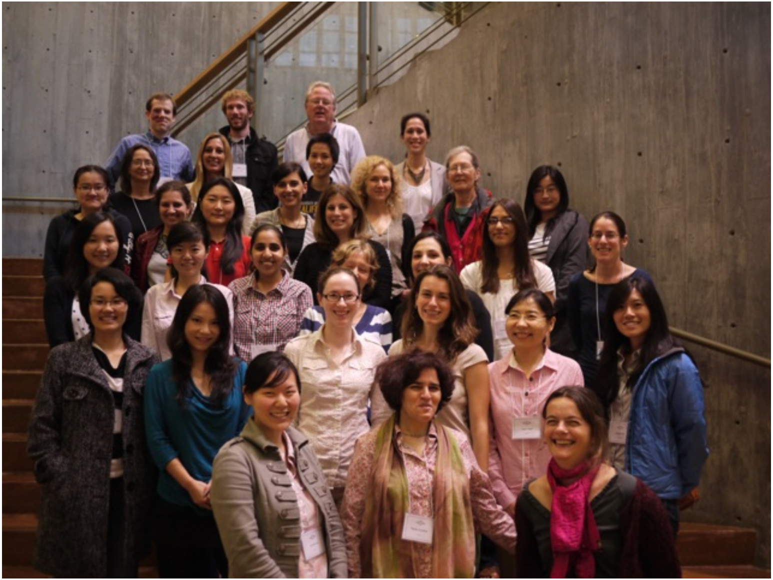


Women in Probability Workshop, Duke, 2012

Conference on New Developments in Probability 2024!
September 26-28 at CRM in Montreal

Infinite-dimensional diffusions under Hörmander's condition

Tai Melcher
University of Virginia

joint with D. Phillips

30 years of Women in Probability
University of North Carolina, Chapel Hill
5 Aug 2024

Hömander's condition

Theorem (Hörmander, 1967)

Suppose that $\{X_1, \ldots, X_k\}$ *is a collection of vector fields* $X_i : \mathbb{R}^n \to \mathbb{R}^n$ *so that, for all* $x \in \mathbb{R}^n$,

$$\mathrm{span}\{X_i, [X_{i_1}, X_{i_2}], \ldots, [X_{i_1}, [\cdots, [X_{i_{r-1}}, X_{i_r}]]]\}(x) = \mathbb{R}^n. \quad \text{(HC)}$$

Then for ξ_t *the solution to the SDE*

$$d\xi_t^x = X_1(\xi_t) \circ dB_t^1 + \cdots + X_k(\xi_t) \circ dB_t^k, \quad \xi_0 = x,$$

$\mathrm{Law}(\xi_t^x)$ *is smooth for each* $t > 0$*, that is,* $\mathrm{Law}(\xi_t^x)$ *is abs cts wrt Lebesgue measure with strictly positive and smooth density:*

$$\mathrm{Law}(\xi_t^x) = p_t(x, \cdot)\, dm, \ \textit{for some } p_t \in C^\infty(\mathbb{R}^n, (0, \infty)).$$

Heisenberg group examples

The elliptic case

Now let

$$\left.\begin{array}{l}\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right)\\ \tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right)\\ \tilde{X}_3(x) = (0, 0, 1)\end{array}\right\} \text{ NOTE: } \forall x \in \mathbb{R}^3,\ \mathrm{span}\{\tilde{X}_1(x), \tilde{X}_2(x), \tilde{X}_3(x)\} = \mathbb{R}^3$$

Consider the solution $\xi_t = (\xi_t^1, \xi_t^2, \xi_t^3)$ to SDE

$$\begin{aligned} d\xi_t &= \tilde{X}_1(\xi_t) \circ dB_t^1 + \tilde{X}_2(\xi_t) \circ dB_t^2 + \tilde{X}_3(\xi_t) \circ dB_t^3 \\ &= \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}\xi_t^2 \end{pmatrix} \circ dB_t^1 + \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}\xi_t^1 \end{pmatrix} \circ dB_t^2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \circ dB_t^3 \end{aligned}$$

with $\xi_0 = 0$.

