Math 553: Asymptotics and Perturbation Methods Fall 2024

Problem Set 9 **Assigned Fri Nov 8** **Due Sat Nov 16**

Matched asymptotic expansions and boundary layer problems

0. Reading: Lecture notes 20-22, B&O Chapter 9 (primarily sections 9.1-9.3).

1. Consider the problem for $y(x)$ on $0 \le x \le 1$ with $\epsilon \to 0$:

$$\epsilon^3 y'' - y' + 10\,\epsilon\, e^{2x} = 0 \qquad y(0) = 7\epsilon^2 \qquad y(1) = 3\epsilon\,.$$

This problem is mostly about the scaling of the solution amplitude (the ϵ^β factors), so let me simplify things a little by telling you that the solution has a boundary layer (only) at $x_* = 1$.

(a) Determine the leading order outer solution.

(b) Determine the leading order inner solution.

(c) Determine the boundary layer correction and write the uniformly valid solution.

2. Use matched asymptotics to determine the leading-order representation of the unique solution on $0 \le x \le 1$ of

$$\epsilon y'' - y' + (y')^3 = 0, \qquad \epsilon \to 0$$
$$y(0) = 0, \qquad y(1) = 1/2$$

Determine all of the possible distinguished limits and layer positions x_* to show that the solution must be unique.

Hint: You do not need to solve the inner problem, but you need to understand the properties of its solns (i.e. oscillatory or monotone or ???)

3. Consider the differential equation

$$\frac{d^4y}{dx^4} + x\frac{dy}{dx} - y = 0 \qquad \text{on } 0 \le x \le L$$

in the limit $L \to \infty$:

(a) Let $y(x) = u(z)$ with $x = Lz$ to write the problem on the domain $0 \le z \le 1$ as a singular perturbation problem for $u(z)$ in terms of $\epsilon = 1/L \to 0$.

For the following two sets of boundary conditions (parts b,c), construct a uniformly valid solution of the problem from part (a) using the following steps (i-iv):

(i) Show that the entire expansion of the outer solution is given by the leading order term, $u_0(z)$

(ii) Determine the first two terms in the expansion of the inner solution, $U(Z) \sim U_0(Z) + \delta U_1(Z)$,

(iii) Use higher-order matching to confirm the matching of the inner and outer solutions,

(iv) Write the uniformly valid solution as the sum of the outer and inner solutions minus the overlap terms from matching. (Maple/Mathematica are recommended since $U_1(Z)$ has several terms from the homogeneous and particular solutions.)

(b) $u(0) = 0 \qquad u'(0) = 1 \qquad u'(1) = 0 \qquad u''(1) = 0.$
This is called a *"corner layer"* because the boundary layer does not change the value of the solution to leading order, but there is a dramatic change with respect to the derivative of u.

(c) $u(0) = 0 \qquad u'(0) = 1 \qquad u(1) = 0 \qquad u'(1) = 0.$
From the leading order uniform solution it might look like $u'(1) = 0$ is not being satisfied, but this is resolved by including $U_1(Z)$.

(continued)

4. Find the values of $y(0)$ and $y'(0)$ to leading order where $y(x)$ satisfies

$$(x + \epsilon y)y' + y = 1, \qquad \epsilon \to 0$$

$$0 \le x \le 1, \qquad y(1) = 2$$

The solution is singular in the limit $\epsilon \to 0$, but is finite for any $\epsilon > 0$. Find the leading order outer solution then consider matching it to an inner solution to get a relation between α, β. Then use dominant balance in the ODE.

Hint: Review Example 1 from L22, the `ODE-1st.pdf` or `LCC-CE` summary sheets may also be helpful.

5. Find the leading-order uniformly-valid asymptotic solution on $0 \le x \le 1$ of

$$\epsilon^2 y'' + \left(x^2 + 2x^3\right) y' - \left(2x^2 + \sqrt{\epsilon}\right) y = 0, \qquad \epsilon \to 0$$

$$y(0) = 1, \qquad y(1) = 1$$

This problem has an outer solution ($\alpha = 0$) and two different distinguished limits giving a boundary layer within a boundary layer (an inner solution and an inner-inner solution) (both with $\alpha > 0$), this is called a *"triple deck"* (or "nested BL") problem.

Hint: For the matching and application of the boundary condition: apply the boundary condition to the inner-inner solution, match that solution to the inner solution, then match the inner solution to the outer solution. Pick some notation to clearly distinguish your inner $Y(X)$ from your inner-inner soln $\mathfrak{Y}(\mathfrak{X})$.

6. Use matched asymptotic expansions to write the two-term uniform asymptotic solution, accurate to up to $O(\epsilon^2)$ errors ($y = y_0 + \epsilon y_1 + O(\epsilon^2)$), for the solution on $0 \le x \le 1$ of

$$x^3 \frac{dy}{dx} = \epsilon((1+\epsilon)x + 2\epsilon^2)y^2 \qquad y(1) = 1 - \epsilon \qquad \epsilon \to 0$$

Note: From the ODE with any ϵ, at $x = 0$ it must be true that $y(0) = 0$. This is incompatible with the regular outer solution and needs a triple deck to resolve the behavior at $x_* = 0$. Use intermediate variables to do the matching.

Hint: You will need the first two terms (Y_0, Y_1) in the outer and inner expansions, but only Y_0 for the inner-inner solution. And $\beta \ne 0$ for the inner-inner!

[from Perturbation Methods in Fluid Mechanics, M. Van Dyke, 1975]
