Math 577: Mathematical Modeling — Spring 2025

Problem Set 2 — **Assigned Wed Jan 22** — **Due Sat Feb 1**

Nondimensionalization and Scaling

0. Reading: Logan section 1.2. The notes for Lecture 4 may also be particularly useful.

1. Logan, Section 1.2.3 Exercises, page 43: Problem 7 ("A rocket blasts off...").

2. (2012) Consider the dimensional problem for $U(X,T)$:

$$\frac{\partial U}{\partial T} + AU^2\frac{\partial U}{\partial X} + B\frac{\partial^4 U}{\partial X^4} = CX^2T^3 \qquad X \geq 0 \qquad T \geq 0$$

$$U_X(0,T) = 0, \qquad U^3(0,T) + DU_{XXX}(0,T) = E,$$

   where A, B, C, D, E are given positive constants.

   (a) X is measured in meters, T is measured in seconds, and U is a density function, measured in kg/m^3. Determine the units for A, B, C, D, E.

   (b) Determine characteristic scales $\underline{L}, \underline{T}, \underline{U}$ in terms of A, B, C, D, E in the dimensional scaling $U = \underline{U}\,u(x,t)$ with $X = \underline{L}\,x$, $T = \underline{T}\,t$ so that all of coefficients in the PDE are normalized. Write the complete nondimensionalized problem and identify the dimensionless parameters. (DO NOT attempt to solve)

3. (2014) Consider the dimensional problem for $X(T)$,

$$M\frac{d^2X}{dT^2} + B\left(\frac{dX}{dT}\right)^3 + KX^2 = AT^2,$$

   where A, B, K, M are given constants. Let $X = \underline{L}\,x(t)$ and $T = \underline{T}\,t$ with $\underline{L, T}$ being characteristic scales. There are <u>**four**</u> ways to select the characteristic scales, so that coefficients of three out of the four terms in the ODE are normalized.

   For each of these four options:

   (a) Determine $\underline{L, T}$ in terms of the given constants. Simplify as much as possible.

   (b) Write the scaled ODE and determine the remaining parameter in the problem. Simplify as much as possible.

   (c) Consider the limit $B \to \infty$. Do the normalized terms in the scaled ODE represent a consistent dominant balance in this limit? (Yes or No)

4. (2020) Consider the dimensional equations for solutions $X(T), Y(T)$,

$$\frac{dX}{dT} = AXY - BY^3 \qquad \frac{dY}{dT} = -CX^2 + DT$$

   where A, B, C, D are given positive dimensional constants.
   Let $X(T) = \underline{X}\,x(t)$, $Y(T) = \underline{Y}\,y(t)$, and $T = \underline{T}\,t$ with $\underline{X}, \underline{Y}, \underline{T}$ being characteristic concentration and time-scales.
   Select the characteristic scales to normalize <u>**all**</u> of the coefficients in <u>**the x-rate equation**</u> and as many as possible in the other equation.

   For each of the <u>three</u> ways to scale the equations:

   (a) Write $\underline{X}, \underline{Y}, \underline{T}$ and the single remaining dimensionless parameter in terms of the given quantities. Simplify your answer as much as possible.

   (b) Write the scaled y-ODE with the dimensionless parameter.

(continued)

5. Consider the dimensional problem for projectile motion on the surface of the Earth:

$$\frac{d^2X}{dT^2} = -\frac{GM}{(R+X)^2} \qquad X(0) = 3\,\mathrm{m} \qquad X'(0) = -5\,\mathrm{m/sec}$$

Let $X(T) = \underline{L}\,x(t)$ and $T = \underline{T}\,t$. Consider the "small Earth" limit with

$$R \to 0 \qquad M = O(1).$$

(a) Choose your characteristic scales for $\underline{L}, \underline{T}$ to normalize as many terms as possible and ensure that none of the scaled coefficients in the problem blow-up in the limit.

(b) Write the scaled (normalized) problem, identify all remaining dimensionless parameters.

(c) Identify the limiting small dimensionless parameter ($\epsilon \to 0$) and write the leading order problem (where the limiting parameter is set to zero).

Hint: For (a) there are three possible choices for the scaled problem, you can use any of them.

6. Consider the dimensional equations describing a system of chemical reactions for the concentrations of three chemicals (X, Y, Z):

$$\begin{aligned} \frac{dX}{dT} &= A - BY & X(0) &= X_0 \\ \frac{dY}{dT} &= CX - DZ & Y(0) &= Y_0 \\ \frac{dZ}{dT} &= EY - FY^2 + GY^3 - HZ & Z(0) &= Z_0 \end{aligned} \tag{1}$$

where A, B, C, D, E, F, G, H and the initial conditions X_0, Y_0, Z_0 are given dimensional constants. By scaling X, Y, Z, T (i.e. $X(T) = \underline{X}\,x(t)$ and similarly for the others) show that these equations can be non-dimensionalized to the form:

$$\begin{aligned} \frac{dx}{dt} &= \alpha - y & x(0) &= \mu \\ \beta\frac{dy}{dt} &= x - z & y(0) &= \sigma \\ \gamma\frac{dz}{dt} &= y - y^2 + \tfrac{1}{3}\delta y^3 - z & z(0) &= \omega \end{aligned} \tag{2}$$

(a) Determine the characteristic scalings $\underline{X}, \underline{Y}, \underline{Z}, \underline{T}$ and the dimensionless parameters (Greek letters) in terms of A–H and X_0, Y_0, Z_0.

(b) If $\gamma = 0$ in (2), show that the problem reduces to a system of two ODE's for x and y: $dx/dt = p(x, y)$ and $dy/dt = q(x, y)$. Find the condition that the initial concentrations μ, σ, ω must satisfy in order to avoid a contradiction for this reduced problem.

(c) If $\beta = 0$ in (2), show that the problem can be reduced to a single ODE for y: $dy/dt = r(y)$. Find the two conditions that the initial concentrations μ, σ, ω must satisfy in order to avoid a contradiction for this reduced problem.

Hint: For parts (b,c), start each from the system in form (2), you will not need anything from part (a).