Heisenberg group examples

The elliptic case

Now let

$$\tilde{X}_1(x) = \left(1, 0, -\tfrac{1}{2}x_2\right)$$
$$\tilde{X}_2(x) = \left(0, 1, \tfrac{1}{2}x_1\right)$$
$$\tilde{X}_3(x) = (0, 0, 1)$$

The solution to the SDE

$$d\xi_t = \tilde{X}_1(\xi_t) \circ dB_t^1 + \tilde{X}_2(\xi_t) \circ dB_t^2 + \tilde{X}_3(\xi_t) \circ dB_t^3,$$

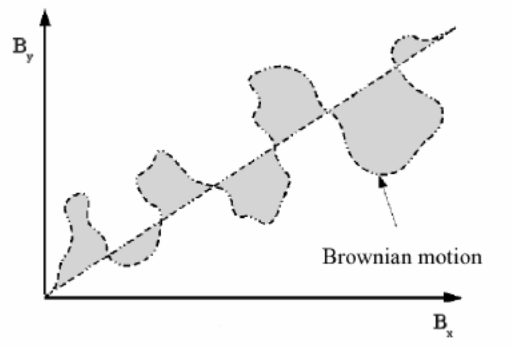
with $\xi_0 = 0$ may be written explicitly as

$$\xi_t = \left(B_t^1, B_t^2, B_t^3 + \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1\right).$$

Heisenberg group examples

Stochastic Lévy area

$$Z_t = \int_0^t B_x(s)dB_y(s) - B_y(s)dB_x(s)$$



(image from Manigo-Bornales)

Heisenberg group examples

Stochastic Lévy area

$$\{Z_t\}_{t\geq 0} = \left\{ \int_0^t B_1(s)dB_2(s) - B_2(s)dB_1(s) \right\}_{t\geq 0}$$
$$\overset{d}{=} \left\{ B\left(\frac{1}{4} \int_0^t (B_s^1)^2 + (B_s^2)^2\, ds \right) \right\}_{t\geq 0}$$

This implies, for example, that for

$$\rho_t(B^1, B^2) := \frac{1}{4} \int_0^t (B_s^1)^2 + (B_s^2)^2\, ds,$$

we may write

$$(Z_t \mid \rho_t(B^1, B^2)) \sim \mathcal{N}(0, \rho_t(B^1, B^2)).$$

Heisenberg group examples

The hypoelliptic case

Again let

$$\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right) = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_3}$$
$$\tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right) = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial x_3}$$
$$\tilde{X}_3(x) = (0, 0, 1) = \frac{\partial}{\partial x_3}$$

Note that $[\tilde{X}_1, \tilde{X}_2] := \tilde{X}_1\tilde{X}_2 - \tilde{X}_2\tilde{X}_1 = \tilde{X}_3$. Thus, we can write

$$\mathrm{span}\{\tilde{X}_1(x), \tilde{X}_2(x), [\tilde{X}_1, \tilde{X}_2](x)\} = \mathbb{R}^3.$$

So $\{\tilde{X}_1, \tilde{X}_2\}$ satisfies *Hörmander's Condition*.

Heisenberg group examples

The hypoelliptic case

Again let

$$\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right) = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_3}$$

$$\tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right) = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial x_3}$$

$$\tilde{X}_3(x) = (0, 0, 1) = \frac{\partial}{\partial x_3}$$

Now consider the SDE

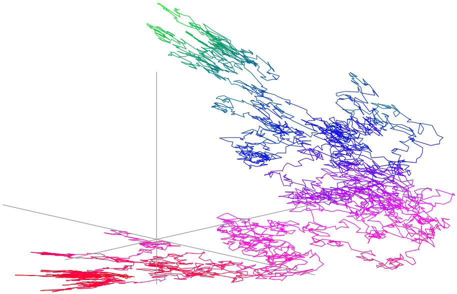
$$dg_t = \tilde{X}_1(g_t) \circ dB_t^1 + \tilde{X}_2(g_t) \circ dB_t^2,$$

which again we may solve explicitly, now as

$$g_t = \left(B_t^1, B_t^2, \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1\right).$$

Heisenberg group examples

$$g_t = \left(B_t^1, B_t^2, \frac{1}{2} \int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1 \right) =: (B_t, Z_t)$$



(image from Nate Eldredge)

Heisenberg group examples

An explicit heat kernel formula

(HC) $\implies \mathrm{Law}(\xi_t)$ and $\mathrm{Law}(g_t)$ are smooth measures on $\mathbb{R}^3$.

In particular for $\nu_t := \mathrm{Law}(g_t) = \mathrm{Law}\,(B_t, Z_t)$,

$$\frac{d\nu_t}{dm}(x, y, z) = \frac{1}{16\pi^2} \int_{\mathbb{R}} e^{i\lambda z/2} \frac{\lambda}{\sinh(\lambda t)} e^{-(x^2+y^2)\lambda \coth(\lambda t)/4}\, d\lambda.$$

One may prove this by showing that

$$\mathbb{E}\left[f(B_t)e^{i\lambda Z_t}\right] = \mathbb{E}\left[f(B_t)e^{-\rho_t(B)\lambda^2}\right]$$

where $\rho_t(B) = \frac{1}{4}\int_0^t |B_s|^2\, ds$. (see Gaveau, Lévy, Yor, Helmes-Schwane,. . .)

