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Infinite-dimensional diffusions under Hörmander’s condition

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Hömander’s condition

Theorem (Hörmander, 1967)

Suppose that \( \{X_1, \ldots, X_k\} \) is a collection of vector fields \( X_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \) so that, for all \( x \in \mathbb{R}^n \),

\[
\text{span}\{X_i, [X_i, X_{i_2}], \ldots, [X_i, [\cdots, [X_{i_{r-1}}, X_{i_r}]]]\}(x) = \mathbb{R}^n. \quad (HC)
\]

Then for \( \xi_t \) the solution to the SDE

\[
d\xi_t^x = X_1(\xi_t) \circ dB_t^1 + \cdots + X_k(\xi_t) \circ dB_t^k, \quad \xi_0 = x,
\]

Law(\( \xi_t^x \)) is smooth for each \( t > 0 \), that is, Law(\( \xi_t^x \)) is abs cts wrt Lebesgue measure with strictly positive and smooth density:

\[
\text{Law}(\xi_t^x) = p_t(x, \cdot) \, dm, \quad \text{for some } p_t \in C^\infty(\mathbb{R}^n, (0, \infty)).
\]
Heisenberg group examples
The elliptic case

Now let

\[
\begin{align*}
\tilde{X}_1(x) &= (1, 0, -\frac{1}{2}x_2) \\
\tilde{X}_2(x) &= (0, 1, \frac{1}{2}x_1) \\
\tilde{X}_3(x) &= (0, 0, 1)
\end{align*}
\]

\[
\left\{ \text{NOTE: } \forall x \in \mathbb{R}^3, \quad \text{span}\{\tilde{X}_1(x), \tilde{X}_2(x), \tilde{X}_3(x)\} = \mathbb{R}^3 \right\}
\]

Consider the solution \( \xi_t = (\xi^1_t, \xi^2_t, \xi^3_t) \) to SDE

\[
d\xi_t = \tilde{X}_1(\xi_t) \circ dB^1_t + \tilde{X}_2(\xi_t) \circ dB^2_t + \tilde{X}_3(\xi_t) \circ dB^3_t
\]

\[
= \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \xi^2_t \end{pmatrix} \circ dB^1_t + \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \xi^1_t \end{pmatrix} \circ dB^2_t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \circ dB^3_t
\]

with \( \xi_0 = 0 \).
Heisenberg group examples

The elliptic case

Now let

\[ \tilde{X}_1(x) = (1, 0, -\frac{1}{2}x_2) \]
\[ \tilde{X}_2(x) = (0, 1, \frac{1}{2}x_1) \]
\[ \tilde{X}_3(x) = (0, 0, 1) \]

The solution to the SDE

\[ d\xi_t = \tilde{X}_1(\xi_t) \circ dB^1_t + \tilde{X}_2(\xi_t) \circ dB^2_t + \tilde{X}_3(\xi_t) \circ dB^3_t, \]

with \( \xi_0 = 0 \) may be written explicitly as

\[ \xi_t = \left( B^1_t, B^2_t, B^3_t + \frac{1}{2} \int_0^t B^1_s dB^2_s - B^2_s dB^1_s \right). \]
Heisenberg group examples

Stochastic Lévy area

\[ Z_t = \int_0^t B_x(s)dB_y(s) - B_y(s)dB_x(s) \]

(image from Manigo-Bornales)
Heisenberg group examples

Stochastic Lévy area

\[ \{Z_t\}_{t \geq 0} = \left\{ \int_0^t B_1(s)dB_2(s) - B_2(s)dB_1(s) \right\}_{t \geq 0} \]

\[ \overset{d}{=} \left\{ B \left( \frac{1}{4} \int_0^t (B_1^1 s)^2 + (B_2^2 s)^2 ds \right) \right\}_{t \geq 0} \]

This implies, for example, that for

\[ \rho_t(B^1, B^2) := \frac{1}{4} \int_0^t (B_1^1 s)^2 + (B_2^2 s)^2 ds, \]

we may write

\[ (Z_t \mid \rho_t(B^1, B^2)) \sim \mathcal{N}(0, \rho_t(B^1, B^2)). \]
Heisenberg group examples

The hypoelliptic case

Again let

\[ \tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right) = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2 \frac{\partial}{\partial x_3} \]

\[ \tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right) = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1 \frac{\partial}{\partial x_3} \]

\[ \tilde{X}_3(x) = (0, 0, 1) = \frac{\partial}{\partial x_3} \]

