Math 577: Mathematical Modeling — Spring 2025

Problem Set 1 — **Assigned Wed Jan 15** — **Due Sat Jan 25**

Dimensional analysis, scaling, and self-similar solutions

0. **Reading**: Logan, Chapter 1: Section 1.1.1–1.1.3 and note that similarity solutions of PDE's are described on pages 19-21 (Example 1.9).

1. Logan, Section 1.1.4 Exercises, page 29: Problem 13. ("Did you ever wonder how fast....")

2. The fluid dynamics problem of how water drips from a faucet depends on several quantities: the acceleration due to gravity g [m/s^2], the density of water ρ [kg/m^3], the viscosity of water μ [kg/(m·s)], the surface tension of water σ [kg/s^2], the radius of the faucet nozzle R [m], and the speed of the water coming out of the faucet U [m/s].

   (a) Use the Buckingham Π theorem to determine the number of dimensionless parameters and write
   $$\Pi = g^A \rho^B \mu^C \sigma^D R^E U^F$$
   to determine the set of equations relating the exponents A-F.

   (b) Solve for A, B, C in terms of D, E, F. (Hint: Use Gaussian elimination on the eqns from (a))

   (c) The choice of dimensionless parameters to describe a problem is not unique. This problem can be described in terms of the set of "historically named" parameters (from engineering/fluid dynamics):
   $$\mathrm{Re} = \frac{\rho U R}{\mu} \qquad \mathrm{Bo} = \frac{\rho g R^2}{\sigma} \qquad \mathrm{Ca} = \frac{\mu U}{\sigma} \qquad \{\text{Reynolds, Bond, and Capillary numbers}\}$$
   Show that the $A - F$ values for this set of 3 parameters satisfy the equations from (b).

   (d) The problem can also be described in terms of the parameters:
   $$\mathrm{We} = \frac{\rho U^2 R}{\sigma} \qquad \mathrm{Oh} = \frac{\mu}{\sqrt{\rho R \sigma}} \qquad \mathrm{Ga} = \frac{\rho^2 g R^3}{\mu^2} \qquad \{\text{Weber, Ohnesorge, and Galileo numbers}\}$$
   Determine D, E, F for each parameter and show how these parameters can be expressed in terms of the parameters from (c).

3. A fictional industrial process operates at the following setting for system parameters: A [kg·mol/(m·s)], B [m·s^2/(kg·mol)], C [kg^3/m^3] and D [mol·s^3] (m=meters, s=seconds, kg=kilograms, mol=moles).

   (a) Count the number of given quantities and dimensional units. How many dimensionless parameters do you expect?

   (b) Carry out the linear algebra calculation to determine the form of the dimensionless parameter(s).

   (c) Due to fictional cutbacks, D has been reduced from $D = 10$ to $D = 1$, but the system should be kept operating in the same state. Explain how to do this by changing one other parameter at a time (i.e. change A but leave the same BC, or change B with same AC, change C with same AB).

4. (2022) Consider the PDE for $u(x,t)$ on $0 \le x < \infty$ with $t > 0$:
   $$t\frac{\partial u}{\partial t} = 4x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + 15t^3 \qquad \text{with the boundary condition at } x = 0\text{:} \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 7t$$

   (a) Determine α, β in the form of the similarity solution, $u(x,t) = t^{-\beta} f(\eta)$ with $\eta = x t^{\alpha}$.

   (b) Write the ODE and BC for $f(\eta)$. Simplify as much as possible.

   (c) Determine the solution $f(\eta)$ and then $u(x,t)$. (Hint: Inhomogeneous Cauchy Euler ODE)

(continued)

5. (2024) Consider the PDE for $u(x,t)$ on $0 \le x < \infty$ with $t > 0$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{x}{t^6}\, u^2 \frac{\partial u}{\partial x} \qquad \text{with BC:} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 7t^5 .$$

(a) This problem is scale invariant. Determine α, β in the form of the similarity solution, $u(x,t) = t^\alpha f(\eta)$ with $\eta = xt^\beta$.

(b) Write the ODE for $f(\eta)$. Simplify as much as possible.

(c) Show that the ODE for $f(\eta)$ is S-scale-invariant under the transformation $f = F\tilde{f}$ and $\eta = S\tilde{\eta}$ (with $F = S^\gamma$ and S being a free parameter).

(d) Show that the f-ODE reduces to an autonomous ODE for $g(s)$ where $f(\eta) = g(s)$ with $s = \ln(\eta)$. Simplify the g-ODE as much as possible.

6. The following nondimensional problem describes the evolution of the height ($z = h(r,t) \ge 0$) of an axisymmetric drop of viscous fluid (like honey) spreading on a dry flat surface ($z = 0$) due to the influence of gravity:

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rh^3 \frac{\partial h}{\partial r}\right) \qquad \text{on } 0 \le r < \infty$$

subject to the conditions:

$$\left.\frac{\partial h}{\partial r}\right|_{r=0} = 0 \quad \text{(symmetry)}, \qquad \iint h(r,t)\, dA = 324\pi \quad \text{(fixed volume)}.$$

(a) This problem is scale invariant; find α, β for its similarity solution, $h(r,t) = t^{-\beta} f(\eta)$ where $\eta = rt^\alpha$.

(b) The ODE for $f(\eta)$ can be written as $\quad -\beta f + \alpha\eta \dfrac{df}{d\eta} = \dfrac{1}{\eta}\dfrac{d}{d\eta}\left(\eta f^3 \dfrac{df}{d\eta}\right).$

If your values of α, β are correct, you can integrate both sides of the equation after multiplying across by η. Use the symmetry boundary condition to set the value of the first constant of integration. Then you can integrate again to get the solution in the form $[f(\eta)]^3 = A - B\eta^2$; <u>find B</u>.

Note that the mathematical form of this problem precludes the solution from going negative; this is called a *positive truncation* and means that $f(\eta)_+ = \max(f(\eta), 0)$. This feature is essential for part (c).

(c) The solution of this problem has a positive height on a finite range $0 \le r \le R(t)$ with $h(R(t), t) = 0$ defining a moving "edge" of the drop (also called the "contact line", "interface" or "moving boundary"). The surface is dry beyond the contact line (no fluid), so the solution is truncated to zero, $h \equiv 0$ for $r > R(t)$.

Use the integral condition on the volume to find A and complete determining the similarity solution $h(r,t)$ and write the moving boundary $R(t)$.

Hint: Note that $\int_0^\infty hr\, dr = \int_0^R hr\, dr$ since $h = 0$ for $r > R$, and find the value of the similarity variable, $\eta = S$, that corresponds to the interface $R(t)$.