \left\{\int_0^{\bar{\alpha} \log \bar{\alpha}} |\zeta_t \cap A(t)|\,dt < \varepsilon_0' |B(L)| l(\delta^{-1})\right\} \cap \Omega_0.$$

By Lemma 3.4 and Markov's inequality, we have, for an appropriate constant C,

$$P\biggl(\int_0^{\bar\alpha \log \bar\alpha} |\zeta_t \cap A(t)|\, dt \ge \varepsilon_0' |B(L)| l(\delta^{-1})\biggr) \le \begin{cases} C/(\varepsilon_0' \log(\delta^{-1})), & \text{in } d = 2,\\ C/(\varepsilon_0' \log L), & \text{in } d \ge 3.\end{cases}$$

Also, $P(\Omega_0) \ge 1 - 1/L^{d+1}$ by Lemma 3.2. So, for small α and δ, $P(\Omega_1) \ge 1 - \varepsilon_0'$. Having defined the event Ω_1, we note that it will enter into our estimates only when handling the term $\hat{Y}_4$. Also, we emphasize that, for the rest of this section, $I = [M, Mr)$.

We now proceed to estimate $\hat{Y}_i$, $i = 1, \dots, 5$. By (6.4), this will give us bounds on $N(B(L), I)$, and hence on $N^L(I)$, as needed for Proposition 6.1.

6.1. *The term $\hat{Y}_1$.* Since the mass of a particle in $\hat{\zeta}_t^{B(L)}$ is not larger than the mass of the corresponding particle in $\zeta_t^{B(L)}$, and the total mass of the particles in $\zeta_t^{B(L)}$ is at most $|B(L)|$, it is easy to see that

$$\text{(6.7)} \qquad \bigl|\hat{\zeta}_t^{B(L)}(I))\bigr| \le \bigl|\hat{\zeta}_t^{B(L)}([M, \infty))\bigr| \le \bigl|\zeta_t^{B(L)}([M, \infty))\bigr| \le \frac{|B(L)|}{M}.$$

Since $T_1^M = \varepsilon_1 M l(M)$, it follows that

$$\int_0^{T_1^M} \bigl|\hat{\zeta}_t^{B(L)}(I)\bigr|\, dt \le \varepsilon_1 |B(L)| l(M).$$

By Lemma 2.1 and the above inequality, there is a Poisson random variable X, with $EX = \alpha\varepsilon_1 |B(L)| l(M)$, such that $P(\hat{Y}_1 \le X) = 1$. The assumptions $L \ge \beta \bar\alpha^{1/2} (\log \bar\alpha)^2$ and $M \le \delta\hat\alpha$ imply that, for small α,

$$\alpha |B(L)| \ge (\log \bar\alpha)^2 \ge \log M.$$

Thus, $EX \ge \varepsilon_1 \log M$, and by Lemma 2.2,

$$P(X \ge 2EX) \le \exp(-c_2 EX) \le M^{-\rho}$$

for some $\rho > 0$. Since we are assuming $2\varepsilon_1 < \varepsilon_0'$, it follows that

$$\text{(6.8)} \qquad P\bigl(\hat{Y}_1 \ge \varepsilon_0' \alpha |B(L)| l(M)\bigr) \le M^{-\rho}.$$

This is the desired bound for small times.

6.2. *The term $\hat{Y}_2$.* The argument for moderate times is somewhat involved, so we give a brief outline before turning to the details. We proceed in a series of steps. In Steps 2a and 2b, we show that $|\hat{\zeta}_t^{B(L)}(I)| \approx E|\hat{\zeta}_t^L(I)|$ and $E|\hat{\zeta}_t^L(I)| \approx E|\zeta_t^L(I)|$ up to appropriate error terms. We demonstrate in Step 2c that these error terms are negligible. We show, in Step 2d, that

$$\int_{T_1^M}^{T_2^M} E\bigl|\zeta_t^L(I)\bigr|\, dt \approx |B(L)| l(M) (\log r)/\gamma_d.$$

In Step 2e, we combine the above results to deduce that with probability close to 1,

$$\int_{T_1^M}^{T_2^M} \left|\hat{\zeta}_t^{B(L)}(I)\right| dt \approx |B(L)| l(M)(\log r)/\gamma_d.$$

We finish the argument by applying the Poisson domination estimate from Section 2, obtaining

$$\hat{Y}_2 \approx \alpha |B(L)| l(M)(\log r)/\gamma_d$$

with probability close to 1.

STEP 2a. To show that $|\hat{\zeta}_t^{B(L)}(I)| \approx E|\hat{\zeta}_t^L(I)|$ for all $t \le T_2^M$ with probability close to 1, we will employ Lemma 3.3 to obtain $E|\hat{\zeta}_t^{B(L)}(I)| \approx E|\hat{\zeta}_t^L(I)|$, and then Proposition 4.1 to obtain $|\hat{\zeta}_t^{B(L)}(I)| \approx E|\hat{\zeta}_t^{B(L)}(I)|$. We first note that a simple computation, using the bounds $M \le \delta\hat{\alpha}$ and $L \ge \beta\bar{\alpha}^{1/2}(\log \bar{\alpha})^2$, shows that

$$T_2^M = o(L^2/(\log L)^3) \quad \text{as } \alpha \to 0. \tag{6.9}$$

So, we may assume $T_2^M \le L^2/(\log L)^3$. By Lemma 3.3(ii),

$$\left| E\left|\hat{\zeta}_t^{B(L)}(I)\right| - E\left|\hat{\zeta}_t^L(I)\right| \right| \le 2|A(t)| p_t \quad \text{for } t \le T_2^M.$$

By (5.26), there exists a constant C such that, for large L, $|A(t)| \le C|B(L)|/\log L$ for all $t \le L^2/(\log L)^3$. Thus, for small α,

$$\left| E\left|\hat{\zeta}_t^{B(L)}(I)\right| - E\left|\hat{\zeta}_t^L(I)\right| \right| \le 2C|B(L)| p_t/\log L \quad \text{for } t \le T_2^M.$$

But, by Proposition 4.1, for an appropriate constant C and small α,

$$P\left(\left| \left|\hat{\zeta}_t^{B(L)}(I)\right| - E\left|\hat{\zeta}_t^{B(L)}(I)\right| \right| > \frac{C|B(L)| p_t}{(\log L)^{1/6}} \text{ for some } t \le T_2^M \right) \le \frac{C \log\log L}{(\log L)^{3/2}}.$$

By the last two estimates and the triangle inequality, it follows that, for an appropriate constant C,

$$P\left(\left| \left|\hat{\zeta}_t^{B(L)}(I)\right| - E\left|\hat{\zeta}_t^L(I)\right| \right| > \frac{C|B(L)| p_t}{(\log L)^{1/6}} \text{ for some } t \le T_2^M \right) \le \frac{C \log\log L}{(\log L)^{3/2}}. \tag{6.10}$$

STEP 2b. Here, we estimate the difference in mass between particles in $\hat{\zeta}_t^L$ and ζ_t^L and then use this information to make precise the approximation $E|\hat{\zeta}_t^L(I)| \approx E|\zeta_t^L(I)|$, the desired bounds being given by (6.14) and (6.15). Let $\Delta_t(x) = n_t(x) - \hat{n}_t(x)$ and $m_t^L = \sum_{x \in B(L)} \Delta_t(x)$. Note that $\Delta_t(x)$ is always nonnegative.