Note that \([\tilde{X}_1, \tilde{X}_2] := \tilde{X}_1 \tilde{X}_2 - \tilde{X}_2 \tilde{X}_1 = \tilde{X}_3\). Thus, we can write

\[ \text{span}\{\tilde{X}_1(x), \tilde{X}_2(x), [\tilde{X}_1, \tilde{X}_2](x)\} = \mathbb{R}^3. \]

So \{\tilde{X}_1, \tilde{X}_2\} satisfies \textit{Hörmander’s Condition}. 
Heisenberg group examples

The hypoelliptic case

Again let

\[
\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right) = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2 \frac{\partial}{\partial x_3}
\]

\[
\tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right) = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1 \frac{\partial}{\partial x_3}
\]

\[
\tilde{X}_3(x) = (0, 0, 1) = \frac{\partial}{\partial x_3}
\]

Now consider the SDE

\[
dg_t = \tilde{X}_1(g_t) \circ dB^1_t + \tilde{X}_2(g_t) \circ dB^2_t,
\]

which again we may solve explicitly, now as

\[
g_t = \left( B^1_t, B^2_t, \frac{1}{2} \int_0^t B^1_s dB^2_s - B^2_s dB^1_s \right).
\]
Heisenberg group examples

\[ g_t = \left( B_t^1, B_t^2, \frac{1}{2} \int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1 \right) =: (B_t, Z_t) \]
(HC) \implies \text{Law}(\xi_t) \text{ and Law}(g_t) \text{ are smooth measures on } \mathbb{R}^3.

In particular for \( \nu_t := \text{Law}(g_t) = \text{Law}(B_t, Z_t) \),

\[
\frac{d\nu_t}{dm}(x, y, z) = \frac{1}{16\pi^2} \int_{\mathbb{R}} e^{i\frac{\lambda z}{2}} \frac{\lambda}{\sinh(\lambda t)} e^{-(x^2+y^2)\lambda \coth(\lambda t)/4} d\lambda.
\]

One may prove this by showing that

\[
\mathbb{E} \left[ f(B_t) e^{i\lambda Z_t} \right] = \mathbb{E} \left[ f(B_t) e^{-\rho_t(B)\lambda^2} \right]
\]

where \( \rho_t(B) = \frac{1}{4} \int_0^t |B_s|^2 \, ds \). (see Gaveau, Lévy, Yor, Helmes-Schwane, . . . )
Heisenberg group geometry

Let $\mathfrak{h} = \text{span}\{X_1, X_2, X_3\} \cong \mathbb{R}^3$ with Lie bracket $[X_1, X_2] = X_3$, and all other brackets are 0. In coordinates, this is

$$ [(x_1, x_2, x_3), (x_1', x_2', x_3')] = [x_1 X_1 + x_2 X_2 + x_3 X_3 , x_1' X_1 + x_2' X_2 + x_3' X_3]$$

$$ = x_1 x_1'[X_1, X_1] + x_1 x_2'[X_1, X_2] + x_1 x_3'[X_1, X_3] + \cdots$$

$$ = (0, 0, x_1 x_2' - x_1' x_2).$$

Via the BCHD formula we may equip $\mathbb{R}^3$ with the group operation

$$ x \cdot x' = x + x' + \frac{1}{2}[x, x']$$

$$ = \left( x_1 + x_1', x_2 + x_2', x_3 + x_3' + \frac{1}{2}(x_1 x_2' - x_2 x_1') \right).$$

Then $\mathbb{R}^3$ with this group operation is the Heisenberg group $\mathbb{H}$, with $\text{Lie}(\mathbb{H}) = \mathfrak{h}$; the $\tilde{X}_i$'s are the unique left inv vfs so that $\tilde{X}_i(0) = X_i$. 
We can define a left invariant Riemannian metric on $\mathbb{H}$ by taking $\{\tilde{X}_i(x)\}_{i=1}^3$ to be an onb at each $x \in \mathbb{H}$, and Riemannian distance

$$\delta(x, y) := \inf \{ \ell(\gamma) : \gamma \text{ path from } x \text{ to } y \}$$

where

$$\ell(\gamma) := \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \, dt.$$ 

Alternatively, we can define the horizontal distance

$$d(x, y) := \inf \{ \ell(\gamma) : \gamma \text{ a horizontal path from } x \text{ to } y \},$$

where $\gamma$ is horizontal if

$$\gamma'(t) \in \text{span}\{\tilde{X}_1(\gamma(t)), \tilde{X}_2(\gamma(t))\}.$$
Heisenberg group geometry