Heisenberg group geometry

Let $\mathfrak{h} = \mathrm{span}\{X_1, X_2, X_3\} \cong \mathbb{R}^3$ with Lie bracket $[X_1, X_2] = X_3$, and all other brackets are 0. In coordinates, this is

$$\begin{aligned}
&[(x_1, x_2, x_3)\,,\,(x_1', x_2', x_3')] \\
&\qquad = [x_1 X_1 + x_2 X_2 + x_3 X_3\,,\, x_1' X_1 + x_2' X_2 + x_3' X_3] \\
&\qquad = x_1 x_1' [X_1, X_1] + x_1 x_2' [X_1, X_2] + x_1 x_3' [X_1, X_3] + \cdots \\
&\qquad = (0, 0, x_1 x_2' - x_1' x_2).
\end{aligned}$$

Via the BCHD formula we may equip $\mathbb{R}^3$ with the group operation

$$\begin{aligned}
x \cdot x' &= x + x' + \frac{1}{2}[x, x'] \\
&= \left(x_1 + x_1', x_2 + x_2', x_3 + x_3' + \frac{1}{2}(x_1 x_2' - x_2 x_1') \right).
\end{aligned}$$

Then $\mathbb{R}^3$ with this group operation is the Heisenberg group $\mathbb{H}$, with $\mathrm{Lie}(\mathbb{H}) = \mathfrak{h}$; the $\tilde{X}_i$'s are the unique left inv vfs so that $\tilde{X}_i(0) = X_i$.

Heisenberg group geometry

We can define a left invariant Riemannian metric on $\mathbb{H}$ by taking $\{\tilde{X}_i(x)\}_{i=1}^3$ to be an onb at each $x \in \mathbb{H}$, and Riemannian distance

$$\delta(x, y) := \inf\{\ell(\gamma) : \gamma \text{ path from } x \text{ to } y\}$$

where

$$\ell(\gamma) := \int_0^1 \|\gamma'(t)\|_{\gamma(t)}\, dt.$$

Alternatively, we can define the horizontal distance

$$d(x, y) := \inf\{\ell(\gamma) : \gamma \text{ a horizontal path from } x \text{ to } y\},$$

where γ is horizontal if

$$\gamma'(t) \in \mathrm{span}\{\tilde{X}_1(\gamma(t)), \tilde{X}_2(\gamma(t))\}.$$

Heisenberg group geometry

We can write

$$\mathfrak{h} := \mathrm{span}\{X_1, X_2\} \times \mathrm{span}\{X_3\} =: H \times C$$

and define $\omega := [\cdot, \cdot]|_{H\times H} : H \times H \to C$ so that

$$[(h, c)\,,\,(h', c')] = (0, \omega(h, h')) = (0, h_1 h_2' - h_1' h_2).$$

and

$$(h, c) \cdot (h', c') = \left(h + h', c + c' + \frac{1}{2}\omega(h, h') \right).$$

Then

$$(HC) \iff \omega(H, H) = C,$$

and we may write

$$g_t := (B_t, Z_t) := \left(B_t, \frac{1}{2} \int_0^t B_s^1 dB_s^2 - B_s^2\, dB_s^1 \right)$$
$$= \left(B_t, \frac{1}{2} \int_0^t \omega\,(B_s, dB_s) \right).$$

Heisenberg group geometry

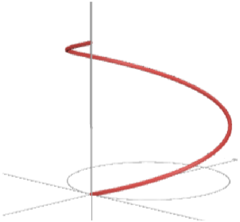
For the horizontal distance on the Heisenberg group,

$$K_1\left(\|h\|_H+\sqrt{|c|}\right) \leq d(e,(h,c)) \leq K_2\left(\|h\|_H+\sqrt{|c|}\right).$$

In particular, $d(0,(h,c))<\infty$ for all $(h,c)\in\mathbb{H}$. Many explicit expressions are known, for example,

$$d(0,(0,c))=\sqrt{\frac{\pi}{2}|c|}$$

and geodesics (length minimizers) are well understood.



Hörmander's condition and horizontal distance

In the same way, given a collection of vfs $\{X_1, \ldots, X_k\}$ satisfying (HC) on $\mathbb{R}^n$, can define the horizontal distance analogously.

Chow-Rashevskii: (HC) $\implies d(x, y) < \infty$ for all $x, y \in \mathbb{R}^n$.
Moreover, the horizontal topology will be equivalent to the Euclidean one.

