Math 553: Asymptotics and Perturbation Methods Fall 2024

Problem Set 6 **Assigned Fri Oct 4** **Due** **Sat Oct 19**

The method of steepest descents for integrals

0. Reading: Bender and Orszag, section 6.6

1. Recall the definition of the Airy function given in class as a complex contour integral,

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_C e^{xt - t^3/3}\,dt$$

where C was any contour starting from the descent region in quadrant III and ending in the descent region in quadrant II.
Find the **first three** non-trivial terms[1] in the expansion of $\mathrm{Ai}(x)$ as $x \to 0$ using the following steps:

   (a) Write $\mathrm{Ai}(x)$ with curve C being constructed from the rays $\theta = 2\pi/3$ and $\theta = 4\pi/3$.

   (b) Write the integrals for the n^{th} derivative of $\mathrm{Ai}(x)$ at $x = 0$ for general $n = 0, 1, 2, \cdots$ (Taylor series coefficients)

   (c) Write the first three non-zero terms using only $\Gamma(2/3)$ and other constants (not other Γ's). Hint: $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and see page 569.

2. For the linearized KdV equation for dispersive water waves,

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad \text{IC: } u(x,0) = e^{-|x|} \qquad -\infty < x < \infty$$

   (a) Determine the Fourier transform of the initial condition, so the solution of the PDE can be written as

$$u(x,t) = \int_{-\infty}^{\infty} C(k) e^{ikx - ik^3 t}\,dk$$

   Hint: Do you need to break-up this calculation into cases for $x \gtrless 0$ or $k \gtrless 0$ or can these get re-combined?

   (b) Find the leading order asymptotics of the solution as (i) $x \to +\infty$ and (ii) $x \to -\infty$.
   Draw all contours used and show all work needed to justify your answers.
   Hint: There will be contributions from residues of pole singularities.

   (c) The results from (b) are good for large $|x|$, but they are not valid near the origin ($x \to 0$); the solution does not really blow-up. What is the result for $u(0,t)$ you can derive for $t \to \infty$ using the method of stationary phase? [2] This breaks-down too, what is the value of the true solution at $u(0,0)$?

3. Consider the differential equation for $y(x)$ on $0 \le x < \infty$:

$$\frac{d^4 y}{dx^4} = y + \frac{x}{4}\frac{dy}{dx}$$

   (a) Express the solutions in terms of a generalized Laplace transform integral: $y(x) = \int_C Y(t) e^{xt}\,dt$. Find $Y(t)$ and determine the descent regions to eliminate boundary terms.

   (b) Pick contours to provide **four** linearly independent solutions for this fourth-order ODE.

   (c) Apply the method of steepest descents to obtain the leading order behavior for each of the four solutions as $x \to +\infty$. Hint: One grows, three decay as $x \to \infty$.

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[1] This is not a steepest descents problem (it is mostly a HW#4-type problem), but it uses some of the same elements: choice of contours, parametrizing contour integrals and evaluating Laplace-type integrals.

[2] Often PDE solutions are studied in the limit $(x,t) \to \infty$ with $x = vt$ for fixed $v = O(1)$ with different cases for v in $-\infty < v < \infty$ in the limit of $t \to \infty$.