We can write

$$\mathfrak{h} := \text{span}\{X_1, X_2\} \times \text{span}\{X_3\} =: H \times C$$

and define

$$\omega := [\cdot, \cdot]|_{H \times H} : H \times H \to C$$

so that

$$[(h, c), (h', c')] = (0, \omega(h, h')) = (0, h_1h'_2 - h'_1h_2).$$

and

$$(h, c) \cdot (h', c') = \left( h + h', c + c' + \frac{1}{2}\omega(h, h') \right).$$

Then

$$(HC) \iff \omega(H, H) = C,$$

and we may write

$$g_t := (B_t, Z_t) := \left( B_t, \frac{1}{2} \int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1 \right)$$

$$= \left( B_t, \frac{1}{2} \int_0^t \omega(B_s, dB_s) \right).$$
Heisenberg group geometry

For the horizontal distance on the Heisenberg group,

\[ K_1 \left( \|h\|_H + \sqrt{|c|} \right) \leq d(e, (h, c)) \leq K_2 \left( \|h\|_H + \sqrt{|c|} \right). \]

In particular, \( d(0, (h, c)) < \infty \) for all \((h, c) \in \mathbb{H}\). Many explicit expressions are known, for example,

\[ d(0, (0, c)) = \sqrt{\frac{\pi}{2}} |c| \]

and geodesics (length minimizers) are well understood.
Hörmander’s condition and horizontal distance

In the same way, given a collection of vfs \( \{X_1, \ldots, X_k\} \) satisfying (HC) on \( \mathbb{R}^n \), can define the horizontal distance analogously.

\[
\text{Chow-Rashevskii: (HC) } \implies d(x, y) < \infty \text{ for all } x, y \in \mathbb{R}^n.
\]

Moreover, the horizontal topology will be equivalent to the Euclidean one.
Smooth measures in $\infty$ dim

A measure $\mu$ on $\mathbb{R}^n$ is said to be smooth if

**Definition 1** $\mu$ is abs cts wrt Lebesgue measure and the RN derivative is strictly positive and smooth – that is,

$$\mu = \rho \, dm, \text{ for some } \rho \in C^\infty(\mathbb{R}^n, (0, \infty)).$$

**Definition 2** for any multi-index $\alpha$, there exists a function $g_\alpha \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mu)$ such that

$$\int_{\mathbb{R}^n} (-D)^\alpha f \, d\mu = \int_{\mathbb{R}^n} fg_\alpha \, d\mu, \text{ for all } f \in C^\infty_c(\mathbb{R}^n).$$

$\Rightarrow$ **Definition 1** $\iff$ **Definition 2**
So as a first step, we’d need to know how a measure behaves under (infinitesimal) translations.

**Definition** A measure $\mu$ on $\Omega$ is *quasi-invariant* under a transformation $T : \Omega \to \Omega$ if $\mu$ and $\mu \circ T^{-1}$ are mutually absolutely continuous.

In particular, we’re generally interested in quasi-invariance under transformations of the type

$$T = T_h = \text{translation by an element } h \in \Omega_0 \subset \Omega,$$

where $\Omega_0$ is some distinguished subset of $\Omega$. 
A typical $\infty$-dim setting

Take $H$ a Hilbert space with Gaussian meas $\mu$ with covariance $Q$:

$$\hat{\mu}(k) := \int_H e^{i\langle k, h \rangle_H} d\mu(h) = e^{-\langle Qk, k \rangle_H / 2}.$$  

$Q$ is necessarily a non-neg, sym, trace class operator.

**Theorem (Cameron-Martin-Maruyama)**

$\mu$ is quasi-invariant under translation by elements of $H_\mu = Q^{1/2}H$.

That is, for $k \in H_\mu$ and $d\mu^k := d\mu(\cdot - k)$,

$$\mu^k \ll \mu \quad \text{and} \quad \mu^k \gg \mu.$$  

Moreover, if $k \notin H$, then $\mu^k \perp \mu$.

$H_\mu$ is called the Cameron-Martin space for $(H, \mu)$. 

A typical $\infty$-dim setting

Facts about the CM space $H_\mu$

- $H_\mu$ is a Hilbert space equipped with inner product
  \[
  \langle h, k \rangle_\mu := \langle Q^{-1/2} h, Q^{-1/2} k \rangle_H,
  \]
  densely embedded in $H$, and, when $\dim(H) = \infty$, $\mu(H_\mu) = 0$.