Smooth measures in ∞ dim

A measure μ on $\mathbb{R}^n$ is said to be *smooth* if

Definition^1 μ is abs cts wrt Lebesgue measure and the RN derivative is strictly positive and smooth – that is,

$$\mu = \rho\, dm, \text{ for some } \rho \in C^\infty(\mathbb{R}^n, (0,\infty)).$$

Definition^2 for any multi-index α, there exists a function $g_\alpha \in C^\infty(\mathbb{R}^n) \cap L^{\infty-}(\mu)$ such that

$$\int_{\mathbb{R}^n} (-D)^\alpha f\, d\mu = \int_{\mathbb{R}^n} f g_\alpha\, d\mu, \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

- Definition^1 $\iff$ Definition^2

A first step to smoothness: Quasi-invariance

So as a first step, we'd need to know how a measure behaves under (infinitesimal) translations.

Definition A measure μ on Ω is *quasi-invariant* under a transformation $T : \Omega \to \Omega$ if μ and $\mu \circ T^{-1}$ are mutually absolutely continuous.

In particular, we're generally interested in quasi-invariance under transformations of the type

$$T = T_h = \text{translation by an element } h \in \Omega_0 \subset \Omega,$$

where Ω_0 is some distinguished subset of Ω.

A typical ∞-dim setting

Take H a Hilbert space with Gaussian meas μ with covariance Q:

$$\hat{\mu}(k) := \int_H e^{i\langle k,h\rangle_H} \, d\mu(h) = e^{-\langle Qk,k\rangle_H/2}.$$

Q is necessarily a non-neg, sym, trace class operator.

Theorem (Cameron-Martin-Maruyama)

μ is quasi-invariant under translation by elements of $H_\mu = Q^{1/2}H$.

That is, for $k \in H_\mu$ and $d\mu^k := d\mu(\cdot - k)$,

$$\mu^k \ll \mu \quad \textit{and} \quad \mu^k \gg \mu.$$

Moreover, if $k \notin H$, then $\mu^k \perp \mu$.

H_μ is called the Cameron-Martin space for (H, μ).

A typical ∞-dim setting

Facts about the CM space H_μ

- H_μ is a Hilbert space equipped with inner product

$$\langle h, k\rangle_\mu := \langle Q^{-1/2}h, Q^{-1/2}k\rangle_H,$$

  densely embedded in H, and, when $\dim(H) = \infty$, $\mu(H_\mu) = 0$.
- The inclusion map $\iota : H_\mu \hookrightarrow H$ is Hilbert-Schmidt

$$\|\iota\|_{HS}^2 := \sum_i \|h_i\|_H^2 < \infty.$$

- In fact, given a Hilbert space H with a Hilbert subspace K such that the inclusion $K \hookrightarrow H$ is HS, immediately implies the existence of a Gaussian measure on H with covariance determined by K.

∞-dim Heisenberg gps with $\dim(C) < \infty$

(Driver-Gordina, '08) Take

- H a Hilbert space with Gaussian measure μ
- C a fin-dim inner product space
- $\omega : H \times H \to C$ is a continuous anti-symmetric bilinear form

Then we can make $\mathfrak{g} := H \times C$ into a Lie algebra with bracket

$$[(h, c), (h', c')] := (0, \omega(h, h'))$$

and Lie group $G := H \times C$ with group operation

$$(h, c) \cdot (h', c') = \left(h + h', c + c' + \frac{1}{2}\omega\left(h, h'\right)\right).$$

∞-dim Heisenberg gps with $\dim(C) < \infty$

Let H_μ denote the Cameron-Martin space for (H, μ).

- $G_{CM} := H_\mu \times C$ inherits a group structure from $\omega|_{H_\mu \times H_\mu}$, and is a dense subgp of $G = H \times C$
- $\omega : H \times H \to C$ cts $\implies$ $\omega : H_\mu \times H_\mu \to C$ Hilbert-Schmidt

$$\|\omega\|^2_{HS(H_\mu \times H_\mu, C)} := \sum_{i,j,\ell} \langle \omega(e_i, e_j), f_\ell \rangle_C^2 < \infty.$$

  - ω HS on H_μ is in some sense the necessary assumption, rather than ω cts on H
- For example, ω HS $\implies$ stochastic Lévy area for $\{B_t\}_{t \geq 0}$ BM on H

$$Z_t := \frac{1}{2} \int_0^t \omega(B_s, dB_s)$$

is well-defined in C.

∞-dim Heisenberg gps with $\dim(C) < \infty$

$(B_t, B_t^C + Z_t)$ and (B_t, Z_t) are still solutions to the "elliptic" and "hypoelliptic" SDEs, respectively.

Driver-Gordina proved regularity properties (e.g., qi and 1st order ibp) for $\mathrm{Law}((B_t, B_t^C + Z_t))$ on G. Dobbs-M proved smoothness via arbitrary ibp formulae.

What about $\nu_t := \mathrm{Law}\left((B_t, Z_t)\right)$?

