Math 577: Mathematical Modeling Spring 2025

Problem Set 3 **Assigned Wed Jan 29** **Due Sat Feb 8**

Regular and Singular Perturbation Problems

-1. Computer-Assisted Algebra Policy: You may always use Maple or Mathematica (or other computer software) to help check your work on problem sets. For problems marked as "Computer-Algebra Recommended" (CAR), parts of the algebra may be somewhat long and tedious to work out by hand, so you can use computer-based assistance to carry out the work. For these problems, you must include copies of the PDF pages of the complete steps and computer-generated solutions. For other problems (Non-CAR), you should write out your solution process by hand.

0. Reading: Logan Sections 3.1, 3.2, and notes from Lectures 4-6.
Also see the `L05-demo.mws`, `L05-expansion.mws` and `L05-iteration.mws` (expansion and iteration methods for the projectile motion example) files posted on Canvas for a guide on how to do the algebra for perturbation expansions via `Maple`.

1. (Computer-Algebra Recommended) Consider the problem for the vertical motion of a projectile:

$$\frac{d^2x}{dt^2} = -\frac{1}{(1+\epsilon x)^2}, \qquad x(0) = 2, \qquad x'(0) = \tfrac{2}{3}\epsilon \qquad \text{with } \epsilon \to 0.$$

(a) Obtain the first three terms in the expansion of the solution, $x \sim x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t)$.
Note: the initial height is $O(1)$ but the initial speed is slow, $O(\epsilon)$.

(b) Let T be the time when the projectile lands on the ground, $x(T) = 0$. Use (a) to determine $T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3)$. Is the time to land longer or shorter compared to landing on a "flat Earth"?[1]
Note: if you don't keep enough terms in the AE's for x and T your answers for T_1, T_2 may not be correct.

(c) The solution from Lecture 5[2] can be used with IC's $x(0) = \beta = O(1)$ and $x'(0) = \alpha = O(1)$ to yield

$$x^{\alpha\beta}(t) = \left(\beta + \alpha t - \frac{1}{2}t^2\right) + \epsilon\left(\beta t^2 + \frac{\alpha}{3}t^3 - \frac{1}{12}t^4\right) + O(\epsilon^2)$$

Show that this solution reproduces your answer from part (a) if $\beta = 2$ and $\alpha = \frac{2}{3}\epsilon$ are plugged in and everything is re-sorted by powers of ϵ^n.

(d) The above solution is good for initial velocities being $O(1)$ or smaller. It does not work for singular initial velocities. Solve the problem with initial conditions $x(0) = 1/2$ and $x'(0) = 1/\epsilon$ using a singular scaling[3] for the solution, $x(t) = X(t)/\epsilon$. Find $X(t) \sim X_0(t) + \epsilon X_1(t) + \epsilon^2 X_2(t)$.

2. Consider the limit of $\epsilon \to 0$ for the equation

$$\left(x - \frac{4}{\epsilon}\right)^3 = 9\epsilon^4 x^2 - 2\epsilon^2 x.$$

Solve for the single real-valued solution the by the iteration method to determine the first three terms in the asymptotic expansion: $x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2$.
Hint: To find δ_0 you don't need to expand the LHS. Consider options for the dominant balance...

(continued)

[1] The projectile is moving in a vertical line, so T isnt directly influenced by the curvature of the ground.
[2] See the Maple worksheets
[3] $x = \delta X$ with $\delta = 1/\epsilon$. Convince yourself that this scaling yields a non-singular problem for $X(t)$.

3. (2010) Consider the algebraic equation for x with $\epsilon \to 0$:

$$\epsilon^9 x^3 - 4\epsilon^4 x^2 - 28\epsilon x - 56 - 8\epsilon^3 = 0.$$

Does this problem have a regular solution? Determine the leading order nontrivial term in the expansion of each of the three solutions. Show your checks for all six possibilities[4] for the leading order dominant balance.

4. (2012) Consider the algebraic equation for x:

$$\epsilon x^3 - \frac{3x^2}{\epsilon^6} + \frac{21x}{\epsilon^8} + \frac{105}{\epsilon^5} = 0 \qquad \epsilon \to 0$$

Determine the leading order nontrivial term in the expansion of each of the three solutions.

5. Consider the problem for $x(t)$ on $t \geq 0$ with $\epsilon \to 0$:

$$\frac{dx}{dt} - 5\epsilon x^2 = 2e^{3\epsilon t} \qquad x(0) = 3e^{-\epsilon}.$$

(a) Find the first three terms in the expansion of the solution, $x(t) \sim x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t)$. (Computer-assisted algebra optional)

(b) Determine the range of times, $0 \leq t < O(\epsilon^\alpha)$, for which the terms in the expansion satisfy asymptotic ordering, i.e. $x_0 \gg \epsilon x_1 \gg \epsilon^2 x_2 \gg \cdots$.
Hint: $x_n \sim$ [the largest term in x_n].

(We will discuss how to fix related problems with competing limits ($\epsilon \to 0$ vs. $t \to \infty$) and the break-down of asymptotic ordering a bit later. For this problem, we see that the solution from (a) is valid only for a limit range of times.)

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[4] $6 = \binom{4}{2}$ combinations. Hint: Why don't you have to check $10 = \binom{5}{2}$ combinations? (Don't do 10, the other 4 can't work!)