- The inclusion map $\iota : H_\mu \hookrightarrow H$ is Hilbert-Schmidt
  \[
  \|\iota\|_{HS}^2 := \sum_i \|h_i\|_H^2 < \infty.
  \]

- In fact, given a Hilbert space $H$ with a Hilbert subspace $K$ such that the inclusion $K \hookrightarrow H$ is HS, immediately implies the existence of a Gaussian measure on $H$ with covariance determined by $K$. 
∞-dim Heisenberg gps with $\text{dim}(C) < \infty$

(Driver-Gordina, ’08) Take

- $H$ a Hilbert space with Gaussian measure $\mu$
- $C$ a fin-dim inner product space
- $\omega : H \times H \to C$ is a continuous anti-symmetric bilinear form

Then we can make $\mathfrak{g} := H \times C$ into a Lie algebra with bracket

$$[(h, c), (h', c')] := (0, \omega(h, h'))$$

and Lie group $G := H \times C$ with group operation

$$(h, c) \cdot (h', c') = \left(h + h', c + c' + \frac{1}{2} \omega(h, h')\right).$$
Let $H_\mu$ denote the Cameron-Martin space for $(H, \mu)$.

$\triangleright$ $G_{CM} := H_\mu \times C$ inherits a group structure from $\omega|_{H_\mu \times H_\mu}$, and is a dense subgp of $G = H \times C$

$\triangleright$ $\omega : H \times H \to C$ cts $\implies \omega : H_\mu \times H_\mu \to C$ Hilbert-Schmidt

$$\|\omega\|_{HS(H_\mu \times H_\mu, C)}^2 := \sum_{i,j,\ell} \langle \omega(e_i, e_j), f_\ell \rangle_C^2 < \infty.$$ 

$\circ \omega$ HS on $H_\mu$ is in some sense the necessary assumption, rather than $\omega$ cts on $H$

$\triangleright$ For example, $\omega$ HS $\implies$ stochastic Lévy area for $\{B_t\}_{t \geq 0}$

BM on $H$

$$Z_t := \frac{1}{2} \int_0^t \omega(B_s, dB_s)$$

is well-defined in $C$. 

∞-dim Heisenberg gps with $\text{dim}(C) < \infty$
∞-dim Heisenberg gps with \( \dim(C) < \infty \)

\((B_t, B_t^C + Z_t)\) and \((B_t, Z_t)\) are still solutions to the “elliptic” and “hypoelliptic” SDEs, respectively.

Driver-Gordina proved regularity properties (e.g., qi and 1st order ibp) for \( \text{Law}((B_t, B_t^C + Z_t)) \) on \( G \). Dobbs-M proved smoothness via arbitrary ibp formulae.

What about \( \nu_t := \text{Law}((B_t, Z_t)) \)?
∞-dim Heisenberg gps with $\dim(C) < \infty$

Theorem (Baudoin-Gordina-M, ’13)
Assume $\omega(H_\mu, H_\mu) = C$. Then $\nu_t$ is qi under left and right translations by elts of $G_{CM} = H_\mu \times C$. Moreover,

$$\left\| \frac{d(\nu_t \circ R_{(h,c)}^{-1})}{d\nu_t} \right\|_{L^q(G,\nu_t)} \leq \exp \left( C \left( q, t, \frac{\|\omega\|_{HS}^2}{\rho_2} \right) d^2(0,(h,c)) \right)$$

where $d$ is the horizontal distance on $G_{CM}$ and

$$\rho_2 := \inf \left\{ \sum_{i,j=1}^{\infty} \left( \sum_{\ell=1}^{N} \langle \omega(e_i,e_j), f_\ell \rangle c_{\ell} \right)^2 : \sum_{\ell=1}^{N} x_{\ell}^2 = 1 \right\},$$

and similarly for the RN derivative under left translation.
\( \infty \)-dim Heisenberg gps with \( \dim(C) < \infty \)

The proof relied on several elts, including

1. final inequalities (particularly generalized CD inequalities à la Baudoin, Bonnefont, Garofalo) involving coefficients with

\[
\|\omega\|_{HS(H_{\mu} \times H_{\mu}, C)}^2 := \sum_{i,j=1}^{\infty} \sum_{\ell=1}^{N} \langle \omega(e_i, e_j), f_\ell \rangle_C^2,
\]

where continuity of \( \omega : H \times H \to C \implies \|\omega\|_{HS}^2 < \infty \), and

\[
\rho_2 := \inf \left\{ \sum_{i,j=1}^{\infty} \left( \sum_{\ell=1}^{N} \langle \omega(e_i, e_j), f_\ell \rangle_c x_\ell \right)^2 : \sum_{\ell=1}^{N} x_\ell^2 = 1 \right\}
\]

\[
= \inf_{\|c\|_C = 1} \sum_{i,j=1}^{\infty} \langle \omega(e_i, e_j), c \rangle_C^2.
\]

