Math 553: Asymptotics and Perturbation Methods Fall 2024

Problem Set 4 **Assigned Fri Sep 20** **Due Sat Sep 28**

Watson's lemma

0. $\boxed{\text{Reading}}$: Bender and Orszag, section 6.4.

1. Bender and Orszag, page 309, problem 6.26a,b.

2. Bender and Orszag, page 309, problem 6.28b,c,d.

3. Bender and Orszag, page 309, problem 6.29.

4. Bender and Orszag, page 310, problem 6.42.
   Starting from the integral for $\Gamma(x)$, write up the whole Laplace's-method process for this moving-maximum problem. Be sure to show how many terms in each expansion are needed in order to obtain the $1/288$ coefficient for the $1/x^2$ term (computer -aided algebra recommended!) – namely, explain which integrand terms, with $b_{m,n}u^m x^n$, contribute to the value of the k term in the $\sum_k c_k/x^k$ sum in the answer for $\Gamma(x)$

5. Asymptotics of integrals is often used to help understand the behavior of solutions to PDE problems. The formula[1]

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right] f(y)\, dy$$

   gives the solution of the Dirichlet initial-boundary value problem for the heat equation on $0 \le x < \infty$ and $t \ge 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 0 \qquad u(x,0) = f(x)\,.$$

   If the initial condition function, $f(x)$, does not satisfy the boundary condition at the origin ($u(0,t) = 0$), i.e. $f(0) \neq 0$, then the solution $u(x,t)$ must work out what to do with the discontinuity between these conditions. If you plug-in $x = 0$ into the above integral, it reduces to $u(0,t) = 0$ for all times. So at $x = 0$ the initial condition $u(0,0) = f(0)$ is not really satisfied.
   Lets consider the specific case $f(x) = e^{-x}$.

   For $x > 0$ the solution u will be positive and $u \to 0$ for $x \to \infty$, with $u = 0$ at $x = 0$. There will be a maximum of u at some x_* at each time. Find the leading order behavior of $x_*(t)$ for $t \to 0$.

   Hints: Let $\lambda = 1/(4t) \to \infty$. To find x_* you will have to do two Laplace-type integrals and solve an algebraic equation.

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[1] The formula can be obtained from the method of images