∞-dim Heisenberg gps with $\dim(C) < \infty$

Theorem (Baudoin-Gordina-M, '13)

Assume $\omega(H_\mu, H_\mu) = C$. Then ν_t is qi under left and right translations by elts of $G_{CM} = H_\mu \times C$. Moreover,

$$\left\| \frac{d(\nu_t \circ R_{(h,c)}^{-1})}{d\nu_t} \right\|_{L^q(G,\nu_t)} \leq \exp\left(C\left(q, t, \frac{\|\omega\|_{HS}^2}{\rho_2}\right) d^2(0,(h,c)) \right)$$

where d is the horizontal distance on G_{CM} and

$$\rho_2 := \inf\left\{ \sum_{i,j=1}^{\infty} \left(\sum_{\ell=1}^{N} \langle \omega(e_i, e_j), f_\ell \rangle_C x_\ell \right)^2 : \sum_{\ell=1}^{N} x_\ell^2 = 1 \right\},$$

and similarly for the RN derivative under left translation.

∞-dim Heisenberg gps with $\dim(C) < \infty$

The proof relied on several elts, including

1. fxnal inequalities (particularly generalized CD inequalities à la Baudoin, Bonnefont, Garofalo) involving coefficients with

$$\|\omega\|^2_{HS(H_\mu \times H_\mu, C)} := \sum_{i,j=1}^{\infty} \sum_{\ell=1}^{N} \langle \omega(e_i, e_j), f_\ell \rangle_C^2,$$

where continuity of $\omega : H \times H \to C \implies \|\omega\|^2_{HS} < \infty$, and

$$\rho_2 := \inf \left\{ \sum_{i,j=1}^{\infty} \left(\sum_{\ell=1}^{N} \langle \omega(e_i, e_j), f_\ell \rangle_C x_\ell \right)^2 : \sum_{\ell=1}^{N} x_\ell^2 = 1 \right\}$$

$$= \inf_{\|c\|_C = 1} \sum_{i,j=1}^{\infty} \langle \omega(e_i, e_j), c \rangle_C^2.$$

Note that (HC) $\implies \rho_2 > 0$.

∞-dim Heisenberg gps with $\dim(C) < \infty$

The proof relied on several elts, including

2. convergence of the horizontal distance by fin-dim approx gps

$$d_n(e, g) \to d(e, g)$$

as $n \to \infty$. We made critical use of the estimate for $g = (h, c)$

$$\|h\|_{H_\mu} + K_1(\omega)\sqrt{\|c\|_C} \leq d(e, (h, c)) \leq \|h\|_{H_\mu} + K_2(\omega, N)\sqrt{\|c\|_C}$$

where $N := \dim(C)$.

∞-dim Heisenberg gps with $\dim(C) < \infty$

Theorem (Driver-Eldredge-M, '16)

For $c \in C$, define the HS operator $\Omega_c : H_\mu \to H_\mu$ by

$$\langle \Omega_c h, k \rangle_{H_\mu} := \langle \omega(h,k), c \rangle_C,$$

and let $\rho_t(B)$ be the random linear transformation on C defined by

$$\langle \rho_t(B) c, c' \rangle_C := \frac{1}{4} \int_0^t \langle \Omega_c B_t, \Omega_{c'} B_t \rangle_{H_\mu} \, dt.$$

Then $\nu_t(dh, dc) = \gamma_t(h,c) \, \mu_t(dh) \, m(dc)$ where m is Lebesgue measure on C and

$$\gamma_t(h,c) := \mathbb{E} \left[\frac{\exp\left(-\frac{1}{2} \langle \rho_t^{-1}(B) c, c \rangle_C\right)}{\sqrt{(2\pi)^N \det \rho_t(B)}} \middle| B_t = h \right]$$

is a smooth density.

Other ∞-dim hypoelliptic results

- Hörmander generators on ∞-dim configuration space: Lugiewicz-Zegarlinski ('07), Inglis-Papageorgiou ('09), Kontis-Ottobre-Zegarlinski ('16)
- evolution equations: Baudoin-Teichmann ('05), Forster-Lütkebohmert-Teichmann ('08)
- SPDEs: Hairer-Mattingly ('04), Mattingly-Pardoux ('04), Bakhtin-Mattingly ('07), Agrachev-Kuksin-Sarychev-Shirikyan ('07), Glatt-Holtz-Herzog-Mattingly ('18), ...

∞-dim Heisenberg gps with $\dim(C) < \infty$

pros and cons: The explicit heat kernel for ν_t found in Driver-Eldredge-M was of course a stronger result, but the techniques were fairly specific to the step 2 structure. The fxnal inequality approach à la Baudoin-Gordina-M is perhaps more robust.

Unresolved questions from (Baudoin-Gordina-M):

- If we remove $\dim(C) < \infty$, can we still prove $d_n \to d$?
- What does ρ_2 mean?

