The key idea when working with absolute value functions is remembering that they are piecewise functions: they have different rules on different portions of their domains. That is,

$$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

You will need to solve the inequalities $f(x) \geq 0$ and $f(x) < 0$: this can be done by graphing (if you can do this by hand, say if $f(x)$ is linear or quadratic) or by applying the Intermediate Value Theorem and making a sign chart with test points: recall a function can only change signs at roots (zeroes) or at points of discontinuity, so first find all roots of $f(x)$ by solving $0 = f(x)$, then find any points of discontinuity (including points excluded from the domain). These roots and points of discontinuity divide the domain of $f(x)$ intervals on which the sign of $f(x)$ is constant, so determining the sign of $f(x)$ at any "test point" in each interval tells you the sign of $f(x)$ everywhere on that interval.

There are lots of free sources online covering prerequisite knowledge of absolute value and piecewise functions: for example, check out `https://www.khanacademy.org/math/algebra/x2f8bb11595b61c86:absolute-value-piecewise-functions`.

1. Let $f(x) = \dfrac{|x^2+5x+4|}{x^2+6x+8}$. Compute each of the following limits, if they exist. If not and you can still say something definitive about the behavior of the function, do so. Otherwise, write DNE.

   (a) $\lim_{x\to -1} f(x)$
   (b) $\lim_{x\to \infty} f(x)$
   (c) $\lim_{x\to -4^+} f(x)$
   (d) $\lim_{x\to -4} f(x)$
   (e) $\lim_{x\to -2^-} f(x)$

2. Let $q(x) = \dfrac{3x^2-9x-12}{|x^2+3x+2|}$. Compute each of the following limits, if they exist. If not and you can still say something definitive about the behavior of the function, do so. Otherwise, write DNE.

   (a) $\lim_{x\to -2^-} q(x)$
   (b) $\lim_{x\to -2} q(x)$
   (c) $\lim_{x\to -\infty} q(x)$
   (d) $\lim_{x\to 4^+} q(x)$
   (e) $\lim_{x\to -1^-} q(x)$

3. Let $g(x) = |24x - x^3 + 5x^2|$. Compute each of the following derivatives, if they exist. If not, write DNE and justify why not. (You may use derivative rules.)
   (Recall $g'_+(c)$ denotes the righthand derivative of $g(x)$ at $x = c$: that is, $g'_+(c) = \lim_{h\to 0^+} \dfrac{g(c+h)-g(c)}{h}$, while $g'_-(c)$ denotes the lefthand derivative at c.)

   (a) $g'_-(-4)$
   (b) $g'_+(-3)$
   (c) $g'(-1)$
   (d) $g'_+(0)$
   (e) $g'(0)$
   (f) $g'_-(9)$

Answers:

1. Key step: $f(x) = \begin{cases} \frac{(x+4)(x+1)}{(x+4)(x+2)}, & x \leq -4, x \geq -1 \\ -\frac{(x+4)(x+1)}{(x+4)(x+2)}, & -4 < x < -1 \end{cases}$
   (a) $\lim_{x \to -1} f(x) = 0$
   (b) $\lim_{x \to \infty} f(x) = 1$
   (c) $\lim_{x \to -4^+} f(x) = -\frac{3}{2}$
   (d) $\lim_{x \to -4} f(x)$ DNE
   (e) $\lim_{x \to -2^-} f(x) = -\infty$ (DNE)

2. Key step: $q(x) = \begin{cases} \frac{3(x-4)(x+1)}{(x+2)(x+1)} & x \leq -2, x \geq -1 \\ -\frac{3(x-4)(x+1)}{(x+2)(x+1)} & -2 < x < -1 \end{cases}$
   (a) $\lim_{x \to -2^-} q(x) = \infty$ (DNE)
   (b) $\lim_{x \to -2} q(x) = \infty (DNE)$
   (c) $\lim_{x \to -\infty} q(x) = 3$
   (d) $\lim_{x \to 4^+} q(x) = 0$
   (e) $\lim_{x \to -1^-} q(x) = 15$

3. Key step: $g(x) = -x(x-8)(x+3)$. Thus, $g(x) = \begin{cases} 24x - x^3 + 5x^2 & x \leq -3, 0 \leq x \leq 8 \\ -(24x - x^3 + 5x^2) & -3 < x < 0, x > 8 \end{cases}$
   (a) $g'_-(-4) = -64$
   (b) $g'_+(-3) = 33$
   (c) $g'(-1) = -11$
   (d) $g'_+(0) = 24$
   (e) $g'(0)$ DNE
   (f) $g'_-(9) = 129$