Note that \( (HC) \implies \rho_2 > 0 \).
\(\infty\)-dim Heisenberg gps with \(\dim(C) < \infty\)

The proof relied on several elts, including

2. convergence of the horizontal distance by fin-dim approx gps

\[ d_n(e, g) \to d(e, g) \]

as \(n \to \infty\). We made critical use of the estimate for \(g = (h, c)\)

\[ \|h\|_{H_\mu} + K_1(\omega) \sqrt{\|c\|_C} \leq d(e, (h, c)) \leq \|h\|_{H_\mu} + K_2(\omega, N) \sqrt{\|c\|_C} \]

where \(N := \dim(C)\).
Theorem (Driver-Eldredge-M, ’16)

For \( c \in C \), define the HS operator \( \Omega_c : H_\mu \to H_\mu \) by

\[
\langle \Omega_c h, k \rangle_{H_\mu} := \langle \omega(h, k), c \rangle_C,
\]

and let \( \rho_t(B) \) be the random linear transformation on \( C \) defined by

\[
\langle \rho_t(B)c, c' \rangle_C := \frac{1}{4} \int_0^t \langle \Omega_c B_t, \Omega_{c'} B_t \rangle_{H_\mu} \, dt.
\]

Then \( \nu_t(dh, dc) = \gamma_t(h, c) \mu_t(dh) \, m(dc) \) where \( m \) is Lebesgue measure on \( C \) and

\[
\gamma_t(h, c) := \mathbb{E} \left[ \frac{\exp \left( -\frac{1}{2} \langle \rho_t^{-1}(B)c, c \rangle_C \right)}{\sqrt{(2\pi)^N \det \rho_t(B)}} \bigg| B_t = h \right]
\]

is a smooth density.
Other $\infty$-dim hypoelliptic results

- Hörmander generators on $\infty$-dim configuration space:
  Lugiewicz-Zegarlinski (’07), Inglis-Papageorgiou (’09),
  Kontis-Ottobre-Zegarlinski (’16)

- evolution equations: Baudoin-Teichmann (’05),
  Forster-Lütkebohmert-Teichmann (’08)

- SPDEs: Hairer-Mattingly (’04), Mattingly-Pardoux (’04),
  Bakhtin-Mattingly (’07), Agrachev-Kuksin-Sarychev-Shirikyan
  (’07), Glatt-Holtz-Herzog-Mattingly (’18), . . .
\(\infty\)-dim Heisenberg gps with \(\dim(C) < \infty\)

**pros and cons:** The explicit heat kernel for \(\nu_t\) found in Driver-Eldredge-M was of course a stronger result, but the techniques were fairly specific to the step 2 structure. The final inequality approach à la Baudoin-Gordina-M is perhaps more robust.

Unresolved questions from (Baudoin-Gordina-M):
- If we remove \(\dim(C) < \infty\), can we still prove \(d_n \to d\)?
- What does \(\rho_2\) mean?

Unresolved question from (Driver-Eldredge-M):
- If we remove \(\dim(C) < \infty\), does anything still make sense? wrt what reference measure?
Suppose $\omega : H_\mu \times H_\mu \to C$ is HS

$$\|\omega\|_{HS}^2 := \sum_{i,j,\ell} \langle \omega(e_i, e_j), f_\ell \rangle_C^2 < \infty.$$ 

For each $\ell$, define $x_\ell := \sum_{i,j} \langle \omega(e_i, e_j), f_\ell \rangle_C^2$.

$\omega$ HS $\implies x_\ell$ is summable, and so $x_\ell \to 0$. This implies that

$$\rho_2 = \inf_{\|c\|_C = 1} \sum_{i,j} \langle \omega(e_i, e_j), c \rangle_C^2 \leq \inf_{\ell} x_\ell = 0.$$ 

Similarly, $\rho_2 > 0$ prohibits $\|\omega\|_{HS} < \infty$. So when $\dim(C) = \infty$, you can’t have $\rho_2 > 0$ and $\|\omega\|_{HS} < \infty$.