We first show that

$$\sum_{x \in B(L)} P(\Delta_t(x) \ge \varepsilon_2 M) \le \alpha |B(L)| t/\varepsilon_2 M \tag{6.11}$$

for $\varepsilon_2 > 0$; this gives us control over the number of sites in $B(L)$ which have lost mass of order of magnitude M. In Step 2d, we will specify ε_2.

To see (6.11), we note that

$$Em_t^L = E\biggl(\sum_{x\in B(L)}\sum_{y\in \mathbb{Z}^d} 1\bigl\{\zeta_t^y = x,\, \hat{\zeta}_t^y = \varnothing\bigr\}\biggr)$$
$$= (1-e^{-\alpha t})\sum_{x\in B(L)}\sum_{y\in \mathbb{Z}^d} P\bigl(\zeta_t^0 = x-y\bigr) = |B(L)|(1-e^{-\alpha t}).$$

Since $1-e^{-u}\le u$, one has

$$Em_t^L \le \alpha|B(L)|t. \tag{6.12}$$

By Markov's inequality,

$$\sum_{x\in B(L)} P(\Delta_t(x)\ge \varepsilon_2 M) \le \sum_{x\in B(L)} E\bigl(\Delta_t(x)\bigr)/\varepsilon_2 M = Em_t^L/\varepsilon_2 M. \tag{6.13}$$

Together with (6.12), this implies (6.11).

We now consider which sites $x\in B(L)$ have $\Delta_t(x) < \varepsilon_2 M$, which leads to the decomposition

$$\bigl|\hat{\zeta}_t^L(I)\bigr| \le \sum_{x\in B(L)} 1\bigl\{\Delta_t(x) < \varepsilon_2 M, \hat{n}_t(x)\in I\bigr\} + \sum_{x\in B(L)} 1\bigl\{\Delta_t(x)\ge \varepsilon_2 M\bigr\}.$$

Since $\Delta_t(x) = n_t(x) - \hat{n}_t(x)$, we have

$$\bigl\{\Delta_t(x) < \varepsilon_2 M,\ \hat{n}_t(x)\in I\bigr\} \subset \bigl\{n_t(x)\in [M, M(r+\varepsilon_2))\bigr\},$$

which implies

$$\sum_{x\in B(L)} 1\bigl\{\Delta_t(x) < \varepsilon_2 M,\ \hat{n}_t(x)\in I\bigr\} \le \bigl|\zeta_t^L\bigl([M, M(r+\varepsilon_2))\bigr|.$$

Combining the last two inequalities, and taking expectations, we obtain

$$E\bigl|\hat{\zeta}_t^L(I)\bigr| \le E\bigl|\zeta_t^L([M, M(r+\varepsilon_2))\bigr| + E\biggl(\sum_{x\in B(L)} 1\bigl\{\Delta_t(x)\ge \varepsilon_2 M\bigr\}\biggr).$$

The last expectation equals the left side of (6.11); substituting in this bound gives

$$E\bigl|\hat{\zeta}_t^L(I)\bigr| \le E\bigl|\zeta_t^L\bigl([M, M(r+\varepsilon_2))\bigr)\bigr| + \alpha|B(L)|t/\varepsilon_2 M, \tag{6.14}$$

which is the upper bound we desire.

We argue similarly for an inequality in the reverse direction:

$$E\bigl|\hat{\zeta}_t^L(I)\bigr| \ge E\biggl(\sum_{x\in B(L)} 1\bigl\{\Delta_t(x) < \varepsilon_2 M, n_t(x)\in \bigl[M(1+\varepsilon_2), Mr\bigr)\bigr\}\biggr)$$
$$\ge E\biggl(\sum_{x\in B(L)} 1\bigl\{n_t(x)\in \bigl[M(1+\varepsilon_2), Mr\bigr)\bigr\}\biggr)$$
$$- E\biggl(\sum_{x\in B(L)} 1\bigl\{\Delta_t(x)\ge \varepsilon_2 M\bigr\}\biggr).$$

By (6.11), this implies

$$E\big|\hat{\zeta}_t^L(I)\big| \geq E\big|\zeta_t^L\big([M(1+\varepsilon_2), Mr)\big)\big| - \alpha|B(L)|t/\varepsilon_2 M. \tag{6.15}$$

STEP 2c. Our goal in this step is to show that the "error terms" in (6.10), (6.14) and (6.15) are negligible. To do this, we set

$$e_{L,M}(t) = C|B(L)|p_t/(\log L)^{1/6} + \alpha|B(L)|t/\varepsilon_2 M,$$

and show that for fixed ε_2, and small enough δ and α,

$$\frac{1}{|B(L)|l(M)} \int_{T_1^M}^{T_2^M} e_{L,M}(t)\,dt < \varepsilon_0'. \tag{6.16}$$

Verification of (6.16) involves just straightforward computation. We restrict ourselves to the case $d = 2$, since the reasoning for $d \geq 3$ is the same except for the absence of the factors of $\log M$ in the following estimates.

The left side of (6.16), for $d = 2$, equals

$$\frac{C}{\log M(\log L)^{1/6}} \int_{T_1^M}^{T_2^M} p_t\,dt + \frac{\alpha}{\varepsilon_2 M \log M} \int_{T_1^M}^{T_2^M} t\,dt. \tag{6.17}$$

By (5.6) and the definitions of T_1^M and T_2^M,

$$\begin{aligned} \int_{T_1^M}^{T_2^M} p_t\,dt &\leq C\big((\log T_2^M)^2 - (\log T_1^M)^2\big) \\ &= C\big(\log(K_1/\varepsilon_1)\big)\big(\log(K_1 M \log M) + \log(\varepsilon_1 M \log M)\big) \end{aligned} \tag{6.18}$$

for appropriate C, since $M \geq \delta^{-1}$ is large for small δ. So, $\int_{T_1^M}^{T_2^M} p_t\,dt \leq C \log M$, where C depends on ε_1 and K_1. On account of this, the first term in (6.17) is bounded above by $C/(\log L)^{1/6}$.

The second term in (6.17) is bounded above by

$$\frac{\alpha}{\varepsilon_2 M \log M}(K_1 M \log M)^2 \leq \frac{\delta\gamma_2 K_1^2}{\varepsilon_2 \log \bar{\alpha}} \log\left(\frac{\delta\gamma_2\bar{\alpha}}{\log\bar{\alpha}}\right) \leq \frac{\delta\gamma_2 K_1^2}{\varepsilon_2}$$

for small α, the first inequality following from $M \leq \delta\gamma_2\bar{\alpha}/\log\bar{\alpha}$. Together with the bound in the previous paragraph, this shows that (6.17) is bounded above by

$$C/(\log L)^{1/6} + \delta\gamma_2 K_1^2/\varepsilon_2.$$

For fixed ε_2, and small δ and α, this implies (6.16).

