Green’s Functions for ODE BVP’s

1. [Math 551 Test 1] Friday, October 13, 2023, in class.

The test is closed-book – no textbooks, calculators or computer-aided algebra. You will be given a copy of the Basic Math Summary sheet and you can bring a one-page (2 sides) letter-sized sheet of notes that you have written yourself.

Tests from previous years have been posted on Sakai to help guide your studying (see the Tests/Test1 folder and note that Test 1’s from past year’s did not include Green’s functions, but greens.pdf gives a compilation of these problems). (also see current HW solutions)

No Homework will be due the week of the test to allow you to focus on studying for the test.

Study well (start early) and do your best!

0. Read Haberman, section 9.3 (pages 379–398) and Lecture notes 9, 10 (+10-recap in 11).

- My secondary variable \( s \) is the same as Haberman’s \( x_0 \).
- Hint: Phase shifting of LCC solns: For finding the \( G_{\pm} \) for an LCC equation, solutions like \( G_{-}(x) = Ax^2 + Be^x + C \sin(x) \) may be very convenient for imposing BC’s at \( x = 0 \) but could be a bit messy for BC’s at \( x = 2 \) (for example). For solns of LCC eqns, you can shift \( x \to x - x_0 \) to generate an equivalent set of solns that would give cleaner algebra for BC’s at \( x = x_0 \). For example, \( G_{+}(x) = D(x-2)^2 + Ee^{x-2} + F \sin(x-2) \). This works ONLY for LCC equations, not CE or other ODE’s.
- Hint: Impose all the BC’s given and simplify the \( G_{\pm} \) first to minimize the remaining algebra for the last \( n \) constants from the Jump conditions.

1. For the boundary value problem

\[
\frac{d^3u}{dx^3} = f(x), \quad u'(0) = 0, \quad u(1) = 0, \quad u''(1) - 2u'(1) = 0
\]

(a) Determine the piecewise-defined Green’s function.

(b) Evaluate the integral \( \int_0^1 G(x, s)f(s)\,ds = u(x) \) for the solution of

\[
\frac{d^3u}{dx^3} = 60x^2, \quad u'(0) = 0, \quad u(1) = 0, \quad u''(1) - 2u'(1) = 0
\]

to confirm that you can obtain the exact solution, \( u(x) = x^5 + 5x^2 - 6 \).

Hint: If you are not getting this solution, you might check if you’ve accidentally swapped \( G(x, s) \) with \( G(s, x) \) (this problem is not self-adjoint) or swapped the pieces \( G_{\pm} \).

(c) Use integration by parts on \( \langle Lu, G \rangle_2 = 0 \) on the problem

\[
\frac{d^3u}{dx^3} = 0, \quad u'(0) = 5, \quad u(1) = 7, \quad u''(1) - 2u'(1) = 6
\]

to get the solution of in terms of properties of \( G_{\pm} \). Show all work.

(continued)
2. (from Fall 2013) Consider the inhomogeneous problem for \( u(x) \) on \( 0 \leq x \leq 1 \)

\[
\frac{d^2 u}{dx^2} - 3 \frac{du}{dx} = f(x), \quad u(0) = a, \quad \frac{du}{dx} \bigg|_{x=0} = b.
\]

(a) Determine the piecewise-defined Green’s function.
(b) Write the solution \( u(x) \) in terms of the Green’s function.

3. (Optional, extra credit) Consider the inhomogeneous boundary value problem on \( 0 \leq x \leq \pi/2 \):

\[
\frac{d^2 u}{dx^2} + u = f(x) \quad u(0) = 0 \quad u\left(\frac{\pi}{2}\right) = 0.
\]  \quad (1)

(a) Determine the eigenfunctions and eigenvalues of

\[
\frac{d^2 \phi}{dx^2} + \phi = -\lambda \phi, \quad \phi(0) = 0, \quad \phi\left(\frac{\pi}{2}\right) = 0.
\]  \quad (2)

(b) Write the solution of problem (1) in terms of an expansion using the eigenfunctions,

\[
u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x).
\]  \quad (3)

(c) Work-out the piecewise-defined Green’s function for problem (1) via the delta function and jump conditions.

(d) Consider the Green’s function to be a function of \( x \) with \( s \) as a fixed parameter, \( G(x) \). Calculate the coefficients in the eigenfunction expansion of the piecewise-defined Green’s function obtained in part (c) using the usual orthogonal projection approach,

\[
G(x) = \sum_{k=1}^{\infty} g_k \phi_k(x) \quad \text{with} \quad g_k = g_k(s)
\]  \quad (4)

(e) What is the relation between \( c_k \) from (b) and \( g_k(s) \) from (d)?

“Mercer’s theorem” states that the piecewise-defined version of the Green’s function is the same as the bilinear eigenfunction expansion form. This problem (if you got it to work out) verifies this for the ODE BVP (1).