(*Notice that this is only an issue when $\dim(C) = \infty$.)
Moving to \( \dim(C) = \infty \)

So, needing (something like) \( \rho_2 > 0 \) and \( \|\omega\|_{HS} < \infty \) necessitates the existence of another Hilbert space \( Z \) densely embedded in \( C \), so that \( \omega : H_\mu \times H_\mu \to Z \) with

\[
\inf_{\|z\|_Z=1} \sum_{i,j} \langle \omega(e_i, e_j), z \rangle_Z^2 > 0
\]

(and so \( \omega \) is not HS into \( Z \)) with a HS inclusion map \( \iota : Z \to C \) so that \( \|\iota\omega\|_{HS(H_\mu \times H_\mu, C)} < \infty \).

Such a \((C, Z)\) necessarily supports a Gaussian measure on \( C \) with Cameron-Martin space \( Z \).
∞-dim Heisenberg gps with $\dim(C) = \infty$

The main assumptions

Let $(W, H_\mu, \mu)$ and $(C, C_\nu, \nu)$ be (Hilbert) Gaussian measure spaces. Let $\omega : H_\mu \times H_\mu \to C_\nu$ be a skew-symmetric bilinear map.

We assume that

\[ \|\omega\|_{\mu}^2 := \sup_{\|z\|_{C_\nu} = 1} \sum_{i,j} \langle \omega(e_i, e_j), z \rangle_{C_\nu}^2 < \infty, \]  

\[ (A1) \]

\[ \|\omega\|_{H_\mu \otimes C_\nu}^2 := \sup_{\|h\|_{H} = 1} \sum_{i,\ell} \langle \omega(h, e_i), f_\ell \rangle_{C_\nu}^2 < \infty, \]  

\[ (A2) \]

and

\[ [\omega]_{\mu}^2 := \inf_{\|z\|_{C_\nu} = 1} \sum_{i,j} \langle \omega(e_i, e_j), z \rangle_{C_\nu}^2 > 0. \]  

\[ (A3) \]
$\infty$-dim Heisenberg gps with $\text{dim}(C) = \infty$

The main assumptions: weakly HS v. HS

(A1) $\iff$ $\omega : H_\mu \times H_\mu \to C_\nu$ is “weakly Hilbert-Schmidt”:

$\omega$ extends to bdd linear operator $\tilde{\omega} : H_\mu \otimes H_\mu \to C_\nu$ so that

$$\tilde{\omega}(h \otimes k) = \omega(h, k)$$

and

$$\|\tilde{\omega}\|_{\mathcal{L}(H_\mu \otimes H_\mu, C_\nu)}^2 = \|\tilde{\omega}^*\|_{\mathcal{L}(C_\nu, H_\mu \otimes H_\mu)}^2 = \sup_{\|z\|_{C_\nu} = 1} \|\tilde{\omega}^* z\|_{H_\mu \otimes H_\mu}^2 = \sup_{\|z\|_{C_\nu} = 1} \sum_{i,j} \langle e_i \otimes e_j, \tilde{\omega}^* z \rangle_{H_\mu \otimes H_\mu}^2$$

$$= \sup_{\|z\|_{C_\nu} = 1} \sum_{i,j} \langle \omega(e_i, e_j), z \rangle_{C_\nu}^2 = \|\omega\|_{\mu}^2.$$ 

Note this $\implies$ $\tilde{\omega} : H_\mu \otimes H_\mu \to C_\nu \overset{i}{\longrightarrow} C$ is HS.

$\iff$ $\omega : H_\mu \times H_\mu \to C_\nu \overset{i}{\longrightarrow} C$ is HS.
The main assumptions: Lower bound

\[ (A3) \quad \longleftrightarrow \quad \tilde{\omega}^* : C_\nu \to H_\mu \otimes H_\mu \text{ is bounded below:} \]

\[ \|\tilde{\omega}^* z\|^2_{H_\mu \otimes H_\mu} = \sum_{i,j} \langle \tilde{\omega}^* z, e_i \otimes e_j \rangle^2_{H_\mu \otimes H_\mu} \]

\[ = \sum_{i,j} \langle z, \omega(e_i, e_j) \rangle^2_{C_\nu} \geq |\omega|_{\mu}^2 \|z\|^2_{C_\nu}. \]

Functional analysis lemma:
For \( H, K \) Hilbert spaces and \( A : H \to K \) a bdd linear operator,

\[ A^* \text{ is bounded below iff } A \text{ is surjective.} \]

Thus, \( \tilde{\omega} : H_\mu \otimes H_\mu \to C_\nu \) is surjective. In particular, this implies that \( \text{span}(\omega(H_\mu \times H_\mu)) \) is dense in \( C_\nu \). That is, \((A3) \leftrightarrow (HC)\).
We now define a Lie algebra structure on $\mathfrak{g}_{CM} := H_\mu \times C_\nu$ and a group structure on $G_{CM} := H_\mu \times C_\nu$ via $\omega : H_\mu \times H_\mu \rightarrow C_\nu$ as before.

