Math 553: Asymptotics and Perturbation Methods — Fall 2024

Problem Set 1 — **Assigned Fri Aug 30** — **Due Sat Sep 7**

Introduction to Asymptotics

-2. Homework policy: Homework is to be submitted via `www.gradescope.com`.

You may discuss progress on problems with other students, but your final work must be written up independently. Show enough steps for me to be able to follow your solution process.
Unexcused late homeworks will be penalized. Any extensions or excuses must be requested before the due date.

-1. (Computer-aided algebra (CAA) policy) Use of a computer algebra program (`Maple`, `Mathematica`, etc) is recommended for solving some of the more algebraically intensive problems on various homeworks. If you work out problems with computer-aided algebra, you must turn in a copy of the worksheet you used to generate your solution (pdf of commands and results).

0. Reading: Bender and Orszag is not the ideal book for this introductory background, but here are some relevant pages: pp. 123–127. For further background, see Miller (Chap 1) or Holmes (Chap 1).

1. Powers of ϵ are gauge functions in many problems, $\delta_n = \epsilon^n$. In some problems "*exponentially small terms*" (e.s.t.) also arise, like $\delta_* = e^{-1/\epsilon}$ for $\epsilon \to 0^+$. The fact that EST's are smaller than all powers, $\delta_* \ll \delta_n$, is an important result used in many problems.

   (a) Try proving this using $\lim_{\epsilon\to 0^+} \delta_*(\epsilon)/\delta_n(\epsilon)$ via L'Hopital's rule. Does this work?

   (b) Show that you can work this out using: (a) $\epsilon^n = e^{n\ln\epsilon}$, and (b) $\epsilon = 1/\lambda$ with $\lambda \to \infty$.

   (c) On the other hand, logarithms are "weakly large": show that $1 \ll \ln(\lambda) \ll \lambda^n$ for any $n > 0$ for $\lambda \to \infty$ (so $|\ln(\epsilon)| = \ln(1/\epsilon) \ll \epsilon^{-n}$ for any $n > 0$).

2. Consider the behavior of the function defined by the integral $f(x) = \displaystyle\int_0^\infty \frac{e^{-3t}}{1+5xt}\,dt$ for $x \to 0$.

   (a) Calculate the n^{th} derivative at $x = 0$, $f^{(n)}(0)$, for $n = 0, 1, 2, \cdots$ (general case).
   Hint: Leibniz's rule and $\int_0^\infty t^k e^{-st}\,dt = k!/s^{k+1}$.

   (b) Write the Taylor series for $f(x)$ expanded at $x = 0$. Show that this is a divergent series with the radius of convergence being zero.

   (c) A finite radius of convergence is associated with the occurrence of a singularity of the function limiting the radius of the largest disk of analytic behavior that could be drawn around the expansion point. $f(x)$ is finite for $x \ge 0$; explain why is the integral singular for all negative x.

   (d) Let $S_N(x) = \sum_{n=0}^N f_n x^n$ be the finite-truncation of the Taylor series. Let $|f_{N+1}x^{N+1}| = |f_N x^N|$ be the truncation condition to determine $N = N_*(x)$ at a given x value.

   (e) For $x = 1/10$, compute a semilog plot of $|S_N(x) - f(x)|$ vs. N for $N = 1, 2, \cdots, 10$ to show that N_* is close to the optimal asymptotic truncation. (Hint: Use Maple or etc for the numerical value of $f(1/10) \approx 0.?????5258467$.)

3. Let $g(y) = \exp(-(x-y)^2/(4t))$ with x, t being fixed constants. Determine the first four non-zero terms in the expansion of g for $y \to 0$. Let $x = \eta\sqrt{4t}$ to re-write the answer in terms of (η, y, t).

   The above expansion is one part of the justification to show that the solution of the initial-value problem for the heat equation, $u(x,t) = \int_{-\infty}^\infty f(y)e^{-(x-y)^2/(4t)}\,dy/\sqrt{4\pi t}$, can be written as an asymptotic expansion for $t \to \infty$ in the form:

$$u \sim \frac{e^{-\eta^2}}{\sqrt{4\pi t}}\left[\int_{-\infty}^\infty f(y)\,dy + \frac{\eta}{t^{1/2}}\int_{-\infty}^\infty y f(y)\,dy + \frac{2\eta^2-1}{4t}\int_{-\infty}^\infty y^2 f(y)\,dy + \frac{h_3(\eta)}{t^{3/2}}\int_{-\infty}^\infty y^3 f(y)\,dy + \cdots\right]$$

   Determine the $\boxed{h_3(\eta)}$ coefficient function.