STEP 2d. Here, we show that $\int_{T_1^M}^{T_2^M} E|\zeta_t^L(I)|\,dt \approx |B(L)|l(M)(\log r)/\gamma_d$. Let $1 \leq a < b < \infty$ and set

$$f_M(s) = MP\big(n_{sMl(M)} \in [Ma, Mb)\big), \qquad s > 0.$$

It follows easily from (1.11) and (1.12), that the functions $f_M(\cdot)$ are uniformly bounded on the interval $[\varepsilon_1, K_1]$, and that

$$f_M(s) \to g_{a,b}(s) \quad \text{as } M \to \infty,$$

where $g_{a,b}(s)$ is given by (6.5). By the bounded convergence theorem,

$$\lim_{M\to\infty} \int_{\varepsilon_1}^{K_1} f_M(s)\,ds = \int_{\varepsilon_1}^{K_1} g_{a,b}(s)\,ds. \tag{6.19}$$

Furthermore, by the comments after (6.5), for a sufficiently close to 1 and b sufficiently close to r, with $r > 1$,

$$\left| \int_{\varepsilon_1}^{K_1} g_{a,b}(s)\,ds - \frac{\log r}{\gamma_d} \right| < \varepsilon_0'. \tag{6.20}$$

We also note that

$$\int_{T_1^M}^{T_2^M} E\left|\zeta_t^L\big([Ma, Mb)\big)\right|dt = |B(L)| \int_{T_1^M}^{T_2^M} P\big(n_t \in [Ma, Mb)\big)\,dt, \tag{6.21}$$

and that the change of variables $t = sMl(M)$ gives

$$\int_{T_1^M}^{T_2^M} P\big(n_t \in [Ma, Mb)\big)\,dt = l(M) \int_{\varepsilon_1}^{K_1} f_M(s)\,ds. \tag{6.22}$$

By (6.21), (6.22) and (6.19), for given $a > 0$ and $b > 0$, there exists $\delta > 0$ such that, for $M \geq \delta^{-1}$,

$$\begin{aligned} &\left| \int_{T_1^M}^{T_2^M} E\left|\zeta_t^L([Ma, Mb))\right|dt - |B(L)|l(M) \int_{\varepsilon_1}^{K_1} g_{a,b}(s)\,ds \right| \\ &\qquad \leq \varepsilon_0'|B(L)|l(M). \end{aligned} \tag{6.23}$$

If we set $a = 1$ and $b = r + \varepsilon_2$, where ε_2 is small, then (6.20) and (6.23) imply that, for large L,

$$\begin{aligned} &\int_{T_1^M}^{T_2^M} E\left|\zeta_t^L\big([M, M(r+\varepsilon_2))\big)\right|dt \\ &\qquad \leq |B(L)|l(M)(\log r)/\gamma_d + 2\varepsilon_0'|B(L)|l(M). \end{aligned} \tag{6.24}$$

Similarly, if we set $a = 1 + \varepsilon_2$ and $b = r$, we obtain, for large L,

$$\begin{aligned} &\int_{T_1^M}^{T_2^M} E\left|\zeta_t^L\big([M(1+\varepsilon_2), Mr)\big)\right|dt \\ &\qquad \geq |B(L)|l(M)(\log r)/\gamma_d - 2\varepsilon_0'|B(L)|l(M). \end{aligned} \tag{6.25}$$

Inequalities (6.24) and (6.25) are the desired bounds.

STEP 2e. We now use the estimates we have obtained in the previous steps, along with Lemmas 2.1 and 2.2, to complete our treatment of moderate times. We consider the upper bound. First, we choose $\varepsilon_2 > 0$ and $\delta > 0$ such that for small enough α, and $M \in [\delta^{-1}, \delta\hat{\alpha})$, (6.16) and (6.24) hold. On account of (6.10) and (6.14), the probability that

$$\int_{T_1^M}^{T_2^M} \left|\hat{\zeta}_t^{B(L)}(I)\right| dt \geq \int_{T_1^M}^{T_2^M} \left(E\left|\zeta_t^L\left([M, M(r+\varepsilon_2)\right)\right| + e_{L,M}(t)\right) dt \tag{6.26}$$

is at most $C(\log\log L)/(\log L)^{3/2}$. By (6.16) and (6.24), the right side of (6.26) is bounded above by

$$|B(L)|l(M)\left[(\log r)/\gamma_d + 3\varepsilon_0'\right].$$

It follows from Lemma 2.1 that there is a Poisson random variable X, with expectation $EX = \alpha|B(L)|l(M)[(\log r)/\gamma_d + 3\varepsilon_0']$, such that

$$P\left(\hat{Y}_2 \geq X\right) \leq C(\log\log L)/(\log L)^{3/2}. \tag{6.27}$$

Since $L \geq \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$ and $M \leq \delta\hat{\alpha}$, one has $EX \geq \beta^2(\log\bar{\alpha})^4(\log r)/\gamma_d \geq \log M$ for small α. Therefore, using Lemma 2.2, there exists, for given $\varepsilon > 0$, a $\rho > 0$ such that

$$P\left(X \geq (1+\varepsilon)EX\right) \leq M^{-\rho}. \tag{6.28}$$

Also, the inequalities $L \geq \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$ and $M \leq \delta\hat{\alpha}$ imply that $M \leq L^2$ for small α. Thus,

$$\frac{\log\log M}{(\log M)^{3/2}} \geq \frac{\log\log(L^2)}{\left(\log(L^2)\right)^{3/2}} \geq \frac{\log\log L}{2^{3/2}(\log L)^{3/2}}$$

for small α. Therefore, for small enough ε, (6.27) and (6.28) imply that for small δ and α,

$$P\left(\hat{Y}_2 \geq \alpha|B(L)|l(M)\left[(\log r)/\gamma_d + 4\varepsilon_0'\right]\right) \leq C(\log\log M)/(\log M)^{3/2} \tag{6.29}$$

for a constant C. This is the desired upper bound for $\hat{Y}_2$.

Similar reasoning may be applied to the lower bound, with (6.25) replacing (6.24) in the above estimates. One then obtains the bound

$$P\left(\hat{Y}_2 \leq \alpha|B(L)|l(M)\left[(\log r)/\gamma_d - 4\varepsilon_0'\right]\right) \leq C(\log\log M)/(\log M)^{3/2}. \tag{6.30}$$

6.3. *A lemma for large times estimates.* In order to show that $\hat{Y}_3$ and $\hat{Y}_4$ are small, we need accurate estimates on the size of $|\hat{\zeta}_t^{B(L)}(I)|$ for times $t \in [K_1 M l(M), \bar{\alpha}\log\bar{\alpha}]$. Unfortunately, the estimates that have proven useful for smaller times are not adequate for these larger times. Instead, we use Lemma 6.1, which provides the more accurate information we need. We first state and prove this result, and then turn to the estimation of $\hat{Y}_3$ and $\hat{Y}_4$.