Note that we don’t require that $\omega$ extend to a cts bilinear map $H \times H \rightarrow C$. However, (A2) is sufficient to say that

$$(h, z) \cdot (x, c) = (h + x, c + z + \frac{1}{2} \omega(h, x))$$

defines a measurable group action of $G_{CM}$ on “$G$” := $H \times C$, which is sufficient to discuss, for example, quasi-invariance under this measurable transformation.
∞-dim Heisenberg gps with \( \dim(C) = \infty \)

An example: Product group

Fix onb \( \{e_i\}_{i=1}^\infty \) and \( \{f_\ell\}_{\ell=1}^\infty \) of \( H_\mu \) and \( C_\nu \), respectively, and define

\[
\omega(e_i, e_j) := \begin{cases} 
  f_\ell & \text{if } i = 2\ell - 1, j = 2\ell \\
  -f_\ell & \text{if } i = 2\ell, j = 2\ell - 1 \\
  0 & \text{otherwise}
\end{cases}
\]

Equivalently, for any \( \{\alpha_i\}, \{\beta_i\} \in \ell^2 \),

\[
\omega \left( \sum_i \alpha_i e_i, \sum_j \beta_j e_j \right) = \sum_\ell \left( \alpha_{2\ell-1} \beta_{2\ell} - \alpha_{2\ell} \beta_{2\ell-1} \right) f_\ell.
\]

For any \( \ell \in \mathbb{N} \), \( \text{span}\{e_{2\ell-1}, e_{2\ell}, f_\ell\} \) is a subgroup of \( G_{CM} \) which is isomorphic to \( \mathbb{H} \), so essentially \( H_\mu \times C_\nu \cong \mathbb{H}^\infty \).
\(\infty\)-dim Heisenberg gps with \(\text{dim}(C) = \infty\)

Horizontal distance again

Remember that arguments in the \(\text{dim}(C) < \infty\) case relied critically on the estimate

\[
d(e,(h,c)) \leq \|h\|_{H_\mu} + K \sqrt{\|c\|_C}.
\]

It turns out these estimates won't necessarily hold in this setting. . . .
Recall the “$\mathcal{H}_\infty$” example: given on $\{e_i\}$ and $\{f_\ell\}$ for $H_\mu$ and $C_\nu$ respectively, define $\omega : H_\mu \times H_\mu \to C_\nu$ as

$$\omega \left( \sum_i \alpha_i e_i, \sum_j \beta_j e_j \right) = \sum_{\ell=1}^\infty (\alpha_{2\ell-1}\beta_{2\ell} - \alpha_{2\ell}\beta_{2\ell-1}) f_\ell.$$

Let $z = \sum_\ell z_\ell f_\ell \in C_\nu$ for some $\{z_\ell\} \in \ell^2$. Then

$$d(e, (0, z)) = \sqrt{\sum_\ell d_{\mathcal{H}}(e, (0, z_\ell))^2} = \sqrt{\frac{\pi}{2} \sum_{\ell=1}^\infty |z_\ell|},$$

and so

$$\{z : d(e, (0, z)) < \infty\} \cong \ell^1 \subsetneq \ell^2 \cong C_\nu.$$
So the product group example suggests that we need to define

\[ \text{dom}(d) := \{(h, z) \in G_{CM} : d(e, (h, z)) < \infty \}. \]

We can’t hope for \( d(e, (h, c)) \leq K(\|h\|_{H_\mu} + \sqrt{\|c\|_{C_\nu}}) \). But:

- the inclusion \((\text{dom}(d), d) \to (G_{CM}, \|\cdot\|_{H_\mu} + \sqrt{\|\cdot\|_{C_\nu}})\) is continuous and

\[ \|h\|_{H_\mu} + \sqrt{\|c\|_{C_\nu}} \leq K(\|\omega\|_{\mu})d(e, (h, c)). \]

- \( \text{dom}(d) \) is a **topological group** wrt the topology induced by \( d \). (direct proof since you can’t use equivalence to the topology coming from the Euclidean norms.)