Unresolved question from (Driver-Eldredge-M):

- If we remove $\dim(C) < \infty$, does anything still make sense? wrt what reference measure?

Moving to $\dim(C) = \infty$

Suppose $\omega : H_\mu \times H_\mu \to C$ is HS

$$\|\omega\|_{HS}^2 := \sum_{i,j,\ell} \langle \omega(e_i, e_j), f_\ell \rangle_C^2 < \infty.$$

For each ℓ, define $x_\ell := \sum_{i,j} \langle \omega(e_i, e_j), f_\ell \rangle_C^2$.

ω HS $\implies$ x_ℓ is summable, and so $x_\ell \to 0$. This implies that

$$\rho_2 = \inf_{\|c\|_C = 1} \sum_{i,j} \langle \omega(e_i, e_j), c \rangle_C^2 \leq \inf_\ell x_\ell = 0.$$

Similarly, $\rho_2 > 0$ prohibits $\|\omega\|_{HS} < \infty$. So when $\dim(C) = \infty$, you can't have $\rho_2 > 0$ and $\|\omega\|_{HS} < \infty$.

(*Notice that this is only an issue when $\dim(C) = \infty$.)

Moving to $\dim(C) = \infty$

So, needing (something like) $\rho_2 > 0$ and $\|\omega\|_{HS} < \infty$ necessitates the existence of another Hilbert space Z densely embedded in C, so that $\omega : H_\mu \times H_\mu \to Z$ with

$$\inf_{\|z\|_Z=1} \sum_{i,j} \langle \omega(e_i, e_j), z \rangle_Z^2 > 0$$

(and so ω is **not** HS into Z) with a HS inclusion map $\iota : Z \to C$ so that $\|\iota\omega\|_{HS(H_\mu \times H_\mu, C)} < \infty$.

Such a (C, Z) necessarily supports a Gaussian measure on C with Cameron-Martin space Z.

∞-dim Heisenberg gps with $\dim(C) = \infty$

The main assumptions

Let (W, H_μ, μ) and (C, C_ν, ν) be (Hilbert) Gaussian measure spaces. Let $\omega : H_\mu \times H_\mu \to C_\nu$ be a skew-symmetric bilinear map.

We assume that

$$\|\omega\|_\mu^2 := \sup_{\|z\|_{C_\nu}=1} \sum_{i,j} \langle \omega(e_i, e_j), z \rangle_{C_\nu}^2 < \infty, \tag{A1}$$

$$\|\omega\|_{H_\mu \otimes C_\nu}^2 := \sup_{\|h\|_H=1} \sum_{i,\ell} \langle \omega(h, e_i), f_\ell \rangle_{C_\nu}^2 < \infty, \tag{A2}$$

and

$$\lfloor \omega \rfloor_\mu^2 := \inf_{\|z\|_{C_\nu}=1} \sum_{i,j} \langle \omega(e_i, e_j), z \rangle_{C_\nu}^2 > 0. \tag{A3}$$

∞-dim Heisenberg gps with $\dim(C) = \infty$

The main assumptions: weakly HS v. HS

(A1) $\quad\longleftrightarrow\quad \omega : H_\mu \times H_\mu \to C_\nu$ is "weakly Hilbert-Schmidt":
ω extends to bdd linear operator $\widetilde{\omega} : H_\mu \otimes H_\mu \to C_\nu$ so that

$$\widetilde{\omega}(h \otimes k) = \omega(h, k)$$

and

$$\begin{aligned}
\|\widetilde{\omega}\|^2_{\mathcal{L}(H_\mu \otimes H_\mu, C_\nu)} &= \|\widetilde{\omega}^*\|^2_{\mathcal{L}(C_\nu, H_\mu \otimes H_\mu)} = \sup_{\|z\|_{C_\nu}=1} \|\widetilde{\omega}^* z\|^2_{H_\mu \otimes H_\mu} \\
&= \sup_{\|z\|_{C_\nu}=1} \sum_{i,j} \langle e_i \otimes e_j, \widetilde{\omega}^* z\rangle^2_{H_\mu \otimes H_\mu} \\
&= \sup_{\|z\|_{C_\nu}=1} \sum_{i,j} \langle \omega(e_i, e_j), z\rangle^2_{C_\nu} = \|\omega\|^2_\mu.
\end{aligned}$$

Note this $\implies \widetilde{\omega} : H_\mu \otimes H_\mu \longrightarrow C_\nu \xrightarrow{i} C$ is HS.

$\iff \omega : H_\mu \times H_\mu \longrightarrow C_\nu \xrightarrow{i} C$ is HS.

