0. Reading: Haberman, Sections 3.1–3.5 and Lecture notes 3-4.1

1. Haberman page 110, Problem 3.3.4. Note that \( f(x) \) is defined on \( 0 \leq x \leq L \).
   (a) First, sketch the periodic extension, \( f(x+nL) = f(x) \) (any integer \( n \)), on the range \( -2L \leq x \leq 2L \).
   (b) Second, predict the form of the Fourier coefficients (What is \( \alpha \) in \( c_k = O(1/k^\alpha) \) for \( k \to \infty \)) based on the smoothness (“degree of continuity”) of the extended \( f(x) \). (DO NOT calculate the integrals for the \( c_k \) for this part.)
   (c) Finally, evaluate the integrals for the \( c_k \) coefficients. For this \( f(x) \) there is a singular case (a special value of \( k = k_s \)) for which the general trig formula does not work (i.e. the formula will have a “division by zero” issue). Identify this case and calculate the value of \( c_k \), separately.

2. Consider \( f(x) = x \sin(x) \) defined on \( 0 \leq x \leq \pi \).
   (a) Sketch the periodic extensions on \(-2\pi \leq x \leq 2\pi \) corresponding to (i) the Fourier cosine and (ii) the Fourier sine series of \( f(x) \). Predict the forms of the coefficients (What is \( \alpha \) in \( c_k = O(1/k^\alpha) \) for \( k \to \infty \)) based on the smoothness of \( f(x) \) (without calculating any integrals).
   (b) Calculate the coefficients for the Fourier cosine series. Identify and calculate the singular case \( k_s \) in the formula for the \( c_k \) coefficients. Compare the \( c_k \) with your prediction from (a).
   (c) Repeat (b) for the Fourier sine series of \( f \).

3. How to deal with discontinuous functions \([\text{like } u(c^-) \neq u(c^+)] \) with \( c^+ = c + \epsilon, c^- = c - \epsilon \) as \( \epsilon \to 0 \):
   (a) Haberman page 120, Problem 3.4.1(a).
      Hint: Break up the LHS integral into sub-intervals \( \int_a^b = \int_a^{c^-} + \int_{c^+}^b \) and use IBP on each separately.
   (b) Haberman page 121, Problem 3.4.3(b). Hint: Let \( f(x) = \sum c_k \sin(k\pi x/L) \) and assume \( f'(x) = \sum b_k \cos(k\pi x/L) \). Then use (a) to rewrite the integral formula for coeffs \( b_k \) in terms of \( c_k \).

4. Adjoint problems: operators and boundary conditions – For each of the following complete linear differential operators on \( 0 \leq x \leq 1 \), use standard \( L^2 \) inner product and the adjoint relation, \( \langle v, Lu \rangle = \langle L^*v, u \rangle \), to determine the differential operator and boundary conditions for the adjoint problem. For (a-e) state if the formal operator and/or the complete operator is self-adjoint.
   (a) \( Ly \equiv \frac{d^2y}{dx^2} + 5y(0) + y'(0) = 0 \quad y(1) + 2y'(1) = 0 \).
   (b) \( Ly \equiv \frac{d^2y}{dx^2} + y(0) + 4y'(1) = 0 \quad y(1) = 0 \).
   (c) \( Ly \equiv e^{-2x} \frac{d^2y}{dx^2} - \sin(x) \frac{dy}{dx} \quad y'(0) = 0 \quad y(1) = 0 \).
   (d) \( Ly \equiv \frac{d^3y}{dx^3} + 2 \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} \quad y(0) - 4y''(0) = 0 \quad y(1) = 0 \quad y'(1) + 5y''(1) = 0 \).
   (e) \( Ly \equiv \frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y \quad \alpha_1 y(0) + \alpha_2 y'(0) = 0 \quad \beta_1 y(1) + \beta_2 y'(1) = 0 \quad (\alpha_1^2 + \alpha_2^2 \neq 0, \beta \text{’s same}) \)
   (f) \( Ly \equiv A(x) \frac{d^2y}{dx^2} + B(x) \frac{dy}{dx} + C(x)y \quad \text{What relation between } A(x) \text{ and } B(x) \text{ makes } L \text{ formally self-adjoint?} \)

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1Policy on use of computers: You are encouraged to use computer programs like Maple, Mathematica, MATLAB, ... to help check your calculations. However, since computers/calculators cannot be used on the exams, you need to practice your calculation skill; the homework problems are the best preparation for the exams!