

## WKB and WKB with turning points: connection problems

0. Reading: B&O Sections 10.1, 10.2, 10.4, and 10.5. Especially 10.5.

1. Consider the limit of large eigenvalues for the Sturm-Liouville eigenvalue equation<sup>1</sup>

$$\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + (q(x) + \lambda r(x)) y = 0$$

with  $p(x) > 0$  and  $r(x) > 0$ .

(a) Convert into normal (Schrodinger equation) form  $d^2u/dx^2 = Q(x, \lambda)u$ .<sup>2</sup>

(b) Let  $u(x) = \exp(S(x)/\delta)$  and consider the limit  $\lambda \rightarrow +\infty$ . Find  $\delta(\lambda)$  and expand out  $S(x)$  far enough to include the first dependence on  $q(x)$ .

(c) Use (b) to write an asymptotic approximation of the leading order general solution  $y(x)$ . Simplify as much as possible.

2. Consider the problem on  $-1 \leq x < \infty$ ,<sup>3</sup>

$$\frac{d^2y}{dx^2} - \lambda^2 x(x^2 + 1)^2 y = 0 \quad y(-1) = 0 \quad y(x \rightarrow \infty) \rightarrow 0$$

Find the eigenfunctions and eigenvalues for  $\lambda \rightarrow \infty$ .

3. Bender and Orszag, page 542, problem 10.36.<sup>4</sup> Determine the eigensolutions  $\{y_n(x), E_n\}$  for  $E \rightarrow \infty$ .

Hints:

(a) Use the WKB solution formula with  $Q(x)$  where it can be applied.

(b) Use steepest descents asymptotics for Ai, Bi where they apply (see BO p. 508, 529 for the leading order terms)

(c) This problem is on a finite interval, so the formulas worked out in section 10.5 do not give the solution (they work for solutions on the whole real line only!). The same general approach applies, but you will not be able to get rid of some of the terms in the “outer” solutions.

(d) Consider two cases: (i) even solutions  $y(-x) = y(x)$ , (ii) odd solutions  $y(-x) = -y(x)$  (why these two cases are sufficient?). Use this to reduce the problem to something solvable on  $0 \leq x \leq \pi$ . (You should work out final answers for both cases.)

(e) Careful choices about how to write your solution in terms of applying boundary conditions, and limits of integration, can possibly reduce the algebra involved to only about 6 pages, for example

$$y'' + y = 0, \quad y(3) = 0 \quad \rightarrow \quad y(x) = A \sin(x - 3).$$

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<sup>1</sup>No turning points, just a basic WKB question

<sup>2</sup>Note that  $Q$  contains the expansion parameter.

<sup>3</sup>A warm-up problem

<sup>4</sup>The main challenge.