LEMMA 6.1. *There exists a constant C_1 such that, for small $\delta > 0$, $\alpha > 0$, and $M \in [\delta^{-1}, \delta\hat{\alpha})$,*

$$E\left(\int_{bMl(M)}^{\infty} \left|\hat{\zeta}_t^L(I)\right| dt\right) \le C_1 |B(L)| l(M)/b \quad \textit{for } b \in [1, M]. \tag{6.31}$$

The basic voter model η_t^z and the coalescing random walk $\zeta_s^{A,t}$ were defined in Section 2 using the percolation substructure. Here, we need a variant of the basic voter model, which we call the voter model with killing, denoted by $\hat{\eta}_t^A$. Its definition is similar to that of η_t^A. Namely, we let $\hat{\eta}_t^A$ denote the collection of sites y such that there is a path up from some $(x, 0)$, $x \in A$, to (y, t), with no mutation event on the path. It is apparent from the construction that $\hat{\eta}_t^A \subset \eta_t^A$ and that

$$\left|\hat{\eta}_t^x\right| = \sum_{y \in \mathbb{Z}^d} 1\left\{\hat{\zeta}_t^{y,t} = x\right\}.$$

From this and the definition of $\hat{n}_t(x)$, one sees that

$$\left|\hat{\eta}_t^x\right| =_d \hat{n}_t(x). \tag{6.32}$$

On account of (6.32), $P(|\hat{\eta}_t^x| \in [a, b)) = P(\hat{n}_t(x) \in [a, b))$ for any $1 \le a \le b < \infty$. By translation invariance, these probabilities do not depend on x.

PROOF OF LEMMA 6.1. We begin by setting $T = bMl(M)$, $b \in [1, M]$, and note that

$$\begin{aligned} E\left(\int_T^{\infty} \left|\hat{\zeta}_t^L(I)\right| dt\right) &= \sum_{x \in B(L)} \int_T^{\infty} P(\hat{n}_t(x) \in I)\, dt \\ &= |B(L)| \int_T^{\infty} P\left(\left|\hat{\eta}_t^O\right| \in I\right) dt \\ &= |B(L)| E\left(\int_T^{\infty} 1\left\{\left|\hat{\eta}_t^O\right| \in I\right\} dt\right). \end{aligned} \tag{6.33}$$

So, to obtain (6.31), we wish to estimate the total "occupation time" after time T for $|\hat{\eta}_t^O|$ of I. We do this as follows. Given that $|\hat{\eta}_s^O| \in I$, we wait a certain further amount of time u_M to see whether $\hat{\eta}_{s+u_M}^O = \varnothing$, which will happen with a certain probability. If this does not happen, we wait until the first time t after $s + u_M$ when $|\hat{\eta}_t^O|$ enters I and then repeat the procedure. Since the probability of the nth such event decreases geometrically in n, one obtains good enough bounds for (6.31). We turn now to the details of this argument.

First, let $u_M = 3rMl(M)/\gamma_d$. Also, let $\sigma_0 = T$ and inductively define, for $n \ge 1$,

$$\begin{aligned} \tau_n &= \inf\left\{t \ge \sigma_{n-1} : \left|\hat{\eta}_t^O\right| \in [M, Mr]\right\}, \\ \sigma_n &= \tau_n + u_M. \end{aligned}$$

Since $|\hat{\eta}_t^O| \le |\eta_t^O|$, the asymptotics (1.11) imply that there is a constant C such that

$$P(\tau_1 < \infty) \le P(\eta_{\sigma_0}^O \ne \varnothing) = p_T \le C/bM \tag{6.34}$$

for large M. Our choice of u_M guarantees that for large M, and all $A \subset \mathbb{Z}^d$ with $|A| \le rM$,

$$P(\hat{\eta}_{u_M}^A \ne \varnothing) \le P(\eta_{u_M}^A \ne \varnothing) \le 1/2. \tag{6.35}$$

To see that this is the case, we note that

$$P(\eta_{u_M}^A \ne \varnothing) \le \sum_{x\in A} P(\eta_{u_M}^x \ne \varnothing) \le rMp_{u_M}.$$

For $d = 2$, $p_t \sim (\log t)/\gamma_2 t$ as $t \to \infty$ by (1.11), and so, as $M \to \infty$,

$$rMp_{u_M} \sim rM\frac{\log(M\log M)}{3rM\log M} = \frac{1}{3}\left(1 + \frac{\log\log M}{\log M}\right).$$

Thus, $rMp_{u_M} \to 1/3$ as $M \to \infty$. For $d \ge 3$, the asymptotics (1.11) give $p_{u_M} \sim 1/3rM$, so again $rMp_{u_M} \to 1/3$ as $M \to \infty$. This verifies (6.35).

By the Markov property and (6.35), each time $|\hat{\eta}_t^O|$ hits I, there is probability at least 1/2 that $\hat{\eta}_t^O$ will die out within u_M time units. Thus,

$$P(\tau_n < \infty) \le P(\tau_1 < \infty)(1/2)^{n-1}.$$

Furthermore, at most u_M can be added to the total occupation time of I by $|\hat{\eta}_t^O|$ during each interval $[\tau_n, \tau_{n+1})$. Therefore,

$$\int_{bMl(M)}^{\infty} 1\{|\hat{\eta}_t^O| \in I\}\,dt \le u_M \sum_{n=1}^{\infty} 1\{\tau_n < \infty\}$$

and, consequently,

$$E\left(\int_{bMl(M)}^{\infty} 1\{|\hat{\eta}_t^O| \in I\}\,dt\right) \le u_M P(\tau_1 < \infty)\sum_{n=1}^{\infty}(1/2)^{n-1} \le \frac{C}{bM}2u_M,$$

where we have used (6.34) in the last inequality. The proof of (6.31) is completed by plugging in the value of u_M and applying (6.33). □

6.4. *The term $\hat{Y}_3$.* The idea is to show, over the time period $[T_2^M, T_3^M]$, that $|\hat{\zeta}_t^{B(L)}(I)| \approx E|\hat{\zeta}_t^{B(L)}(I)|$ and that the latter quantity is approximately $E|\hat{\zeta}_t^L(I)|$, where, as before, $I = [M, Mr)$. Lemma 6.1 can be employed to show that this is small. Lemmas 2.1 and 2.2 then give the desired bound.

We first note that $T_3^M \le T_4^M \le \bar{\alpha}\log\bar{\alpha}$ and recall from (3.16) that $\bar{\alpha}\log\bar{\alpha} \le 2L^2/\beta^2(\log L)^3$. As in (5.26), one can check that, for $t \le T_3^M$,

$$|A(t)| \le C|B(L)|/\log L,$$

where $A(t)$ is given in (3.9). Thus, by combining Lemma 3.3 and Proposition 4.1, we see that there is a constant C such that, for small α, the probability of the complement of the event

$$\left\{ \left|\hat{\zeta}_t^{B(L)}(I)\right| \le E\left|\hat{\zeta}_t^L(I)\right| + \frac{C|B(L)|p_t}{(\log L)^{1/6}} \text{ for all } t \le T_3^M \right\} \tag{6.36}$$

is at most $C(\log\log L)/(\log L)^{3/2}$. On the event in (6.36),

$$\int_{T_2^M}^{T_3^M} \left|\hat{\zeta}_t^{B(L)}(I)\right| dt \le \int_{T_2^M}^{T_3^M} \left(E\left|\hat{\zeta}_t^L(I)\right| + \frac{C|B(L)|p_t}{(\log L)^{1/6}} \right) dt.$$