∞-dim Heisenberg gps with $\dim(C) = \infty$

The main assumptions: Lower bound

(A3) $\longleftrightarrow$ $\widetilde{\omega}^* : C_\nu \to H_\mu \otimes H_\mu$ is bounded below:

$$\|\widetilde{\omega}^* z\|^2_{H_\mu \otimes H_\mu} = \sum_{i,j} \langle \widetilde{\omega}^* z, e_i \otimes e_j \rangle^2_{H_\mu \otimes H_\mu}$$

$$= \sum_{i,j} \langle z, \omega(e_i, e_j) \rangle^2_{C_\nu} \geq \lfloor \omega \rfloor^2_\mu \|z\|^2_{C_\nu}.$$

Functional analysis lemma:

For H, K Hilbert spaces and $A : H \to K$ a bdd linear operator,

A^* is bounded below iff A is surjective.

Thus, $\widetilde{\omega} : H_\mu \otimes H_\mu \to C_\nu$ is surjective. In particular, this implies that $\mathrm{span}(\omega(H_\mu \times H_\mu))$ is dense in C_ν. That is, (A3)$\leftrightarrow$(HC).

∞-dim Heisenberg gps with $\dim(C) = \infty$

We now define a Lie algebra structure on $\mathfrak{g}_{CM} := H_\mu \times C_\nu$ and group structure on $G_{CM} := H_\mu \times C_\nu$ via $\omega : H_\mu \times H_\mu \to C_\nu$ as before.

Note that we don't require that ω extend to a cts bilinear map $H \times H \to C$. However, (A2) is sufficient to say that

$$(h, z) \cdot (x, c) = \left(h + x, c + z + \frac{1}{2}\omega(h, x)\right)$$

defines a measurable group action of G_{CM} on "G"$:= H \times C$, which is sufficient to discuss, for example, quasi-invariance under this measurable transformation.

∞-dim Heisenberg gps with $\dim(C) = \infty$

An example: Product group

Fix onb $\{e_i\}_{i=1}^{\infty}$ and $\{f_\ell\}_{\ell=1}^{\infty}$ of H_μ and C_ν, respectively, and define

$$\omega(e_i, e_j) := \begin{cases} f_\ell & \text{if } i = 2\ell - 1, j = 2\ell \\ -f_\ell & \text{if } i = 2\ell, j = 2\ell - 1 \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, for any $\{\alpha_i\}, \{\beta_i\} \in \ell^2$,

$$\omega\left(\sum_i \alpha_i e_i\,,\, \sum_j \beta_j e_j\right) = \sum_\ell \Big(\alpha_{2\ell-1}\beta_{2\ell} - \alpha_{2\ell}\beta_{2\ell-1}\Big) f_\ell.$$

For any $\ell \in \mathbb{N}$, $\mathrm{span}\{e_{2\ell-1}, e_{2\ell}, f_\ell)\}$ is a subgroup of G_{CM} which is isomorphic to $\mathbb{H}$, so essentially $H_\mu \times C_\nu \cong \mathbb{H}^\infty$.

∞-dim Heisenberg gps with $\dim(C) = \infty$

Horizontal distance again

Remember that arguments in the $\dim(C) < \infty$ case relied critically on the estimate

$$d(e, (h, c)) \leq \|h\|_{H_\mu} + K\sqrt{\|c\|_C}.$$

It turns out these estimates won't necessarily hold in this setting. . . .

Horizontal distance

Product group example

Recall the "$\mathbb{H}^\infty$" example: given onb $\{e_i\}$ and $\{f_\ell\}$ for H_μ and C_ν respectively, define $\omega : H_\mu \times H_\mu \to C_\nu$ as

$$\omega\left(\sum_i \alpha_i e_i \, , \, \sum_j \beta_j e_j\right) = \sum_{\ell=1}^{\infty}\Big(\alpha_{2\ell-1}\beta_{2\ell} - \alpha_{2\ell}\beta_{2\ell-1}\Big) f_\ell .$$

Let $z = \sum_\ell z_\ell f_\ell \in C_\nu$ for some $\{z_\ell\} \in \ell^2$. Then

$$d(e,(0,z)) = \sqrt{\sum_\ell d_{\mathbb{H}}(e,(0,z_\ell))^2} = \sqrt{\frac{\pi}{2}\sum_{\ell=1}^{\infty}|z_\ell|},$$

and so

$$\{z : d(e,(0,z)) < \infty\} \cong \ell^1 \subsetneq \ell^2 \cong C_\nu .$$

Horizontal distance

So the product group example suggests that we need to define

$$\mathrm{dom}(d) := \{(h, z) \in G_{CM} : d(e, (h, z)) < \infty\}.$$

We can't hope for $d(e, (h, c)) \leq K(\|h\|_{H_\mu} + \sqrt{\|c\|_{C_\nu}})$. But:

- the inclusion $(\mathrm{dom}(d), d) \to (G_{CM}, \|\cdot\|_{H_\mu} + \sqrt{\|\cdot\|_{C_\nu}})$ is continuous and

$$\|h\|_{H_\mu} + \sqrt{\|c\|_{C_\nu}} \leq K(\|\omega\|_\mu) d(e, (h, c)).$$

- $\mathrm{dom}(d)$ is a topological group wrt the topology induced by d. (direct proof since you can't use equivalence to the topology coming from the Euclidean norms.)
- for all $g \in \mathrm{dom}(d)$, $d_n(e, g) \to d(e, g)$ (relies on new soft analysis arguments)
- same proof also gives existence of length minimizers

The stochastic Lévy area

Note that (A1) $\implies$ $i\omega : H_\mu \times H_\mu \to C$ is HS $\implies$

$$Z_t := \frac{1}{2} \int_0^t \omega(B_s, dB_s)$$

exists in C as before. So we have

$$g_t := (B_t, Z_t) = \left(B_t, \frac{1}{2} \int_0^t \omega(B_s, dB_s) \right)$$

is the solution to our SDE as before.

Let $\nu_t := \mathrm{Law}(B_t, Z_t)$, and note that ν_t lives on $G = H \times C$.

Results for hkm

QI

Proof of the same generalized CD inequalities with $\rho_2 \to \lfloor \omega \rfloor_\mu^2$ and $\|\omega\|_{HS} \to \|\omega\|_{H_\mu \otimes C_\nu} \cdots$

Theorem (M-Phillips, '24)

Assume (A1), (A2), and (A3). Then ν_t is quasi-invariant under left and right "translations" by elts of $\mathrm{dom}(d) \subsetneq G_{CM}$. Moreover,

$$\left\| \frac{d(\nu_t \circ R_g^{-1})}{d\nu_t} \right\|_{L^q(G,\nu_t)} \leq \exp\left(C\left(q, t, \frac{\|\omega\|_{H_\mu \otimes C_\nu}^2}{\lfloor \omega \rfloor_\mu^2}\right) d^2(e,g) \right)$$

and similarly for the RN derivative under left translation.

The stochastic Lévy area

We can actually use the same arguments as in (Driver-Eldredge-M) to say more about the distribution of Z_t (and more generally g_t). For $c \in C$, define the HS operator $\Omega_c : H_\mu \to H_\mu$ by

$$\langle \Omega_c h, k \rangle_{H_\mu} := \langle \omega(h,k), c \rangle_C,$$

(so $\Omega_c h = \omega(h,\cdot)^* c$) and let $\rho_t(B)$ be the random linear transformation on C defined by

$$\langle \rho_t(B) c, c' \rangle_C := \frac{1}{4} \int_0^t \langle \Omega_c B_t, \Omega_{c'} B_t \rangle_{H_\mu} \, dt.$$

More explicitly, this is

$$\begin{aligned} \langle \rho_T(B) c, c' \rangle_C &= \frac{1}{4} \int_0^T \langle \omega(B_s,\cdot)^* c, \omega(B_s,\cdot)^* c' \rangle_{H_\mu} \, ds \\ &= \left\langle \left(\frac{1}{4} \int_0^T \omega(B_s,\cdot)\omega(B_s,\cdot)^* \, ds \right) c, c' \right\rangle_C . \end{aligned}$$

The stochastic Lévy area

Theorem (M-Phillips, '24+)

The random linear operator on C

$$\rho_t(B) = \frac{1}{4}\int_0^t \omega(B_s,\cdot)\omega(B_s,\cdot)^*\, ds$$

is a.s. trace-class, and

$$\mathbb{E}\left[e^{i\langle c, Z_t\rangle_C}\right] = \mathbb{E}\left[e^{-\frac{1}{2}\langle \rho_t(B)c, c\rangle_C}\right].$$

That is,

$$(Z_t \mid \rho_t(B)) \sim \mathcal{N}_C(0, \rho_t(B)).$$

Other work/questions

- (M-Phillips, '24+) log Sobolev inequality for cylinder functions
- (Phillips, '24) Taylor isomorphism theorems for these and higher step inf-dim nilpotent Lie groups (also requires $d_n \to d$)
- results hold more generally for (H, H_μ, μ) replaced with a general abstract Wiener space
- Hilbert structure on C only really used here to discuss the distribution of Z_t, can be bypassed
- Q: understanding non-degeneracy of $\rho_T(B)$, hopefully quantitatively using $\lfloor \omega \rfloor_\mu$, to extend results of Driver-Eldredge-M
- Q: is $\mathrm{dom}(d)$ the real CM space? (e.g., converse of qi)

$$\mathrm{dom}(d) \subsetneq G_{CM} \subsetneq G$$