Math 553: Asymptotics and Perturbation Methods Fall 2024

Problem Set 2 **Assigned Fri Sep 6** **Due Sat Sep 14**

Regular and Singular Perturbations Problems

0. Reading: Bender and Orszag is not the ideal book for this introductory background, but here are some relevant pages: pp. 317–325. For further background, see Hinch (Chaps 1, 2).

1. Consider the equation $(x - 4/\epsilon)^3 = 2x - 17$ in the limit $\epsilon \to 0$.

   (a) Solve by iteration to determine first three terms in the expansion of the positive real-valued solution $x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2$. Hint: You can determine δ_0 without expanding the cubic term.

   (b) To consider the accuracy of your expansion compared to the exact solution (call it $x(\epsilon)$), how many terms in the expansion must you keep so that the error vanishes in the limit, $|x(\epsilon) - x_{\text{AE}}| \to 0$ as $\epsilon \to 0$. Describe the accuracy of the leading order estimate $x \sim \delta_0 x_0$.

2. Consider the equation $\epsilon^{14}x^4 - 6\epsilon^5x^3 - 24\epsilon^2x^2 + 30\epsilon^3x + 120 + 15\epsilon + 60\epsilon^3 = 0$ in the limit $\epsilon \to 0$.

   (a) Show that there are no regular solutions.

   (b) Determine the δ_0's for all possible dominant balances and identify which of these are distinguished limits (from consistent dominant balances). Hint: Before you start, consider why you will only need to consider $\binom{5}{2} = 10$ options rather than $\binom{7}{2} = 21$.

   (c) Determine the leading order nontrivial term in the expansion of each of the four solutions.

3. Consider the system of equations for $\epsilon \to 0$: $\begin{cases} \epsilon x - 4y = 1 \\ \epsilon^2 x + y = 2 \end{cases}$

   This system has a unique solution for every $\epsilon > 0$, but DO NOT solve it directly. Explain why setting $\epsilon = 0$ does not lead to an acceptable leading order solution. Rescale the solution as $x = \delta(\epsilon)X(\epsilon)$ and $y = \sigma(\epsilon)Y(\epsilon)$ with $X, Y = O(1)$. Determine the dominant balance. Note that for systems of equations, the dominant balance may occur within a single equation with the other terms and equations being subdominant. Determine the first two terms in the expansions of $x(\epsilon), y(\epsilon)$.

4. Consider the matrix $\mathbf{A}(\epsilon) = \begin{pmatrix} e^{-3\epsilon} & -5 + 3\epsilon \\ -2 - 7\epsilon & -2\cos(8\epsilon) \end{pmatrix}$.

   For $\epsilon \to 0$, use matrix perturbation theory to find the first two terms in the expansion of each eigenvalue. (DO NOT try to expand the determinant $|\mathbf{A}(\epsilon) - \lambda\mathbf{I}| = 0$)

5. Separation of variables for solutions to the heat equation subject to Robin boundary conditions produces equations for the eigenvalues in the form

   $$\tan(\lambda) = -\lambda$$

   Introduce an ϵ and obtain the first three terms in the expansion of λ for $\lambda \to \infty$ using the following steps: Re-write the problem as $\sin(\lambda)/\lambda = -\cos(\lambda)$, consider $\epsilon \sin(\lambda)/\lambda = -\cos(\lambda)$ with $\epsilon \to 0$ and show your terms are ordered, $\lambda_0 \gg \lambda_1 \gg \lambda_2$, with respect to some limiting parameter, even when you restore $\epsilon = 1$.
   Reading B+O pages 319–321, 324-325, and problem 7.8 on page 361 may be helpful.
   Note that the numerical values for the series of λ starts with $\lambda = \{2.0287, 4.9131, 7.9786, \cdots\}$ – the asymptotic approximation does very well even when λ isn't very large.
   Hint: You should be able to work out λ_0, λ_1 by hand, but you may appreciate CAA for getting λ_2.