Lemma 6.1, with $b = K_1$, implies that

$$\int_{T_2^M}^{T_3^M} E\left|\hat{\zeta}_t^L(I)\right| dt \le C_1|B(L)|l(M)/K_1.$$

Also, using the asymptotics (5.6), it is easy to see that there is a constant C such that, for large M,

$$\int_{T_2^M}^{T_3^M} p_t\, dt = q\big(T_3^M\big) - q\big(T_2^M\big) \le Cl(M)\log\log M.$$

Recall that $M \ge \delta^{-1}$. Therefore, on the event in (6.36),

$$\int_{T_2^M}^{T_3^M} \left|\hat{\zeta}_t^{B(L)}(I)\right| dt \le |B(L)|l(M)\left[\frac{C_1}{K_1} + \frac{C\log\log M}{(\log L)^{1/6}}\right]. \tag{6.37}$$

We recall that K_1 was chosen so that $K_1 > C_1/\varepsilon_0'$ and also that $M \le \delta\hat{\alpha}$ and $L \ge \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$ imply $M \le L^2$ for small α. It follows that for small α, the right side of (6.37) is no larger than $2\varepsilon_0'|B(L)|l(M)$. Consequently, there is a constant C such that

$$P\left(\int_{T_2^M}^{T_3^M} \left|\hat{\zeta}_t^{B(L)}(I)\right| dt > 2\varepsilon_0'|B(L)|l(M)\right) \le \frac{C\log\log L}{(\log L)^{3/2}}.$$

Applying the Poisson domination estimate and Lemma 2.2, and again using $M \le L^2$, it follows that for small α and δ,

$$P\big(\hat{Y}_3 > 3\varepsilon_0'\alpha|B(L)|l(M)\big) \le \frac{C(\log\log M)}{(\log M)^{3/2}} \tag{6.38}$$

for all $M \in [\delta^{-1}, \delta\hat{\alpha})$. This is the desired upper bound for $\hat{Y}_3$.

6.5. *The term* $\hat{Y}_4$. Here, we make use of the event Ω_1 given below (6.6). Since $\Omega_1 \subset \Omega_0$ and $T_4^M \le L^2$, Lemma 3.3(i) implies that

$$\Big| \left|\hat{\zeta}_t^{B(L)}(I)\right| - \left|\hat{\zeta}_t^L(I)\right| \Big| \le |\zeta_t \cap A(t)| \quad \text{for } t \le T_4^M$$

on Ω_1. Also, by definition,

$$\int_{T_3^M}^{T_4^M} |\zeta_t \cap A(t)|\, dt < \varepsilon_0'|B(L)|l(\delta^{-1})$$

on Ω_1. Combining these bounds gives

$$\int_{T_3^M}^{T_4^M} \left|\hat{\zeta}_t^{B(L)}(I)\right| dt \le \int_{T_3^M}^{T_4^M} \left|\hat{\zeta}_t^{L}(I)\right| dt + \varepsilon_0'|B(L)|l(\delta^{-1}) \tag{6.39}$$

on Ω_1.

By Lemma 6.1, with $b = (\log M)^2$,

$$E\left(\int_{T_3^M}^{T_4^M} \left|\hat{\zeta}_t^{L}(I)\right| dt\right) \le C_1|B(L)|l(M)/(\log M)^2$$

for large M. By Markov's inequality, this implies

$$P\left(\int_{T_3^M}^{T_4^M} \left|\hat{\zeta}_t^{L}(I)\right| dt > \varepsilon_0'|B(L)|l(M)\right) \le C_1/\varepsilon_0'(\log M)^2. \tag{6.40}$$

Combining (6.39) and (6.40) and using $M \ge \delta^{-1}$, we have, for small δ,

$$P\left(\int_{T_3^M}^{T_4^M} \left|\hat{\zeta}_t^{B(L)}\right| dt > 2\varepsilon_0'|B(L)|l(M), \Omega_1\right) \le C_1/\varepsilon_0'(\log M)^2.$$

It therefore follows from the Poisson domination estimate and Lemma 2.2 that there is a constant C such that

$$P\left(\hat{Y}_4 > 3\varepsilon_0'\alpha|B(L)|l(M),\ \Omega_1\right) \le \frac{C}{\varepsilon_0'(\log M)^2} \tag{6.41}$$

for small α and δ, and $M \ge \delta^{-1}$. This is the desired upper bound for $\hat{Y}_4$.

6.6. *The term* $\hat{Y}_5$. By the conservation of mass,

$$M\left|\hat{Y}_{t,\infty}^{B(L)}([M, Mr))\right| \le \sum_{x\in B(L)} 1\{\hat{\zeta}_t^x \ne \varnothing\}.$$

The right side has expected value $e^{-\alpha t}|B(L)|$. Therefore, setting $t = T_4^M = \bar{\alpha}\log\bar{\alpha}$,

$$E\hat{Y}_5 \le \exp(-\alpha T_4^M)|B(L)|/M = \alpha|B(L)|/M.$$

Thus, Markov's inequality implies

$$P\left(\hat{Y}_5 \ge \varepsilon_0'\alpha|B(L)|\right) \le 1/\varepsilon_0' M, \tag{6.42}$$

which is the desired upper bound on $\hat{Y}_5$.

6.7. *Conclusion.* In order to complete the proof of Proposition 6.1, we need only assemble the various estimates we have derived, and check that they imply (6.2). The bound $P(\Omega_1) > 1 - \varepsilon_0'$ was given using Lemmas 3.2 and 3.4 immediately after the definition of Ω_1. By the upper and lower bounds (6.29) and (6.30), there exists a constant C such that for small α and δ and $M \in [\delta^{-1}, \delta\hat{\alpha}]$,

$$(6.43)\qquad P\left(\left|\hat{Y}_2 - \frac{\alpha|B(L)|l(M)\log r}{\gamma_d}\right| > 4\varepsilon_0'\alpha|B(L)|l(M)\right) \le \frac{C\log\log M}{(\log M)^{3/2}}.$$

Also, by the upper bounds (6.8), (6.38), (6.41) and (6.42), there exists a constant C such that for small α and δ, and $M \in [\delta^{-1}, \delta\hat{\alpha}]$,

$$(6.44)\qquad P\big(\hat{Y}_1 + \hat{Y}_3 + \hat{Y}_4 + \hat{Y}_5 > 8\varepsilon_0'\alpha|B(L)|l(M), \Omega_1\big) \le \frac{C\log\log M}{(\log M)^{3/2}}.$$

Summing up $\hat{Y}_i$ as in (6.4), one obtains

$$P\left(\left|N\big(B(L), I\big) - \frac{\alpha|B(L)|l(M)\log r}{\gamma_d}\right| > 12\varepsilon_0'\alpha|B(L)|l(M), \Omega_1\right) \le \frac{C\log\log M}{(\log M)^{3/2}}$$

for suitable C. Normalization by $|B(L)|$ implies (6.2) for $\varepsilon_0 = 12\varepsilon_0'$, which completes the proof of Proposition 6.1.