- for all \( g \in \text{dom}(d) \), \( d_n(e, g) \to d(e, g) \) (relies on new soft analysis arguments)

- same proof also gives existence of length minimizers
The stochastic Lévy area

Note that (A1) \( \Rightarrow \) \( i\omega : H_\mu \times H_\mu \to C \) is HS \( \Rightarrow \)

\[
Z_t := \frac{1}{2} \int_0^t \omega(B_s, dB_s)
\]

exists in \( C \) as before. So we have

\[
g_t := (B_t, Z_t) = \left( B_t, \frac{1}{2} \int_0^t \omega(B_s, dB_s) \right)
\]

is the solution to our SDE as before.

Let \( \nu_t := \text{Law}(B_t, Z_t) \), and note that \( \nu_t \) lives on \( G = H \times C \).
Proof of the same generalized CD inequalities with $\rho_2 \to [\omega]_{\mu}^2$ and $\|\omega\|_{HS} \to \|\omega\|_{H_{\mu} \otimes C_{\nu}} \ldots$

Theorem (M-Phillips, '24)

Assume (A1), (A2), and (A3). Then $\nu_t$ is quasi-invariant under left and right “translations” by elts of $\text{dom}(d) \subseteq G_{CM}$. Moreover,

$$\left\| \frac{d(\nu_t \circ R_g^{-1})}{d\nu_t} \right\|_{L^q(G,\nu_t)} \leq \exp \left( C \left( q, t, \frac{\|\omega\|_{H_{\mu} \otimes C_{\nu}}^2}{[\omega]_{\mu}^2} \right) d^2(e, g) \right)$$

and similarly for the RN derivative under left translation.
The stochastic Lévy area

We can actually use the same arguments as in (Driver-Eldredge-M) to say more about the distribution of \( Z_t \) (and more generally \( g_t \)).

For \( c \in C \), define the HS operator \( \Omega_c : H_\mu \to H_\mu \) by

\[
\langle \Omega_c h, k \rangle_{H_\mu} := \langle \omega(h, k), c \rangle_C,
\]

(so \( \Omega_c h = \omega(h, \cdot)^* c \)) and let \( \rho_t(B) \) be the random linear transformation on \( C \) defined by

\[
\langle \rho_t(B)c, c' \rangle_C := \frac{1}{4} \int_0^t \langle \Omega_c B_t, \Omega_{c'} B_t \rangle_{H_\mu} \, dt.
\]

More explicitly, this is

\[
\langle \rho_T(B)c, c' \rangle_C = \frac{1}{4} \int_0^T \langle \omega(B_s, \cdot)^* c, \omega(B_s, \cdot)^* c' \rangle_{H_\mu} \, ds
\]

\[
= \left\langle \left( \frac{1}{4} \int_0^T \omega(B_s, \cdot) \omega(B_s, \cdot)^* \, ds \right) c, c' \right\rangle_C.
\]
The stochastic Lévy area

**Theorem (M-Phillips, ’24+)**

The random linear operator on $C$ is a.s. trace-class, and

$$
\rho_t(B) = \frac{1}{4} \int_0^t \omega(B_s, \cdot)\omega(B_s, \cdot)^* \, ds
$$

is a.s. trace-class, and

$$
\mathbb{E} \left[ e^{i \langle c, Z_t \rangle} \right] = \mathbb{E} \left[ e^{-\frac{1}{2} \langle \rho_t(B)c, c \rangle} \right].
$$

That is,

$$
(Z_t | \rho_t(B)) \sim \mathcal{N}_C(0, \rho_t(B)).
$$
Other work/questions

- (M-Phillips, '24+) log Sobolev inequality for cylinder functions
- (Phillips, '24) Taylor isomorphism theorems for these and higher step inf-dim nilpotent Lie groups (also requires $d_n \to d$)
- Results hold more generally for $(H, H_\mu, \mu)$ replaced with a general abstract Wiener space
- Hilbert structure on $C$ only really used here to discuss the distribution of $Z_t$, can be bypassed
- Q: understanding non-degeneracy of $\rho_T(B)$, hopefully quantitatively using $[\omega]_\mu$, to extend results of Driver-Eldredge-M
- Q: is $\text{dom}(d)$ the real CM space? (e.g., converse of qi)

$$\text{dom}(d) \subsetneq G_{CM} \subsetneq G$$