Math 553: Asymptotics and Perturbation Methods Fall 2024

Problem Set 10 **Assigned Fri Nov 15** **Due Mon Nov 25**

Perturbation methods for oscillator problems (The last HW)

0. Reading: Lecture notes 23, 24 and B&O: Chapter 11.

1. Use multiple-scales to obtain the leading order solution of

$$\frac{d^2y}{dt^2} + 9y = -8\epsilon \sin\left(6t + \tfrac{\pi}{6}\right)\frac{dy}{dt} \qquad y(0) = 0 \qquad y'(0) = 12$$

   (a) Writing $Y_0(t,T) = A(T)\sin(\omega_0 t) + B(T)\cos(\omega_0 t)$, determine the amplitude ODE's, $dA/dT = F(A,B)$ and $dB/dT = G(A,B)$.

   (b) Solve the amplitude equations and use the initial conditions to determine $y(t)$ to leading order. Is its amplitude growing or decaying?

2. Solve for the leading order general solution $y(t)$ as $\epsilon \to 0$ of

$$\frac{d^2y}{dt^2} + y = \epsilon y^2$$

   (a) First use a naive regular expansion and expand out to $y(t) \sim y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t)$ to observe what combination of ϵ, t selects the slow timescale T, then use the method of multiple scales to re-solve the problem and derive the amplitude equations for $A(T), B(T)$ that determine the leading order solution $Y_0(t,T) = A(T)\sin(t) + B(T)\cos(t)$.
   Hint: Use of Maple/Mathematica for algebraic assistance may be helpful.

   (b) This problem can also be solved using Poincare-Lindstedt, but NO, don't go through all of that work. Instead we can get those results by using the ODE's for $A(T), B(T)$ from (a): if $A(T) = R(T)\cos(\Theta(T))$ and $B(T) = R(T)\sin(\Theta(T))$, what are the ODE's for $R(T), \Theta(T)$?
   What does this tell you about the PL variable $\tau = (1 + \epsilon\sigma_1 + \epsilon^2\sigma_2)t$?

3. Use the method of multiple scales with $T = \epsilon t$ for

$$\frac{d^2y}{dt^2} + y = -\epsilon \tanh\left(\frac{1}{\epsilon}\frac{dy}{dt}\right), \qquad y(0) = 0, \qquad y'(0) = 1, \qquad \epsilon \to 0.$$

   (a) Show that the leading order solution can be written in the polar form

$$Y_0(t,T) = R(T)\sin(t + \Theta(T)).$$

   Relate the amplitude R and phase Θ to the coefficients A, B in $Y_0 = A(T)\sin(t) + B(T)\cos(t)$. What are the initial conditions for R, Θ?

   (b) Derive and solve the amplitude equations for $R(T)$ and $\Theta(T)$ to obtain the leading order solution $y(t) \sim Y_0(t,T)$.
   Hint: You will need to calculate some terms from a Fourier series. Write the series in terms of the variable $s = t + \Theta$ on $-\pi < s < \pi$, namely $\sum_k a_k \sin(ks) + b_k \cos(ks)$.
   Meta-Hint: Doing the asymptotics of the integrals with the tanh might not be easy – so go ahead and replace the $\tanh(\cdot)$ with the $\mathrm{sgn}(\cdot)$ step function to get an easier piecewise-defined integral that has $O(\epsilon^2)$ error from the exact tanh integral.

   (c) Your amplitude equation from (b) based on the step function approximation should show that the oscillations stops at a finite time. What is that time?
   Should this happen if you keep the correct "low-speed" ($y' = o(\epsilon)$) influence of the damping? What is the leading order version of the ODE in the "low-speed" distinguished limit?

(continued)

(d) This problem models the classic physics example of a mass on a spring on a table with friction given by Coulomb damping (a constant force proportional to weight, opposite to the direction of motion (velocity)), $My'' + Ky = -\mu M g\,\mathrm{sgn}(y')$.
This can be understood as a "hybrid dynamical system" with two piecewise modes of evolution:

$$\begin{cases} y_+'' + y_+ = -\epsilon & \text{when } y_+' > 0 \\ y_-'' + y_- = \epsilon & \text{when } y_-' < 0 \end{cases}$$

and the y_+ and y_- pieces smoothly patching together

$$y_+(t_*) = y_-(t_*) \qquad \text{at time } t_* \text{ when } y_+'(t_*) = y_-'(t_*) = 0.$$

Construct the solution $y(t)$ as a sequence of pieces $\{y_+, y_-, y_+, y_-, \cdots\}$ and show that the rate of decay of the amplitude is the same as given by the amplitude eqn in part (b).

4. Find the leading order <u>large amplitude</u> periodic solution of

$$\frac{d^2y}{dt^2} + y = \epsilon y^2 + \epsilon \cos t \qquad y'(0) = 0$$

Hints: Let $y(t) = \epsilon^\beta Y(t, T)$ with $T = \epsilon^\alpha t$ and $Y \sim Y_0 + \epsilon^\gamma Y_1 + \epsilon^{2\gamma} Y_2$. To find α, β, γ you'll need to look at the equations up to $O(\epsilon^{\beta+2\gamma})$. To get Y_0 you'll need to use Y_0, Y_1 in the equation for Y_2.