We remark that one can, if one wishes, modify Theorem 2 by replacing the approximations $\alpha k(\log r)^2/\pi$, in $d = 2$, and $\alpha(\log r)/\gamma_d$, in $d \ge 3$, used to define $E_L(k)$ by

$$(6.45)\qquad \alpha\int_0^\infty p_t\big(\exp(-r^k p_t) - \exp(-r^{k+1}p_t)\big)\,dt$$

in both cases. By doing this, one is essentially "going back one step" in the analysis of $N^L([r^k, r^{k+1}))$ by using the approximation (1.12) but not (1.11) for p_t. The advantage of using (6.45) is that, according to simulations, convergence is substantially faster in this setting. This modification is used in Bramson, Cox and Durrett (1997) to compare the prediction in Theorem 2 with field data from Hubbell (1995). The justification of the substitution in (6.45) is not difficult, and follows by applying (1.11) and reasoning similar to that between (6.5) and (6.6)

7. Proof of Theorem 3. Our strategy here will be to first estimate the mean of $N^L([a\hat{\alpha}, b\hat{\alpha}])$ (in 7.1), and then to show that $N^L([a\hat{\alpha}, b\hat{\alpha}])$ is close to its mean with probability close to 1 (in 7.2). This will give us (1.6). The first part includes an application of Sawyer's limit (1.15); the second part employs estimates similar to those in Sections 5 and 6. Since we consider only fixed a and b, our probability estimates need not tend to zero at a specific rate (as in Section 6), which simplifies matters here. Throughout this section, we write I for the interval $[a\hat{\alpha}, b\hat{\alpha}]$.

7.1. *Estimation of* $EN^L(I)$. The goal here is to show the limit (7.12). For this, we first rewrite $N^L(I)$. In keeping with the notation given in the introduction, we denote by $\nu^A(x)$, $A \subset \mathbb{Z}^d$, the patch size in A of x, that is,

$$\nu^A(x) = \sum_{z \in A} 1\{\xi_\infty(z) = \xi_\infty(x)\}.$$

As before, ξ_∞ denotes the unique equilibrium distribution of the voter model with mutation ξ_t. Since the patch at site x in $B(L)$ has exactly $\nu^{B(L)}(x)$ members, one has

$$N(B(L), I) = \sum_{x \in B(L)} \frac{1\{\nu^{B(L)}(x) \in I\}}{\nu^{B(L)}(x)}. \tag{7.1}$$

In this subsection, we will estimate the expectation of the right side of (7.1); division by $|B(L)|$ will then produce our estimate on $EN^L(I)$.

Our strategy will be to show that one can replace $\nu^{B(L)}(x)$ in (7.1) with $\nu(x)$, without significant error. Note that once the replacement is made, one has by translation invariance,

$$E\biggl(\sum_{x \in B(L)} \frac{1\{\nu(x) \in I\}}{\nu(x)}\biggr) = |B(L)|\, E\biggl(\frac{1\{\nu(O) \in I\}}{\nu(O)}\biggr). \tag{7.2}$$

Using $h(u) = (1/u)1\{u \in [a, b]\}$, one can write

$$Eh(\nu(O)/\hat{\alpha}) = \hat{\alpha}\, E\biggl(\frac{1\{\nu(O) \in I\}}{\nu(O)}\biggr).$$

By (1.15) and the asymptotics immediately above it, $\nu(O)/\hat{\alpha}$ converges in distribution, as $\alpha \to 0$, to a mean–one exponential random variable. It therefore follows that

$$\hat{\alpha} E\biggl(\frac{1\{\nu(O) \in I\}}{\nu(O)}\biggr) \to \int h(u) e^{-u}\, du = \int_a^b \frac{e^{-u}}{u}\, du \quad \text{as } \alpha \to 0. \tag{7.3}$$

Together with (7.2), this implies that

$$\frac{\hat{\alpha}}{|B(L)|} E\biggl(\sum_{x \in B(L)} \frac{1\{\nu(x) \in I\}}{\nu(x)}\biggr) \to \int_a^b \frac{e^{-u}}{u}\, du \quad \text{as } \alpha \to 0$$

for all L. We want to show the analogous result for $\nu^{B(L)}(x)$ replacing $\nu(x)$, and $L \geq \beta \bar{\alpha}^{1/2} (\log \bar{\alpha})^2$. We will employ (7.3) for this.

To approximate the right side of (7.1) by the left side of (7.2), we use duality. For $x \in \mathbb{Z}^d$, let $\tau(x)$ be the the time at which the random walk starting at x is killed; that is, $\tau(x) = \inf\{t > 0\colon \hat{\zeta}_t^x = \emptyset\}$. Then, $\tau(x)$ is an exponential random variable with mean $\bar{\alpha}$. For $A \subset \mathbb{Z}^d$, let $\tilde{\nu}^A(x)$ be the number of walks starting from A that coalesce with the walk starting from x before either is killed; that is,

$$\tilde{\nu}^A(x) = \sum_{z \in A} 1\{\hat{\zeta}_t^z = \hat{\zeta}_t^x \text{ for some } t < \tau(x) \wedge \tau(z)\}. \tag{7.4}$$

It is easy to see from the percolation substructure, that

$$\tilde{\nu}^A(\cdot) =_d \nu^A(\cdot). \tag{7.5}$$

So, to compute $EN^L(I)$, one can use

$$N(B(L), I) =_d \sum_{x \in B(L)} \frac{1\{\tilde{\nu}^{B(L)}(x) \in I\}}{\tilde{\nu}^{B(L)}(x)} \tag{7.6}$$

in place of (7.1).

Our basic approach will be to estimate the expected value of the right side of (7.6), restricted to $x \in B(\underline{L})$ for appropriate $\underline{L} \le L$, and to subsets $\Omega_2(x)$ with $P(\Omega_2(x)) \approx 1$. For this, we set $\underline{L} = L - w_L(\bar{\alpha} \log \bar{\alpha})$, where w_L is defined in (3.3). By (3.16), for small α and $L = L(\alpha) \ge \beta \bar{\alpha}^{1/2} (\log \bar{\alpha})^2$, $\bar{\alpha} \log \bar{\alpha}$ is at most $2L^2/\beta^2 (\log L)^3$. With this estimate, one can easily check that there is a constant C such that, for small α,

$$w_L(\bar{\alpha} \log \bar{\alpha}) \le CL/(\log L).$$

We now show that the contribution to the right side of (7.6) from $x \notin B(\underline{L})$ is negligible. Since $\tilde{\nu}^{B(L)}(x) \ge a\hat{\alpha}$ if the indicator function $1\{\tilde{\nu}^{B(L)}(x) \in I\}$ is not 0, there are constants C such that

$$\begin{aligned} \hat{\alpha} \sum_{x \in B(L) \setminus B(\underline{L})} \frac{1\{\tilde{\nu}^{B(L)}(x) \in I\}}{\tilde{\nu}^{B(L)}(x)} &\le \big|B(L) \setminus B(\underline{L})\big|/a \\ &\le CL^{d-1} w_L(\bar{\alpha} \log \bar{\alpha})/a \\ &\le C\big|B(L)\big|/a(\log L) \end{aligned}$$

for small α. From this, it follows that

$$\frac{\hat{\alpha}}{|B(L)|} E\Bigg(\sum_{x \in B(L) \setminus B(\underline{L})} \frac{1\{\tilde{\nu}^{B(L)}(x) \in I\}}{\tilde{\nu}^{B(L)}(x)} \Bigg) \to 0 \quad \text{as } \alpha \to 0. \tag{7.7}$$

For $x \in B(\underline{L})$, we restrict ourselves to a "good set" $\Omega_2(x)$, over which $\tilde{\nu}^{B(L)}(x)$ and $\tilde{\nu}(x)$ are identical. For this, we want to specify $\Omega_2(x)$ so that $\tilde{\nu}(x)$ is determined by time $\bar{\alpha} \log \bar{\alpha}$, and no particles from outside $B(L)$ have moved far enough by then to contribute to $\tilde{\nu}(x)$. To this end, we define

$$\Omega_2(x) = \big\{\tau(x) \le \bar{\alpha} \log \bar{\alpha}\big\} \cap \Omega_0,$$

where Ω_0 is given below (3.4). It is not hard to see that

$$\tilde{\nu}^{B(L)}(x) = \tilde{\nu}(x) \quad \text{on } \Omega_2(x) \tag{7.8}$$

for each $x \in B(\underline{L})$.

We check that one may safely neglect $\Omega_2^c(x)$ for each $x \in B(\underline{L})$. Since $\tilde{\nu}^{B(L)}(x) \ge a\hat{\alpha}$ when the indicator function is not 0,

$$\hat{\alpha} E\Bigg(\frac{1\big\{\tilde{\nu}^{B(L)}(x) \in I\big\}}{\tilde{\nu}^{B(L)}(x)}; \Omega_2^c(x) \Bigg) \le \frac{1}{a}\big(P(\tau(x) > \bar{\alpha} \log \bar{\alpha}) + P(\Omega_0)\big).$$

Since $P(\tau(x) > \bar{\alpha}\log\bar{\alpha}) = \alpha$, and $P(\Omega_0) \le 1/L^{d+1}$ by Lemma 3.2, this is at most $(\alpha + 1/L^{d+1})/\alpha$. Thus,

$$\hat{\alpha}E\biggl(\frac{1\{\tilde{\nu}^{B(L)}(x) \in I\}}{\tilde{\nu}^{B(L)}(x)}; \Omega_2^c(x)\biggr) \to 0 \quad \text{as } \alpha \to 0 \tag{7.9}$$

(and hence $L \to \infty$), where the convergence is uniform in $x \in B(\underline{L})$. Exactly the same argument shows that

$$\hat{\alpha}E\biggl(\frac{1\{\tilde{\nu}(x) \in I\}}{\tilde{\nu}(x)}; \Omega_2^c(x)\biggr) \to 0 \quad \text{as } \alpha \to 0 \tag{7.10}$$

uniformly in $x \in B(\underline{L})$.

Turning to the contribution from $\Omega_2(x)$, $x \in B(\underline{L})$, it follows from (7.8), translation invariance and (7.5) that for each $x \in B(\underline{L})$,

$$\begin{aligned} E\biggl(\frac{1\{\tilde{\nu}^{B(L)}(x) \in I\}}{\tilde{\nu}^{B(L)}(x)}; \Omega_2(x)\biggr) &= E\biggl(\frac{1\{\tilde{\nu}(x) \in I\}}{\tilde{\nu}(x)}; \Omega_2(x)\biggr) \\ &= E\biggl(\frac{1\{\nu(O) \in I\}}{\nu(O)}\biggr) - E\biggl(\frac{1\{\tilde{\nu}(x) \in I\}}{\tilde{\nu}(x)}; \Omega_2^c(x)\biggr). \end{aligned}$$

Together with (7.3) and (7.10), this implies that

$$\hat{\alpha}E\biggl(\frac{1\{\tilde{\nu}^{B(L)}(x) \in I\}}{\tilde{\nu}^{B(L)}(x)}; \Omega_2(x)\biggr) \to \int_a^b \frac{e^{-u}}{u}\,du \quad \text{as } \alpha \to 0 \tag{7.11}$$

uniformly in $x \in B(\underline{L})$.

As $L \to \infty$, $|B(\underline{L})|/|B(L)| \to 1$. By combining (7.7), (7.9) and (7.11), we therefore obtain

$$\frac{\hat{\alpha}}{|B(L)|}E\biggl(\sum_{x \in B(L)} \frac{1\{\tilde{\nu}^{B(L)}(x) \in I\}}{\tilde{\nu}^{B(L)}(x)}\biggr) \to \int_a^b \frac{e^{-u}}{u}\,du \quad \text{as } \alpha \to 0$$

for $L = L(\alpha) \ge \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$. Substitution into (7.6) then gives

$$\hat{\alpha}EN^L(I) \to \int_a^b \frac{e^{-u}}{u}\,du \quad \text{as } \alpha \to 0 \tag{7.12}$$

for $L \ge \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$. This is the desired limit.

7.2. *Deviation of $N^L(I)$ from its mean.* We now estimate $N(B(L), I)$ as in Sections 5 and 6. For given $0 < \varepsilon_1 < K_1 < \infty$, define $T_0 = 0$, $T_1 = \varepsilon_1\bar{\alpha}$, $T_2 = K_1\bar{\alpha}$ and $T_3 = \infty$. Letting $\hat{Y}_i = \hat{Y}^{B(L)}_{T_{i-1}, T_i}$, we have, by (2.5),

$$N(B(L), I) =_d \hat{Y}_1 + \hat{Y}_2 + \hat{Y}_3. \tag{7.13}$$

We will examine each of the three terms on the right side of (7.13). We first show that $\hat{Y}_1$ and $\hat{Y}_3$ are small by showing that their expectations are small; this gives us the bounds (7.15) and (7.18). On the other hand, the upper and lower bounds for $\hat{Y}_2$ in (7.23) and (7.24) will follow from (7.12) and Proposition 4.1. Together, (7.15), (7.18), (7.23) and (7.24) imply that for $L = L(\alpha) \ge \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$ and given $\varepsilon > 0$,

$$\lim_{\alpha\to 0} P\left(\left|N(B(L), I) - \frac{|B(L)|}{\hat{\alpha}}\int_a^b \frac{e^{-u}}{u}\,du\right| > \frac{\varepsilon|B(L)|}{\hat{\alpha}}\right) = 0.$$

This implies (1.6) and hence Theorem 3.

We proceed to estimate $\hat{Y}_i$, $i = 1, 2, 3$. In what follows, $\varepsilon_0 > 0$ is assumed to be fixed.

The term $\hat{Y}_1$. By monotonicity and the conservation of mass,

$$\left|\hat{\zeta}_t^{B(L)}(I)\right| \le \left|\hat{\zeta}_t^{B(L)}\big([a\hat{\alpha}, \infty)\big)\right| \le \left|\zeta_t^{B(L)}([a\hat{\alpha}, \infty))\right| \le |B(L)|/a\hat{\alpha}.$$

Thus, since $T_1 = \varepsilon_1\bar{\alpha}$, it follows from (2.6) that

$$E\hat{Y}_1 = \alpha\int_0^{\varepsilon_1\bar{\alpha}} E\left|\hat{\zeta}_t^{B(L)}(I)\right|dt \le \varepsilon_1|B(L)|/a\hat{\alpha}.$$

Setting $\varepsilon_1 = a\varepsilon_0^2$, one obtains

$$E\hat{Y}_1 \le \frac{|B(L)|}{\hat{\alpha}}\varepsilon_0^2 \tag{7.14}$$

and thus, by Markov's inequality,

$$P\big(\hat{Y}_1 \ge \varepsilon_0|B(L)|/\hat{\alpha}\big) \le \varepsilon_0. \tag{7.15}$$

This is the desired bound for $\hat{Y}_1$.

The term $\hat{Y}_3$. Here, we use the elementary bound $\hat{Y}_{t,\infty}^{B(L)}(I) \le |\hat{\zeta}_t^{B(L)}|$. By Lemma 3.3(ii),

$$\begin{aligned} E\hat{Y}_3 \le E\left|\hat{\zeta}_{K_1\bar{\alpha}}^{B(L)}\right| &\le E\left|\hat{\zeta}_{K_1\bar{\alpha}}^{L}\right| + 2\left|A(K_1\bar{\alpha})\right|p_{K_1\bar{\alpha}} \\ &\le \big(|B(L)| + 2|A(K_1\bar{\alpha})|\big)p_{K_1\bar{\alpha}}. \end{aligned} \tag{7.16}$$

Since by assumption $L \ge \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$, one has $K_1\bar{\alpha} \le L^2/(\log L)^3$ for small α, which implies that $|A(K_1\bar{\alpha})| \le C|B(L)|$ for an appropriate constant C. Therefore,

$$E\hat{Y}_3 \le C|B(L)|p_{K_1\bar{\alpha}}$$

for another choice of C. By the asymptotics (1.11), as $\alpha \to 0$,

$$p_{K_1\bar{\alpha}} \sim \begin{cases} (\log(K_1\bar{\alpha}))/\pi K_1\bar{\alpha}, & \text{in } d = 2, \\ 1/\gamma_d K_1\bar{\alpha}, & \text{in } d \ge 3. \end{cases}$$

Recalling the definition of $\hat{\alpha}$, it is clear that we may choose K_1 large enough so that for small α,

$$E\hat{Y}_3 \le \varepsilon_0^2|B(L)|/\hat{\alpha}, \tag{7.17}$$

where $\varepsilon_0 > 0$ is given above. Consequently, by Markov's inequality,

$$P\big(\hat{Y}_3 \ge \varepsilon_0|B(L)|/\hat{\alpha}\big) \le \varepsilon_0. \tag{7.18}$$

This is the desired bound for $\hat{Y}_3$.

The term $\hat{Y}_2$. Here, we show the bounds (7.23) and (7.24). Using (7.13) and the bounds for $EN^L(I)$, $E\hat{Y}_1$ and $E\hat{Y}_3$ in (7.12), (7.14) and (7.17), one has for small α and $L \geq \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$,

$$\left|\frac{\hat{\alpha}}{|B(L)|}E\hat{Y}_2 - \int_a^b \frac{e^{-u}}{u}\,du\right| \leq 3\varepsilon_0.$$

By (2.6), this implies

$$\left|\frac{\alpha\hat{\alpha}}{|B(L)|}\int_{\varepsilon_1\bar{\alpha}}^{K_1\bar{\alpha}} E|\hat{\zeta}_t^{B(L)}(I)|\,dt - \int_a^b \frac{e^{-u}}{u}\,du\right| \leq 3\varepsilon_0. \tag{7.19}$$

We next employ Proposition 4.1. Since $K_1\bar{\alpha} \leq L^2/(\log L)^3$ for small α, the event

$$\left\{\left||\hat{\zeta}_t^{B(L)}(I)| - E|\hat{\zeta}_t^{B(L)}(I)|\right| \leq 8p_t|B(2L)|/(\log L)^{1/6} \quad \text{for all } t \leq K_1\bar{\alpha}\right\} \tag{7.20}$$

has probability at least $1 - C(\log\log L)/(\log L)^{3/2}$ for an appropriate constant C. By (5.6), for an appropriate C and small α,

$$\int_{\varepsilon_1\bar{\alpha}}^{K_1\bar{\alpha}} p_t\,dt = q(K_1\bar{\alpha}) - q(\varepsilon_1\bar{\alpha}) \leq \begin{cases} C\log\bar{\alpha}, & \text{in } d = 2,\\ C, & \text{in } d \geq 3.\end{cases}$$

Consequently, on the event in (7.20),

$$\alpha\left|\int_{\varepsilon_1\bar{\alpha}}^{K_1\bar{\alpha}} |\hat{\zeta}_t^{B(L)}(I)|\,dt - \int_{\varepsilon_1\bar{\alpha}}^{K_1\bar{\alpha}} E|\hat{\zeta}_t^{B(L)}(I)|\,dt\right| \leq \frac{C|B(2L)|}{\hat{\alpha}(\log L)^{1/6}}. \tag{7.21}$$

Combining (7.19) and (7.21), we have, for small α and $L \geq \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$,

$$\left|\alpha\int_{\varepsilon_1\bar{\alpha}}^{K_1\bar{\alpha}} |\hat{\zeta}_t^{B(L)}(I)|\,dt - \frac{|B(L)|}{\hat{\alpha}}\int_a^b \frac{e^{-u}}{u}\,du\right| \leq \frac{4\varepsilon_0|B(L)|}{\hat{\alpha}} \tag{7.22}$$

on the event in (7.20).

We now apply Lemmas 2.1 and 2.2. There is a Poisson random variable X with mean

$$EX = \frac{|B(L)|}{\hat{\alpha}}\left(\int_a^b \frac{e^{-u}}{u}\,du + 4\varepsilon_0\right)$$

such that $\hat{Y}_2 \leq X$ on the event in (7.20). Since $L \geq \beta\bar{\alpha}^{1/2}(\log\bar{\alpha})^2$, $|B(L)|/\hat{\alpha} \to \infty$ as $\alpha \to 0$. Therefore, for given $\varepsilon_0 > 0$, $P(X \geq (1+\varepsilon_0)EX) \to 0$ as $\alpha \to 0$, and it follows that

$$P\left(\hat{Y}_2 \geq (1+\varepsilon_0)\frac{|B(L)|}{\hat{\alpha}}\left(\int_a^b \frac{e^{-u}}{u}\,du + 4\varepsilon_0\right)\right) \to 0 \quad \text{as } \alpha \to 0. \tag{7.23}$$

A similar argument yields the inequality

$$P\left(\hat{Y}_2 \leq (1-\varepsilon_0)\frac{|B(L)|}{\hat{\alpha}}\left(\int_a^b \frac{e^{-u}}{u}\,du - 4\varepsilon_0\right)\right) \to 0 \quad \text{as } \alpha \to 0. \tag{7.24}$$

These are the desired bounds for $\hat{Y}_2$.

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